# First Order Feynman-Kac Formula


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### Abstract

We study the parabolic integral kernel associated with the weighted Laplacian with a potential. For manifold with a pole we deduce formulas and estimates for the derivatives of the Feynman-Kac kernels and their logarithms, these are in terms of a 'Gaussian' term and the semi-classical bridge. Assumptions are on the Riemannian data.



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## 1 Introduction

Let $M$ be a complete connected smooth Riemannian manifold of dimension $n$ and $\Delta$ the Laplace-Beltrami operator with the sign convention that $\Delta$ is negative definite. Let $h$ be a smooth real valued function on $M$, define $\Delta^h = \Delta + 2L_{\nabla h}$. We study the solutions of the parabolic equation $\frac{\partial}{\partial t} f = (\frac{1}{2}\Delta^h - V)f$ where $t > 0$, $V$ a real valued bounded Hölder continuous function on $M$, and $\lim_{t \downarrow 0} f(t,x) = f(x)$. Without loss of generality we may assume that $V \geq 0$. Denote by $P_t^{h,V} f$ its solution, which is also denoted by $P_t^V$ if $h = 0$, by $P_t^h$ if $V = 0$, and by $P_t f$ if both $h$ and $V$ vanish. Their corresponding integral kernels are denoted by the lower case functions: $p_t^{h,V}$, $p_t^V$, $p_t^h$, and $p_t$. Also, if $Z$ is an additional $C^1$ vector field the notations $p_t^{h,Z,V}$ etc. will be used. All stochastic processes are assumed to have infinite life time.

Set $\mathbb{V}_t = e^{-\int_0^t V(x_s) ds}$ where $(x_t)$ is the canonical h-Brownian motion, by an h- Brownian motion we mean a strong Markov processes with generator $\frac{1}{2}\Delta^h$. We deduce the following first order Feynman-Kac formula

$$d(P_T^{h,V} f)(v) = \frac{1}{t}\mathbf{E}\left[\mathbb{V}_T f(x_T)\left(\int_0^t \langle W_s(v), u_s dB_s\rangle - \int_0^t \int_0^r dV(W_s(v))\, ds\, dr\right)\right],$$

where $u_s$ and $W_s$ are respectively the stochastic parallel and stochastic damped parallel translations along its sample paths, under the assumption that $\mathrm{Ric} - 2\,\mathrm{Hess}(h) \geq K$ where $K$ is a constant. This formula can be obtained by filtering out redundant noise from the equation in [EL94b, section 2]. However it is fairly easy to deduce it directly, avoiding some assumptions made in [EL94b]. From this formula, it is clear that $dP_t^{h,V} f$ is uniformly close to $dP_t^h f$. Using a technique rom D. Stroock we will in fact show that, for explicit constants $C_1(t,K)$ and $C_2(t,K)$ depending on $t$ and $K$, $f \geq 0$ with $P_t^{h,V} f(x_0) = 1$,

$$|\nabla P_t^{h,V} f|_{x_0} \leq \frac{1}{\sqrt{t}} \Big( 2C_1 \mathbf{E}\left[ (f(x_t)\mathbb{V}_t \log(f(x_t)\mathbb{V}_t))^+ \right] \Big)^{\frac{1}{2}} + t|\nabla V|_\infty C_2.$$

Choosing $f$ to be the Feynman-Kac kernel leads to the following estimates:

$$|\nabla \log p_t^{h,V}|_{x_0} \leq \frac{\sqrt{2C_1}}{\sqrt{t}} \left( \Big( \sup_{y\in M} \log \frac{p_t^h(y,y_0)}{p_{2t}^h(x_0,y_0)} + 2t(\sup V - \inf V) \Big)^+ \right)^{\frac{1}{2}} + t|\nabla V|_\infty C_2.$$

Together with estimates on $\sup_{y\in M} \log \frac{p_t(y,y_0)}{p_{2t}(x_0,y_0)}$, this leads to the estimate in [She91] which were extended in [Hsu99, ST98, Eng06], all for the case $V = 0$ and $h = 0$. The fundamental solutions of Schrödinger operators have been studied extensively, see e.g. [LY86, Dav88]. Feynman-Kac formula is an effective tool in a range of studies, see [Mol68, Aze74, Sim82, Fre85, Stu93, DvC00, LHB11, Gün12], and is used also in infinite dimensional analysis and for identifying Dirichlet spaces [AKR12, Gro93, Fit89]. Differentiating Feynman-Kac semigroups has been studied in connection with Hamiltonian-Jacobian equation, see [ET81, DPZ97, DP15].

If $M$ is a manifold with a pole $y_0$, by which we mean that the exponential map $\exp_{y_0}$ is a diffeomorphism, it makes sense to compare the Feynman-Kac kernel and its derivatives with the 'Gaussian kernel'. We do not make assumptions on uniform ellipticity of the operator; all assumptions will be on Riemannian data. Let $J_{y_0}$ denote the Jacobian determinant of the exponential map at $y_0$ and $\Phi(y) = \frac{1}{2} J_{y_0}^{\frac{1}{2}}(y) \Delta J_{y_0}^{-\frac{1}{2}}(y)$. The subscript $y_0$ will be omitted from time to time. For $T > 0$ fixed, a semi-classical bridge $\tilde{x}_s$ is a time dependent diffusion with generator $\frac{1}{2}\triangle + \nabla \log k_{T-s}(\cdot, y_0)$ where, for $d$ the Riemannian distance function,

$$k_t(x_0,y_0) := (2\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(x_0,y_0)}{2t}} J^{-\frac{1}{2}}(x_0).$$

The process $r_t = d(\tilde{x}_t, y_0)$ is the $n$-dimensional Bessel bridge. On $\mathbb{R}^n$, the semi-classical bridge agrees with the conditioned Brownian motion. In [ET81], K. D. Elworthy and A. Truman proved the following formula, whose consideration comes from classical mechanics and semi-classical limits,

$$p_T^V(x_0,y_0) = k_T(x_0,y_0)\mathbf{E}\left[ e^{\int_0^T (\Phi - V)(\tilde{x}_s)\,ds} \right]. \tag{1.1}$$

We give a simple proof for the following formula from, see [Wat88] where the methods are similar to that in [ET81, Elw82, ETW85].

$$p_T^{h,V}(x_0,y_0)=e^{h(y_0)-h(x_0)}k_T(x_0,y_0)\mathbf{E}\beta_T^h, \quad \forall x_0\in M,$$
$$\beta_T^h=\exp\left(\int_0^t\left(\Phi-V-\frac{1}{2}|\nabla h|^2-\frac{1}{2}\Delta h\right)(\tilde{x}_s)ds\right).$$

Our method is different, which also allows us to deduce a first order formula. The function $\Phi$ is bounded on manifolds of constant negative curvature. If $-\frac{1}{2}|\nabla h|^2-\frac{1}{2}\Delta h$ is bounded, the formula leads easily to a Gaussian upper bound for the integral kernel $p_t^h$. In this formula, an additional non-gradient type drift $Z$ is also allowed for which we follow a beautiful idea of Watling [Wat88]. The semi-classical bridge will be then be replaced by the semi-classical Riemannian bridge whose Markov generator is $\frac{1}{2}\triangle+\nabla\log k_{T-s}(\cdot,y_0)+\nabla S$, where

$$S(x)=\int_0^1\langle\dot{\gamma}(u),Z(\gamma(u))\rangle du,$$

for $\gamma:[0,1]\to M$ the unique geodesic from $x$ to $y_0$, representing the path average of the radial part of $Z$.

Let $\tilde{u}_t$ denote the solution to the stochastic differential equation (2.2) on the orthonormal frame bundle with $\tilde{u}_0\in\pi^{-1}(x_0)$ and set $\tilde{x}_t=\pi(\tilde{u}_t)$. We often need the condition that Ric $-2\,$Hess$(h)$ is bounded from below. Set $\rho^h=\inf_{|v|=1}\{Ric(v,v)-2\,\mathrm{Hess}\,h(v,v)\}$. Recall $k_t(x_0,y_0):=(2\pi t)^{-\frac{n}{2}}e^{-\frac{d^2(x_0,y_0)}{2t}}J_{y_0}^{-\frac{1}{2}}(x_0)$.

**Theorem** *Assume $y_0$ is a pole, $V\in C^{1,\alpha}\cap BC^1$, $\Phi$ and $f\in L_\infty$ and $\rho^h\ge K$. Then*

$$dp_T^{h,V}(\cdot,y_0)=\frac{1}{T}e^{h(y_0)-h(x_0)}k_T(x_0,y_0)\mathbf{E}\left[\beta_T^h\int_0^T\langle\tilde{W}_r(\cdot),\tilde{u}_r d\tilde{B}_r-(T-r)\nabla V dr\rangle\right]$$

*where $d\tilde{B}_r=dB_r+\tilde{u}_r^{-1}\nabla\log(e^{-h}k_{T-r})(\tilde{x}_r)\,dr$ and $\tilde{W}$ is the solution to (2.4).*

From this theorem we immediately see that, for an explicit constant $C(K,h,|\nabla\log J|_\infty)$, depending only on $K$, $h$, and $|\nabla\log J|_\infty$, the following estimate holds,

$$\begin{aligned}&|\nabla p_T^{h,V}(\cdot,y_0)|_{x_0}\\ &\le Ce^{h(y_0)-h(x_0)}\,k_t(x_0,y_0)\,|\beta_T^h|_\infty\left(\frac{d(x_0,y_0)}{T}+1+|\nabla h|_\infty+\frac{1}{\sqrt{T}}+T|dV|_\infty\right).\end{aligned}$$

Letting $Z_T=\frac{\beta_T^h}{\mathbf{E}(\beta_T^h)}$, we also have:

$$\left|\nabla\log p_T^{h,V}(\cdot,y_0)\right|_{x_0}\le C\,|Z_T|_{L_2(\Omega)}\left(\frac{d(x_0,y_0)}{T}+1+|\nabla h|_\infty+\frac{1}{\sqrt{T}}+T|dV|_\infty\right).$$

There is also a version of the above formula and estimates for Hölder continuous potential $V$, which however involves a term of the form $(\ln T)|V|_\infty$. Note that precise Gaussian estimates for heat kernels and their derivatives using semi-classical bridge were obtained by S. Aida [Aid01] for asymptotically flat Riemannian manifolds with a pole where the derivatives of $\log J$ up to order 4, the Riemmannian curvature and the derivative of the Ricci curvature are assumed to be bounded. Heat kernel formula for Schrödinger type operator acting on sections of vector bundles can be found in M. Ndumu [Ndu09] and M. Braverman [Bra98]. The study of the probability measure induced by the semi-classical bridge has been followed up in [Li16].

## 2 Preliminaries and First Order Feynman-Kac Formula

In this section we introduce the notation and the preliminary results. Denote by $BC^r$ the space of bounded $C^r$ functions on $M$ with bounded derivatives, $C^r_K$ its subspace of functions with compact supports and $C^{1,\alpha}$ the space of functions whose first derivatives are locally Hölder continuous.

The h-Brownian motion we use will be given by the canonical construction below. Let $\{B^i_t\}$ be a family of independent one-dimensional Brownian motions on a filtered probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$, set $B_t = (B^1_t, \dots, B^n_t)$. Let $\{H_i\}$ be canonical horizontal vector fields on the orthonormal frame bundle $OM$ of $M$, associated to an orthonormal basis of $\mathbb{R}^n$. The tilde sign over a vector field on $M$ indicates its horizontal lift to $OM$. If the $h$-Brownian motion is complete, which holds if Ric $-2\,$Hess$(h)$ is bounded from below, then the following stochastic differential equation (SDE) is complete,

$$du_t = \sum_{i=1}^n H_i(u_t) \circ dB^i_t + \widetilde{\nabla h}(u_t)dt. \tag{2.1}$$

Furthermore if $\pi$ is the projection from $OM$ to $M$, $x_t := \pi(u_t)$ is a h-Brownian motion on $M$ starting at $x_0 := \pi(u_0)$. If $h$ vanishes it is sufficient to assume that $\underline{\mathrm{Ric}}_x \geq -\alpha(r(x))$ where $\alpha$ grows at most quadratically and $\mathrm{Ric}_x$ is the Ricci curvature at $x \in M$ and $\underline{\mathrm{Ric}} = \inf_{v \in T_xM: |v|=1}\{\mathrm{Ric}_x(v,v)\}$. The corresponding equation,

$$d\tilde{u}_t = \sum_{i=1}^n H_i(\tilde{u}_t) \circ dB^i_t + \widetilde{\nabla h}(\tilde{u}_t)dt + \widetilde{\nabla \log k_{T-t}}(\tilde{u}_t)dt, \quad t < T, \tag{2.2}$$

gives rise to the semi-classical bridge in the same way, $\tilde{x}_t = \pi(\tilde{u}_t)$.

Let $\mathrm{Ric}^\sharp_x : T_xM \to T_xM$ be the linear map defined by $\langle \mathrm{Ric}^\sharp_x u, v\rangle = \mathrm{Ric}_x(u,v)$. Denote by $(W_t)$ and $(\tilde{W}_t)$ respectively the solutions to the following two equations, the first, along $(x_t)$, being

$$\frac{D}{dt}W_t = -\frac{1}{2}\mathrm{Ric}^{\#}_{x_t}(W_t) + \mathrm{Hess}(h)(W_t), \quad W_0 = \mathrm{id}_{T_{x_0}M} \tag{2.3}$$

and the second, along $(\tilde{x}_t)$, being

$$\frac{D}{dt}\tilde{W}_t = -\frac{1}{2}\mathrm{Ric}^{\#}_{\tilde{x}_t}(\tilde{W}_t) + \mathrm{Hess}(h)(\tilde{W}_t), \quad \tilde{W}_0 = \mathrm{id}_{T_{x_0}M}. \tag{2.4}$$

Here $\frac{D}{dt}W_t = u_t\frac{d}{dt}u_t^{-1}W_t$ and $\frac{D}{dt}\tilde{W}_t = \tilde{u}_t\frac{d}{dt}\tilde{u}_t^{-1}\tilde{W}_t$, and so the first equation, for example, is interpreted as follows: $u_t^{-1}W_t$ solves the equation

$$\frac{d}{dt}w_t = -\frac{1}{2}(u_t^{-1}\mathrm{Ric}^{\sharp}_{x_t}u_t)w_t + u_t{}^{-1}\,\mathrm{Hess}(h)(u_t w_t), \quad w_0 = \mathrm{id}_{\mathbb{R}^n}.$$

Throughout this section we assume that the $h$-Brownian motions do not explode.

If $h$ is a smooth real valued function on $M$ then $\Delta^h = (d+d^*)^2$ where $d^*$ is the $L^2$ adjoint of the exterior differential operator $d$ with respect to the measure $e^{2h}d\mathrm{vol}$ and the initial domain of $\Delta^h$ consists of smooth and compactly supported differential forms. Then $d+d^*$ and all its powers are essentially self-adjoint, [Li92, ch.2] see also [Li95], and we denote by the same notation their closures. Suppose that $V$ is bounded, then the operator $f \to Vf$ is $\Delta^h$ bounded and by the Kato-Rellich theorem, $\frac{1}{2}\Delta^h - V$ is self-adjoint on the domain of $\Delta^h$ and essentially self-adjoint on $C_K^\infty$, for more general potentials see [Ole93, Shu01]. By functional calculus $e^{-t(\frac{1}{2}\Delta^h - V)}$ is a strongly continuous contraction semi-group on $L_2 \cap \mathcal{B}_b$, where the $L^2$ space is defined by the weighted measure $e^{2h}dx$. Furthermore if $f \in L^2(M; e^{2h}dx)$ then $e^{-t(\frac{1}{2}\Delta^h - V)}f$ belongs to the domain of $\Delta^h$. By direct computation $\mathbf{E}\left(f(x_t)\mathbb{V}_t\right)$, where $\mathbb{V}_t = e^{-\int_0^t V(x_r)dr}$, is also a strongly continuous contraction semi-group on $L_2 \cap L_\infty$ with generator $\frac{1}{2}\Delta^h - V$ and core $C_K^\infty$. Consequently

$$e^{-t(\frac{1}{2}\Delta^h - V)}f = \mathbf{E}\left(f(x_t)\mathbb{V}_t\right), \qquad f \in L_2 \cap L_\infty.$$

Furthermore they solve the variation of constant formula:

$$g_t = P_t^h f + \int_0^t P_{t-s}^h(V g_s)\, ds,$$

and consequently they are $C^{1,2}$ functions and solve the parabolic equation. The solution measure has a density $p_t^{h,V}$ with respect to the volume measure. We need the following formulation for the Feynman-Kac formula.

**Lemma 2.1** *If $P_t^{h,V} f$ is a $C^{1,2}$ solution to the parabolic equation, then*

$$\mathbb{V}_s P_{t-s}^{h,V} f(x_s) = P_t^{h,V} f(x_0) + \int_0^s \mathbb{V}_r dP_{t-r}^{h,V} f(u_r dB_r), \quad 0 \le s \le t. \tag{2.5}$$

**Proof** By the assumption on the $h$-Brownian motion, $u_s$ exists for all time. We may therefore apply Itô's formula to $\mathbb{V}_s P^{h,V}_{t-s} f(x_s)$, using $d\mathbb{V}_s = -V(x_s)\mathbb{V}_s ds$ and $dx_s = u_s \circ dB_s + \nabla h(x_s) ds$ to obtain (2.5). □

For $k \in 0, 1, \cdots$ denote by $H^k$ the completion of $H^k_0$, where

$$H^k_0 = \left\{ f \in C^\infty : |f|^2_{H_k} = \sum_{j=0}^{k} |\nabla^{(j)} f|^2_{L^2} < \infty \right\},$$

under the norm $|\cdot|_{H^k}$. Denote by $\overline{C^\infty_K}^{H_k}$ the closure of $C^\infty_K$ under $|\cdot|_{H^k}$. Denote also by $d^\star$ the dual of $d$ in $L^2(M; dx)$, taking $h = 0$, the latter having initial domain $C^\infty_K$. Then the Laplace-Beltrami operator on functions is the closure of $-d^\star d$ and for any complete Riemannian manifold Dom$(d)$ $= \overline{C^\infty_K}^{H^1} = H^1$. For higher order derivatives the corresponding statements hold for manifolds with bounded geometry, see [Aub76]. For $k = 2$, $\overline{C^\infty_K}^{H^2} = H^2$ if the injectivity radius of $M$ is positive and if the Ricci curvature is bounded below, see [Heb99]. We avoid these assumptions.

Recall that $\rho^h(x) = \inf_{v \in ST_xM} \{\mathrm{Ric}(v, v) - 2\,\mathrm{Hess}(h)(v, v)\}$.

**Lemma 2.2** *Fix $T > 0$. Assume that $V$ is bounded with $V \in C^{1,\alpha}$. If $f \in L^2 \cap BC^1$ then for all $0 \le t < T$ and $v \in T_{x_0}M$ we have*

$$\begin{aligned} \mathbb{V}_t \cdot \left( dP^{h,V}_{T-t} f \right)(W_t(v)) = \; & dP^{h,V}_T f(v) + \int_0^t \mathbb{V}_s \cdot \left( \nabla dP^{h,V}_{T-s} f \right)(u_s dB_s, W_s(v)) \\ & + \int_0^t \mathbb{V}_s \cdot dV(W_s(v)) \cdot P^{h,V}_{T-s} f(x_s)\, ds. \end{aligned} \tag{2.6}$$

*If furthermore $|dV|$ is bounded and $\rho^h$ is bounded from below, then for all $t \in [0, T)$,*

$$\mathbf{E}\left[ \mathbb{V}_t (dP^{h,V}_{T-t} f)(W_t(v)) \right] = d(P^{h,V}_T f)(v) + \mathbf{E}\left[ \int_0^t \mathbb{V}_s dV(W_s(v)) P^{h,V}_{T-s} f(x_s)\, ds \right]. \tag{2.7}$$

**Proof** Since $V \in C^{1,\alpha}$, the solution $P^{h,V}_t f$ is three times differentiable in space and we may differentiate both sides of the parabolic equation. Since $P^{h,V}_t f \in$ Dom$(d)$, it follows that $d\Delta(P^{h,V}_t f) = \Delta^{1,h} d(P^{h,V}_t f)$, see [Li92, Chapter 2] or [Gaf59] for $h = 0$ case. Consider the function $(t, \alpha, (x, v)) \in [0, T] \times \mathbb{R}_+ \times TM \mapsto (\alpha, dP^{h,V}_{T-t} f(v))$ and apply Itô's formula to it and to the process $(t, \mathbb{V}_t, W_t(v))$ to obtain

$$\begin{aligned}
&\mathbb{V}_t dP^{h,V}_{T-t} f(W_t(v)) - dP^{h,V}_T f(v)\\
=\;& \int_0^t \mathbb{V}_s \nabla (dP^{h,V}_{T-s} f)(u_s dB_s, W_s(v)) + \int_0^t \nabla \mathbb{V}_s (dP^{h,V}_{T-s} f)(\nabla h(x_s), W_s(v)) ds\\
&+ \int_0^t \mathbb{V}_s \left( \frac{\partial}{\partial s} dP^{h,V}_{T-s} f \right) (W_s(v))\, ds + \frac{1}{2} \int_0^t \mathbb{V}_s \operatorname{tr} \nabla^2 (dP^{h,V}_{T-s} f)(W_s(v))\, ds\\
&+ \int_0^t \mathbb{V}_s dP^{h,V}_{T-s} f \left( \frac{D}{ds} W_s(v) \right) ds - \int_0^t V(x_s) \mathbb{V}_s dP^{h,V}_{T-s} f(W_s(v))\, ds.
\end{aligned} \tag{2.8}$$

Using Bochner's formula, $\Delta^{1,h} = \operatorname{trace} \nabla^2 + 2L_{\nabla h} - \mathrm{Ric}^\sharp$ for the Laplace-Beltrami operator $\Delta^1$ on differential 1-forms, the definition of the Lie derivative and equation (2.3), we thus have

$$\begin{aligned}
&\mathbb{V}_t dP^{h,V}_{T-t} f(W_t(v)) - dP^{h,V}_T f(v)\\
=&\int_0^t \mathbb{V}_s \nabla (dP^{h,V}_{T-s} f)(u_s dB_s, W_s(v)) - \int_0^t V(x_s) \mathbb{V}_s dP^{h,V}_{T-s} f(W_s(v))\, ds\\
&+ \int_0^t \mathbb{V}_s \left( \frac{\partial}{\partial s} dP^{h,V}_{T-s} f \right) (W_s(v))\, ds + \frac{1}{2} \int_0^t \mathbb{V}_s \Delta^{1,h} (dP^{h,V}_{T-s} f)(W_s(v))\, ds.
\end{aligned}$$

We can commute the time and space derivatives and also commute $\Delta^h$ with $d$ to obtain

$$\frac{\partial}{\partial s} d(P^{h,V}_{T-s} f) + \frac{1}{2} \Delta^{1,h} (dP^{h,V}_{T-s} f) = V d(P^{h,V}_{T-s} f) + (dV) P^{h,V}_s f.$$

By substituting into equation (2.8) we then obtain, after some cancellation, the required formula. □

**Lemma 2.3** *Assume that $\rho^h$ is bounded from below and $V$ is a bounded Hölder continuous function. Then for all $f \in L_\infty$ and $v \in T_{x_0} M$,*

$$\begin{aligned}
(dP^{h,V}_t f)(v) =& \frac{1}{t} \mathbf{E} \left[ f(x_t) \int_0^t \langle W_s(v), u_s dB_s \rangle \right]\\
&+ \mathbf{E} \left[ f(x_t) \left( \int_0^t e^{-\int_{t-s}^t V(x_u) du} \frac{V(x_{t-s})}{t-s} \int_0^{t-s} \langle W_r(v), u_r dB_r \rangle \right) ds \right].
\end{aligned}$$

**Proof** Let $f \in BC^1 \cap L^2$ and we first assume that $V$ belongs also to $BC^1$. We differentiate both sides of the variation of constants formula $P^{h,V}_t f = P^h_t f + \int_0^t P^h_{t-s} (V P^{h,V}_s f)\, ds$ to obtain for $v \in T_{x_0} M$,

$$dP^{h,V}_t f(v) = dP^h_t f(v) + \int_0^t dP^h_{t-s} (V P^{h,V}_s f)(v)\, ds$$

$$=dP_t^h f(v) + \int_0^t \frac{1}{t-s}\mathbf{E}\left[(VP_s^{h,V} f)(x_{t-s}) \int_0^{t-s} \langle W_r(v), u_r dB_r\rangle\right] ds.$$

Since the first two terms in the equation make sense, so does the last term, which by the standard Feynman-Kac formula, Lemma 2.1, and the Markov property, has the following expression:

$$\begin{aligned}&\mathbf{E}\left[(VP_s^{h,V} f)(x_{t-s}) \int_0^{t-s} \langle W_r(\cdot), u_r dB_r\rangle\right]\\ &= \mathbf{E}\left[V(x_{t-s})f(x_t)e^{-\int_{t-s}^t V(x_u)du} \int_0^{t-s} \langle W_r(\cdot), u_r dB_r\rangle\right].\end{aligned}$$

The formula follows from the corresponding formula for $P_t^h f$ which is well known and can be easily seen by multiply both sides of (2.5) by the martingale $\int_0^t \langle u_s dB_s, W_s(v)\rangle$:

$$\begin{aligned}\mathbf{E}\left(\int_0^t \langle u_s dB_w, W_s(v)\rangle P_{T-t}^h f(x_t)\right) &= \mathbf{E}\left(\int_0^t dP_{T-r}^h f(u_r dB_r) \int_0^t \langle u_s dB_s, W_s(v)\rangle\right)\\ &= \mathbf{E}\int_0^t dP_{T-r}^h f(W_r(v))dr = t\, dP_T^h f(v),\end{aligned}$$

The last identity follows from the second formula in Lemma 2.2. Then using the normalised geodesic $\sigma : [0,1] \to M$ connecting two points $x$ and $y$ in $M$ so that $f(x) - f(y) = \int_0^1 \frac{d}{ds}[f(\sigma(s))]ds$ and by the formula above, and the fact that $V$ is bounded, $|W_t|$ is bounded to see that $|dP_t^{h,V} f(x) - dP_t^{h,V} f(y)| \le C|f|_\infty d(x,y)$. In particular if $Q_t^{h,V}(x,\cdot)$ denotes the probability measure associated with $DP_t^{h,V} f$, then $|Q_t^{h,V}(x,M) - Q_t^{h,V}(y,M)| \le Cd(x,y)$. Thus the total variation norm of $|Q_t^{h,V}(x,\cdot) - Q_t^{h,V}(y,\cdot)| \le Cd(x,y)$ which means that the required formula holds for a bounded measurable function $f$. For a bounded Hölder continuous $V$ it is sufficient to approximate with regular functions. □

**Proposition 2.4** *Suppose that* Ric $-2\,$Hess$(h) \ge K$ *and that* $V \in C^{1,\alpha} \cap BC^1$. *Then for all* $0 \le t < T$, $v \in T_{x_0}M$ *and* $f \in L_\infty$, *(2.7) holds.*

**Proof** We prove the formula using equation (2.6). By the boundedness of $V$, $dV$ and $f$ it is clear that $\int_0^t \mathbb{V}_s dV(W_s(v)) P_{T-s}^{h,V} f(x_s)\, ds$ is bounded. Since $|W_t(v)|$ is bounded, we use Lemma 2.3 to conclude that $|d(P_{T-t}^{h,V} f| \in L_2$, and so does $\mathbb{V}_t d(P_{T-t}^{h,V} f)(W_t(v))$ which means that the stochastic integral appearing in equation (2.6) is $L^2$-bounded and therefore a true martingale with vanishing expectation. □

**Theorem 2.5** *Assume that* $V \in C^{1,\alpha} \cap BC^1$ *and* $f \in L_\infty$. *Suppose that* $\mathrm{Ric} - 2\,\mathrm{Hess}(h) \geq K$. *Then for all* $0 < t \leq T$ *and* $v \in T_{x_0}M$ *we have*

$$\begin{aligned} dP_T^{h,V} f(v) = &\ \frac{1}{t}\mathbf{E}\left[\mathbb{V}_T f(x_T) \int_0^t \langle W_s(v), u_s dB_s \rangle\right] \\ &- \frac{1}{t}\mathbf{E}\left[\mathbb{V}_T f(x_T) \int_0^t \int_0^r dV(W_s(v))\, ds\, dr\right]. \end{aligned}$$

**Proof** By Lemma 2.1 we have

$$\mathbb{V}_t P_{T-t}^{h,V} f(x_t) = P_T^{h,V} f(x_0) + \int_0^t \mathbb{V}_r d(P_{T-r}^{h,V} f)(u_r dB_r). \tag{2.9}$$

Next we multiply the above equation by the $L^2$ martingale $\int_0^t \langle u_s dB_s, W_s(v)\rangle$ and use Itô's isometry to obtain

$$\mathbf{E}\left[\mathbb{V}_t P_{T-t}^{h,V} f(x_t) \int_0^t \langle u_s dB_s, W_s(v)\rangle\right] = \mathbf{E}\left[\int_0^t \mathbb{V}_r d(P_{T-r}^{h,V} f)(W_r(v))\, dr\right]$$

using the fact that the last term in equation (2.9) is an $L^2$ martingale. We now apply equation (2.7) to deduce that for any $0 < t \leq T$,

$$\begin{aligned} dP_T^{h,V} f(v) = &\ \frac{1}{t}\mathbf{E}\left[\mathbb{V}_t P_{T-t}^{h,V} f(x_t) \int_0^t \langle u_r dB_r, W_r(v)\rangle\right] \\ &- \frac{1}{t}\mathbf{E}\left[\int_0^t \int_0^r \mathbb{V}_s dV(W_s(v)) P_{T-s}^{h,V} f(x_s)\, ds\, dr\right] \end{aligned} \tag{2.10}$$

and the rest follows from the Markov property. □

This extends the formula in [Bis84] for the logarithmic derivative of the heat kernel (on a compact manifold) proved using Malliavin calculus, the latter was extended to non-compact manifolds in [Li92, EL94b] by an elementary stochastic calculus. For manifolds whose second fundamental form of an isometric embedding satisfies suitable conditions, the formula can be deduced from that in [EL94b] using the techniques of filtering our redundant noise. Here we do not make any assumptions on the second fundamental form. See also [EL94a] and [LZ96] for reaction-diffusion equation on $\mathbb{R}^n$ and [DT01, Tha97].

For estimations we need the following lemma, which is [Str00, Lemma 6.45].

**Lemma 2.6** *Suppose* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space and* $\phi \geq 0$ *is a measurable function on* $\Omega$ *with* $\mathbf{E}[\phi] = 1$. *If* $\Psi$ *is a measurable function on* $\Omega$ *such that* $\phi\Psi$ *is integrable then*

$$\mathbf{E}[\phi\Psi] \leq \mathbf{E}[\phi \log \phi] + \log \mathbf{E}[\exp(\Psi)].$$

**Proposition 2.7** *Suppose that* Ric $-2\,$Hess$(h) \geq K$ *and* $V \in C^{1,\alpha} \cap BC^1$. *Then for a non-negative bounded measurable function* $f$ *with* $P_t^{h,V} f(x_0) = 1$ *we have*

$$|\nabla P_t^{h,V} f|_{x_0} \leq \frac{1}{\sqrt{t}}\Big(2C_1(t,K)\mathbf{E}\left[(f(x_t)\mathbb{V}_t \log(f(x_t)\mathbb{V}_t))^+\right]\Big)^{\frac{1}{2}} + t|\nabla V|_\infty C_2(t,K)$$

*for all* $t > 0$ *where*

$$C_1(t,K) := \frac{1-e^{-Kt}}{Kt}, \quad C_2(t,K) := \frac{2}{Kt}\left(1 + \left(\frac{e^{-Kt/2}-1}{Kt/2}\right)\right).$$

**Proof** For $\gamma \in \mathbb{R}$ set

$$\phi := f(x_t)\mathbb{V}_t, \quad \psi_t := \gamma \int_0^t \langle W_s(v_0), u_s dB_s - (t-s)\nabla V(x_s)\,ds\rangle$$

to see, by Theorem 2.5, Fubini's theorem and Lemma 2.6, that for $v_0 \in T_{x_0}M$,

$$\gamma t d(P_t^{h,V} f)(v_0) \leq \mathbf{E}\left[\phi \log \phi\right] + \log \mathbf{E}\left[\exp\left[\gamma \int_0^t \langle W_s(v_0), u_s dB_s - (t-s)\nabla V(x_s)\,ds\rangle\right]\right].$$

Since $\frac{D}{ds}W_s = -\frac{1}{2}\mathrm{Ric}^\sharp(W_s) + \nabla^2 h(W_s,\cdot)^\sharp$ we have $|W_s| \leq e^{-\frac{Ks}{2}}$ so,

$$\begin{aligned}
&\log \mathbb{E}\left[\exp\left[\gamma \int_0^t \langle W_s(v_0), u_s dB_s - (t-s)\nabla V(x_s)ds\rangle\right]\right] \\
\leq\ &\log \mathbb{E}\left[\exp\left(\gamma \int_0^t \langle W_s(v_0), u_s dB_s\rangle + |\,\gamma| \left|\int_0^t (t-s)\langle -\nabla V(\xi_s), W_s(v_0)\rangle ds\right|\right)\right] \\
\leq\ &\gamma^2 \int_0^t e^{-Ks}|v_0|ds + |\gamma||\nabla V|_\infty \int_0^t (t-s)e^{-\frac{Ks}{2}}|v_0|ds \\
\leq\ &\gamma^2 t C_1(t,K)|v_0| + |\gamma||\nabla V|_\infty t^2 C_2(t,K)|v_0|.
\end{aligned}$$

Thus we obtain

$$\gamma t d(P_t^{h,V} f)(v_0) \leq \mathbf{E}\left[(\phi\log\phi)^+\right] + \gamma^2 t C_1(t,K)|v_0| + |\gamma||\nabla V|_\infty t^2 C_2(t,K)$$

which after minimizing over $\gamma$ yields

$$t|\nabla P_t^{h,V} f|_{x_0} \leq \left(2tC_1(t,K)\mathbf{E}\left[(\phi\log\phi)^+\right]\right)^{\frac{1}{2}} + t^2|\nabla V|_\infty C_2(t,K),$$

as claimed. □

If $f \geq 0$ and $P_t^{h,V} f(x_0) > 0$ we define

$$\mathcal{H}_t(f,x_0) := \mathbb{E}\left[\frac{f(x_t)\mathbb{V}_t}{P_t^{h,V} f(x_0)}\left(\log\left(\frac{f(x_t)\mathbb{V}_t}{P_t^{h,V} f(x_0)}\right)\right)^+\right],$$

and replacing the non-negative function $f$ in Proposition 2.7 by $\frac{f}{P_t^{h,V} f(x_0)}$, we deduce the following corollary.

**Corollary 2.8** *Suppose that* Ric $-2\,\mathrm{Hess}(h) \geq K$ *and* $V \in C^{1,\alpha} \cap BC^1$. *Then for any bounded measurable function* $f \geq 0$ *we have*

$$|\nabla \log P_t^{h,V} f|_{x_0} \leq \frac{1}{\sqrt{t}} \Big(2C_1(t,K)\mathcal{H}_t(f,x_0)\Big)^{\frac{1}{2}} + t|\nabla V|_\infty C_2(t,K), \quad t > 0.$$

*and*

$$|\nabla \log p_t^{h,V}|_{x_0} \leq \frac{1}{\sqrt{t}} \sqrt{2C_1(t,K)} \Big( \sup_{y \in M} \left( \log \frac{p_t^h(y,y_0)}{p_{2t}^h(x_0,y_0)} + 2t(\sup V - \inf V) \right)^+ \Big)^{\frac{1}{2}} + t|\nabla V|_\infty C_2(t,K).$$

Indeed, choosing $f(\cdot) = p_t^{h,V}(\cdot, y_0)$, the Feynman-Kac kernel associated to $P_t^{h,V}$, we have

$$\begin{aligned}
\mathcal{H}_t(f,x_0) &\leq \sup_{y\in M} \left( \log \frac{f(y)\mathbf{E}\left[e^{-\int_0^t V(x_s)ds} | x_t = y\right]}{P_t^{h,V} f(x_0)} \right)^+ \mathbb{E}\left[ \frac{f(x_t)\mathbb{V}_t}{P_t^{h,V} f(x_0)} \right] \\
&\leq \sup_{y\in M} \left( \log \frac{p_t^{h,V}(y,y_0) e^{-t \inf V}}{p_{2t}^{h,V}(x_0,y_0)} \right)^+ \\
&= \sup_{y\in M} \left( \log \frac{p_t^h(y,y_0) \mathbb{E}\left[e^{-\int_0^t V(b_s^{t,y,y_0})ds}\right] e^{-t\inf V}}{p_{2t}^h(x_0,y_0)\mathbb{E}\left[e^{-\int_0^{2t} V(b_s^{t,x_0,y_0})ds}\right]} \right)^+ \\
&\leq \sup_{y\in M} \left( \log \frac{p_t^h(y,y_0)}{p_{2t}^h(x_0,y_0)} + 2t(\sup V - \inf V) \right)^+
\end{aligned}$$

where $b_s^{t,x,y}$ is the process obtained by conditioning the $h$-Brownian motion by $x_t = y$ and $x_0 = x$. Given suitable heat kernel upper and lower bounds, or more generally a Harnack inequality for $p_t^h$, we would have an estimate on the logarithmic derivative of $p_t^{h,V}$. These two types of assumptions are naturally related. For the first see [CLY81, LY86, CGT82, GHL08, Dav88, Nor97], [LY86, Corollary 3.1], and [Gri99, Thms 7.4, 7.5, 7.9]. If the Sobolev inequality $|f|^2_{2n/(n-2)} \leq C\int_M |\nabla f|^2\,dx$ holds for $f \in C_K^\infty$ then the heat kernel satisfies the on-diagonal estimate $p_t(x,x) \leq Ct^{-\frac{n}{2}}$ (leading to off-diagonal estimates). In fact, N. Th. Varopoulos proved that the Sobolev inequality is also necessary for the on-diagonal upper bound, [Gri91, Lemma 5.1, Thm 6.1]. See also [Gri99, SC10] for a clear account on the relation between various functional inequalities and the on diagonal Gaussian upper bounds of the type $\mathrm{vol}(B(x,\sqrt{t}))^{-1} e^{-\frac{d(x,y)^2}{t}}$. For the case $h = 0$ and Ric $\geq -K$, $K \geq 0$, a global Harnack inequality exists, [LY86, Thm. 2.2], for positive solutions of the heat equation. For example, [BQ97, Corollary 2],

$$\frac{f_t(x)}{f_{t+s}(y)} \leq \left(\frac{t+s}{t}\right)^{\frac{n}{2}} \exp\left[\frac{(d(x,y)+\sqrt{nK}s)^2}{4s} + \frac{\sqrt{nK}}{2}\min\{(\sqrt{2}-1)d(x,y), \frac{\sqrt{nK}}{2}s\}\right].$$

From this can be deduced the following theorem,

**Theorem 2.9** *Suppose that the Ricci curvature is bounded from below and $V \in C^{1,\alpha} \cap BC^1$. Then for all $T > 0$ there exists a positive constant $C(T)$ such that*

$$|\nabla \log p_t^V(\cdot, y_0)(x_0)| \leq C(T)\left(\frac{1}{t} + \frac{d^2(x_0,y_0)}{t^2} + |V|_\infty\right)^{\frac{1}{2}} + C(T)t|\nabla V|_\infty$$

*for all $t \in (0,T]$ and $x_0, y_0 \in M$.*

When $V = 0$, this recovers the estimates in J. Sheu [She91], see also [Hsu99, MS96, ST98]. For brevity we do not write down the estimate involving $h$ it is however worth noticing that gradient formula for $P_t^h f$ would lead to a parabolic Harnack inequality for $p_t^h$, see [ATW06] for such an estimate.

# 3 On Manifolds with a Pole

Let $n \geq 2$. Assume that $y_0$ is a pole for $M$. That is, we assume that the exponential map $\exp_{y_0}$ is a diffeomorphism between $T_{y_0}M$ and $M$. If the Jacobian $J: M \to \mathbb{R}$ defined by

$$J(y) \equiv J_{y_0}(y) = |\det D_{\exp_{y_0}^{-1}(y)} \exp_{y_0}|$$

is non-singular, then we denote by $J^{-1}$ the reciprocal of $J$ and for $t > 0$ define

$$\Phi(y) = \frac{1}{2} J^{\frac{1}{2}}(y) \Delta J^{-\frac{1}{2}}(y), \qquad k_t(y) = (2\pi t)^{-\frac{n}{2}} e^{-\frac{r^2(y)}{2t}} J^{-\frac{1}{2}}(y)$$

for $y \in M$, where $r$ denotes the distance to the pole $y_0$. The function $J^{\frac{1}{2}}$ is called Ruse's invariant (see for example A. G. Walker [Wal42]). The objective is to obtain probabilistic representation for the kernel $p^V$ and its derivatives, involving the distance function, $J$ and $\Phi$. On the standard $n$-dimensional hyperbolic space, the heat kernel $p_t(x,y)$ depends on $x$ and $y$ through $r = d(x,y)$ and is given by iterative formulas:

$$p_t(x,y) = C(m)\frac{1}{\sqrt{t}}\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m e^{-m^2 t - \frac{r^2}{4t}}, \qquad n = 2m+1,$$

$$p_t(x,y) = C(m) t^{-\frac{3}{2}} e^{-\frac{(2m+1)^2}{4}t}\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m \int_\rho^\infty \frac{s e^{-\frac{s^2}{4t}}}{\cosh s - \cosh r}\, ds, \qquad n = 2m+2.$$

If its sectional curvature is $-R^{-2}$, then [Elw82],

$$J = \left(\frac{R}{r}\sinh\frac{r}{R}\right)^{n-1}$$
$$\Phi = -\frac{(n-1)^2}{8R^2} + \frac{(n-1)(n-3)}{8}\left(r^{-2} - \left(R^2\sinh^2\left(\frac{r}{R}\right)\right)^{-1}\right).$$

In particular $\Phi$ is bounded above.

If $M$ is a model space, i.e. $M$ is a manifold with a pole $p$ such that for every linear isometry $\phi : T_pM \to T_pM$ there exists an isometry $\Phi : M \to M$ such that $\phi(p) = p$ and $d\Phi_p = \phi$. In the geodesic polar coordinates, $(r, \theta) \in (0, \infty) \times S^{n-1}$, the pull back metric in $(0, \infty) \times S^{n-1}$ can be written as $dr^2 + f(r)^2 d\theta^2$. The function $f$ is $C^\infty$ and satisfies $f(0) = 0$, $f'(0) = 1$, $f(r) > 0$ for $r > 0$ and $f''(r) = -R(r)f(r)$ where $R(r)$ is the sectional curvature in a plane containing the radial direction $\partial_r$ at a point $x$ with $d(x, p) = r$. Then $\log J_p(x) = (n-1)\log\frac{f(r)}{r}$. If $R(r) = R$, a constant, then $f(r) = \frac{\sinh(\sqrt{R}r)}{\sqrt{R}}$ and $\log f$ has bounded derivatives of all order. In general,

$$|\nabla \log J| = (n-1)\left|(\log f)'(r) - \frac{1}{r}\right|, \qquad \Delta r = (n-1)(\log f)'(r),$$
$$\Delta(\log J) = (n-1)^2(\log f)'(r)\left((\log f)'(r) - \frac{1}{r}\right) + (n-1)\left((\log f)''(r) + \frac{1}{r^2}\right),$$
$$\Phi = \frac{1}{4}(n-1)\left[\frac{n-2}{r^2} - \frac{n-1}{r}(\log f)'(r) - (\log f)''(r)\right].$$

### 3.1 Girsanov Transform and Zeroth Order Formula

Let $T$ be a positive number and $x_0, y_0 \in M$. The semi-classical bridge process, between $x_0$ and $y_0$ in time $T$, is the diffusion process starting at $x_0$ with generator $\frac{1}{2}\Delta + \nabla \log k_{T-s}$ where $k_t(x_0, y_0) := (2\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(x_0,y_0)}{2t}} J^{-\frac{1}{2}}(x_0)$. If $\tilde{u}_s$ satisfies

$$d\tilde{u}_s = \sum H_i(\tilde{u}_s) \circ dB_s^i + \nabla \widetilde{\log k}_{T-s}(\tilde{u}_s)\, ds$$

with $\tilde{u}_0 = u_0$ and $\tilde{A}_s$ the horizontal lift of $A_s$ then $\tilde{x}_s = \pi(\tilde{u}_s)$ is a semi-classical bridge.

Since $|\nabla r| = 1$ away from $y_0$, Itô's formula implies for $0 \le t < T$ that

$$r(\tilde{x}_t) - r(x_0) = \beta_t + \int_0^t \frac{1}{2}\Delta r(\tilde{x}_s)ds - \int_0^t \frac{r(\tilde{x}_s)}{T-s}ds - \frac{1}{2}\int_0^t dr\left(\nabla \log J(\tilde{x}_s)\right) ds$$

where $\beta_t$ is a standard one-dimensional Brownian motion. It is clear from this and the formula

$$\Delta r = \frac{n-1}{r} + dr(\nabla \log J) \tag{3.1}$$

that $r(\tilde{x}_t)$ is distributed as a Bessel bridge, starting at $r(x_0)$ and ending at $0$ at time $T$, from which it follows that $\lim_{t\uparrow T}\tilde{x}_t = y_0$, almost surely.

We follow a beautiful idea of Watling [Wat88] and use a modification of the construction by Elworthy-Truman to define the semi-classical Riemannian bridge for a given vector field $Z$. For $\gamma_x : [0,1] \to M$ the unique geodesic from $x$ to $y_0$,

$$S(x) = \int_0^1 \langle \dot{\gamma}_x(u), Z(\gamma_x(u))\rangle du,$$

and naturally $S(y_0) = 0$. If $Z = \nabla f$, then $S(x) = f(y_0) - f(x)$ and $\nabla S + \nabla f = 0$.

**Lemma 3.1** *[Watling] If $\tilde{x}_t$ is a $\frac{1}{2}\Delta + Z + \nabla S + \nabla(\log(k_{T-s}))$ diffusion (called the semi-classical Riemannian bridge), then $d(y_0, \tilde{x}_t)$ is the $n$-dimensional Bessel bridge process.*

**Proof** In the formula for $r_t := d(\tilde{x}_t, y_0)$, the additional drift $Z + \nabla S$ appears in the form of $\langle \nabla r, Z + \nabla S\rangle$ and this vanishes:

$$r_t - r_0 = \beta_t + \frac{1}{2}\int_0^t \frac{n-2}{r_s} ds - \int_0^t \frac{r_s}{T-s} ds + \int_0^t \langle \nabla r, Z + \nabla S\rangle_{\tilde{x}_s} ds.$$

Indeed, $S(\gamma_x(t)) = \int_t^1 \langle \dot{\gamma}_x(u), Z(\gamma_x(u))\rangle du$ so $\frac{d}{dt}|_{t=0} S(\gamma_x(t)) = -\langle \dot{\gamma}_x(0), Z(x)\rangle$ on one hand, and $\frac{d}{dt}|_{t=0} S(\gamma_x(t)) = \langle \nabla S(x), \dot{\gamma}_x(0)\rangle$ on the other hand. Finally $\nabla r(x) = \dot{\gamma}_x(0)$, and so $\langle \nabla S + Z, \nabla r\rangle = 0$. □

The following basic lemma will be used repeatedly, where $h \in C^2(M;\mathbb{R})$, $Z$ is a $C^1$ vector field, and $t \in [0,T)$. We now consider a Brownian motion with drift $\nabla h + Z$. Although by the earlier discussion the symmetric drift can be absorbed into $Z$, it is convenient to treat $\nabla h$ differently from $Z$, the former is not involved in the definition of the semi-classical riemannian bridge. Both $\nabla h$ and $Z$ appear in the formulation of the Lemma (Lemma 3.2) below. Recall that

$$\Phi = \frac{1}{2} J^{\frac{1}{2}} \Delta (J^{-\frac{1}{2}}) = \frac{1}{2}|\nabla \log J|^2 - \frac{1}{4}\Delta(\log J).$$

Let us define

$$\Phi^h = \Phi - \frac{1}{2}\Delta h - \frac{1}{2}|\nabla h|^2, \quad \Psi = \frac{1}{2}|\nabla S|^2 + \frac{1}{2}\Delta S + \langle Z, \nabla S - \nabla h\rangle.$$

**Lemma 3.2** *Suppose the $\frac{1}{2}\Delta^h + Z$ diffusion is complete. Then the probability distributions of the $\frac{1}{2}\Delta^h + Z$ diffusion and of the semi-classical Riemannian bridge, c.f. Lemma 3.1, are equivalent on $\mathcal{F}_t$. Furthermore for any bounded measurable function $F$ on the path space $C([0,t];M)$, $\mathbf{E}F(u_\cdot) = \mathbf{E}[F(\tilde{u}_\cdot)M_t]$ where Radon-Nikodym derivative $M_t$ is given by*

$$M_t := \frac{k_T(x_0) e^{(S-h)(x_0)}}{k_{T-t}(\tilde{x}_t) e^{(S-h)(\tilde{x}_t)}} \exp\left[\int_0^t (\Phi^h + \Psi)(\tilde{x}_s)\, ds\right]. \tag{3.2}$$

**Proof** For any $u_0 \in OM$ with $\pi(u_0) = x_0$, let $u_s$ be the solution of the following equation with initial value $u_0$:

$$du_s = \sum H_i(u_s) \circ dB_s^i + (\widetilde{\nabla h} + \widetilde{Z})(u_s)\, ds. \tag{3.3}$$

Then $(u_t)$ exists for all time and $x_t = \pi(u_t)$ has generator $\frac{1}{2}\Delta^h + Z$. Let $\tilde{u}_t$ be the solution to

$$d\tilde{u}_s = \sum H_i(\tilde{u}_s) \circ dB_s^i + (\widetilde{\nabla \log k_{T-s}} + \widetilde{Z} + \widetilde{\nabla S})(\tilde{u}_s)\, ds, \quad \tilde{u}_0 = u_0 \tag{3.4}$$

Then $\tilde{x}_s = \pi(\tilde{u}_s)$ is a semi-classical Riemannian bridge, the Markov process with generator $\frac{1}{2}\Delta + Z + \nabla S + \nabla(\log k_{T-s})$. One simple, and yet crucial observation, is that

$$\frac{1}{2}\Delta^h + Z + \nabla \log(k_{T-s} e^{S-h}) = \frac{1}{2}\Delta + Z + \nabla S + \nabla(\log k_{T-s}),$$

and so the generators of $\tilde{x}_t$ and $(x_t)$ differ by the drift $\nabla \log(k_{T-s} e^{S-h})$. Since both stochastic processes are well defined before time $T$, they are equivalent on $\mathcal{F}_t$ for any $t < T$.

Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $\mathbb{R}^n$ and let

$$\tilde{B}_t^i := B_t^i + \int_0^t d\log(k_{T-s} e^{S-h})(\tilde{u}_s e_i)\, ds.$$

Then $\{\tilde{B}_t^i\}$ are independent Brownian motions on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ where, on each $\sigma$-algebra $\mathcal{F}_t$, $\mathbb{Q}$ is defined to be the probability measure equivalent to $\mathbb{P}$ on each $\mathcal{F}_t$ with $M_t = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$ given by

$$M_t = \exp\left[-\sum_{i=1}^m \int_0^t \langle \nabla \log(k_{T-s} e^{S-h})(\tilde{x}_s), \tilde{u}_s e_i\rangle dB_s^i - \frac{1}{2}\int_0^t |\nabla \log(k_{T-s} e^{S-h})|^2_{(\tilde{x}_s)} ds\right].$$

Now Itô's formula implies

$$\begin{aligned}\log(e^{S-h} k_{T-t})(\tilde{x}_t) = \;& \log(e^{S-h} k_T)(x_0) + \sum_{i=1}^m \int_0^t \langle \nabla \log(k_{T-s} e^{S-h})(\tilde{x}_s), \tilde{u}_s e_i\rangle dB_s^i \\ &+ \int_0^t |\nabla \log(k_{T-s} e^{S-h})(\tilde{x}_s)|^2\, ds + \int_0^t \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta^h + Z\right) \log(k_{T-s} e^{S-h})(\tilde{x}_s)\, ds,\end{aligned}$$

using which we may eliminate the stochastic integral appearing in the formula for $M_t$. Indeed, by the relation (3.1) we see that

$$\begin{aligned}\frac{\partial}{\partial s}\log k_{T-s} &= \frac{n}{2(T-s)} - \frac{r^2}{2(T-s)^2},\\ |\nabla \log(k_{T-s} e^{-h})|^2 &= \frac{r^2}{(T-s)^2} + \frac{1}{4}|\nabla \log J|^2 + \frac{r dr(\nabla \log J)}{T-s} - 2\langle \nabla h, \nabla \log(k_{T-s})\rangle + |\nabla h|^2,\\ \Delta \log(k_{T-s} e^{-h}) &= -\frac{n}{T-s} - \frac{r dr(\nabla \log J)}{T-s} - \frac{1}{2}\Delta(\log J) - \Delta h.\end{aligned}$$

We have the following identity:

$$\begin{aligned}
&\frac{1}{2}|\nabla \log(k_{T-s}e^{-h})|^2 + \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta^h + Z\right)\log(k_{T-s}e^{-h})\\
&= \frac{1}{8}|\nabla \log J|^2 + \frac{1}{2}|\nabla h|^2 - \frac{1}{4}\Delta(\log J) - \frac{1}{2}\Delta h - |\nabla h|^2 + \langle Z, \nabla \log(k_{T-s}e^{-h})\rangle\\
&= \frac{1}{8}|\nabla \log J|^2 - \frac{1}{4}\Delta(\log J) - \frac{1}{2}\Delta h - \frac{1}{2}|\nabla h|^2 + \langle Z, \nabla(\log k_{T-s}e^{-h})\rangle,
\end{aligned}$$

from which we deduce the formula with non-vanishing $S$ ,

$$\begin{aligned}
&\frac{1}{2}|\nabla \log(k_{T-s}e^{-h+S})|^2 + \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta^h + Z\right)\log(k_{T-s}e^{-h+S})\\
=&\frac{1}{2}|\nabla S|^2 + \langle \nabla S, \nabla \log(k_{T-s}e^{-h})\rangle + \frac{1}{2}\Delta^h S + \langle Z, \nabla S\rangle + \langle Z, \nabla \log(k_{T-s}e^{-h})\rangle\\
&+ \frac{1}{8}|\nabla \log J|^2 - \frac{1}{4}\Delta(\log J) - \frac{1}{2}\Delta h - \frac{1}{2}|\nabla h|^2.
\end{aligned}$$

Since $\nabla S + Z$ vanishes on radial directions, see the proof for Lemma 3.1, the first line on the right hand side of the equation is

$$\begin{aligned}
&\frac{1}{2}|\nabla S|^2 - \langle \nabla S + Z, \nabla h\rangle + \frac{1}{2}\Delta S + \langle \nabla h, \nabla S\rangle + \langle Z, \nabla S\rangle\\
&= \frac{1}{2}|\nabla S|^2 + \frac{1}{2}\Delta S + \langle Z, \nabla S - \nabla h\rangle.
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
&\frac{1}{2}|\nabla \log(k_{T-s}e^{-h+S})|^2 + \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta^h + Z\right)\log(k_{T-s}e^{-h+S})\\
&= \frac{1}{2}|\nabla S|^2 + \frac{1}{2}\Delta S + \langle Z, \nabla S - \nabla h\rangle + \Phi - \frac{1}{2}\Delta h - \frac{1}{2}|\nabla h|^2
\end{aligned}$$

and consequently,

$$\begin{aligned}
\log(e^{S-h}k_{T-t})(\tilde{x}_t) =\ & \log(e^{S-h}k_T)(x_0) + \sum_{i=1}^m \int_0^t \langle \nabla \log(k_{T-s}e^{S-h})(\tilde{x}_s), \tilde{u}_s e_i\rangle dB_s^i\\
&+ \frac{1}{2}\int_0^t |\nabla \log(k_{T-s}e^{S-h})(\tilde{x}_s)|^2\, ds + \int_0^t (\Phi^h + \Psi)(\tilde{x}_s)\, ds,
\end{aligned}$$

and

$$M_t = \frac{(e^{S-h}k_T)(x_0)}{(e^{S-h}k_{T-t})(\tilde{x}_t)} \exp\left(\int_0^t (\Phi^h + \Psi)(\tilde{x}_s)\, ds\right),$$

completing the proof. $\square$

In particular, if $f \in \mathcal{B}_b$ then Lemma 3.2 implies that for $0 \le t < T$,

$$\int_M f(y) p_t^{h,Z,V}(x_0, y)\, dy = \mathbf{E}[\mathbb{V}_t f(x_t)] = k_T(x_0, y_0) e^{(S-h)(x_0)} \mathbb{E}\left[\frac{f(\tilde{x}_t)\beta_t^{h,Z}}{k_{T-t}(\tilde{x}_t) e^{(S-h)(\tilde{x}_t)}}\right], \tag{3.5}$$

where

$$\beta_t^{h,Z} = \exp\left(\int_0^t (\Phi^h + \Psi - V)(\tilde{x}_s) ds\right). \tag{3.6}$$

Elworthy and Truman's proof of the following theorem, for the case $h = 0$, was inspired by semiclassical mechanics. They used a semiclassical bridge which arrives at $y_0$ at time $T + \lambda$ and took the limit as $\lambda \downarrow 0$. We give a slightly modified proof, generalising their result for $\Delta^h$, the method of which will later be used to derive a derivative formula. The following generalises a formula by Elworthy-Truman, [Elw82]. Watling [Wat88] treated Brownian motion with a drift $Z$.

**Theorem 3.3** *[ET81, Wat88] Let $V \in C^{0,\alpha} \cap L_\infty$ and suppose that the Brownian motion with drift $\nabla h + Z$ does not explode. Suppose that $\Phi^h + \Psi - V$ is bounded above, or more generally the following convergence $\lim_{t \to T} \left|\beta_t^h - \beta_T^h\right| = 0$. Then*

$$p_T^{h,Z,V}(x_0, y_0) = e^{(h-S)(y_0)-(h-S)(x_0)} k_T(x_0, y_0) \mathbf{E}\left[\exp\left(\int_0^T (\Phi^h + \Psi - V)(\tilde{x}_s) ds\right)\right]. \tag{3.7}$$

**Proof** Let $\phi$ be a smooth function with compact support with $\phi(y_0) = 1$. Denote by $E_t$ the standard Gaussian kernel in the tangent space $T_{y_0}M$. Then, by a change of variables, we see that

$$\begin{aligned}
&\lim_{t\uparrow T} \lim_{r\uparrow t} p_t^{h,Z,V}(\phi k_{T-r})(x_0) = \lim_{t\uparrow T} \lim_{r\uparrow t} \int_M p_t^{h,Z,V}(x_0, y)\phi(y) k_{T-r}(y)\, dy \\
&= \lim_{t\uparrow T} \lim_{r\uparrow t} \int_{T_{y_0}M} (p_t^{h,Z,V}(x_0, \cdot) \cdot \phi \cdot J^{1/2})(\exp_{y_0} v) E_{T-r}(v)\, dv \\
&= (p_T^{h,Z,V}(x_0, \cdot) \cdot \phi \cdot J^{1/2})(\exp_{y_0} 0_{y_0}) = p_T^{h,Z,V}(x_0, y_0)
\end{aligned}$$

where $0_{y_0}$ denotes the origin of the tangent space $T_{y_0}M$, using the fact that $p_t^{h,Z,V}(x_0, \cdot) \cdot \phi \cdot J^{1/2}$ has compact support with $J(\exp_{y_0} 0_{y_0}) = J(y_0) = 1$. Thus, by taking $f = \phi \cdot k_{T-r}$ in equation (3.5) for $r < t$, we observe, since $\tilde{x}_t$ converges to $y_0$ a.s., that

$$\begin{aligned}
p_T^{h,Z,V}(x_0, y_0) &= e^{(S-h)(x_0)} k_T(x_0, y_0) \lim_{t\uparrow T} \lim_{r\uparrow t} \mathbf{E}\left[\phi(\tilde{x}_t) \frac{k_{T-r}(\tilde{x}_t)}{k_{T-t}(\tilde{x}_t) e^{(S-h)(\tilde{x}_t)}} \exp\left(\int_0^t (\Phi^h + \Psi - V)(\tilde{x}_s) ds\right)\right] \\
&= e^{(S-h)(x_0)-(S-h)(y_0)} k_T(x_0, y_0) \mathbf{E}\left[\exp\left(\int_0^T (\Phi^h + \Psi - V)(\tilde{x}_s) ds\right)\right],
\end{aligned}$$

Since $\phi$ has compact support the limit $r \to t$ is trivial to justify. The second follows from the assumption. The proof is complete. $\square$

At this point we compare formula (3.7), valid for all time $t \leq T$, with S. R. S. Varadhan's asymptotic relation [Var67] and the asymptotic expansion of S. Minakshisundaram and A. Pleijel [MP49] for small time. The first states that $\lim_{t\downarrow 0} 2t \log p_t \to -d^2$, uniformly on compact sets. This was proved in [Var67] for operators of the form $\sum_{i,j} a_{i,j}\partial^2 \partial x_i \partial x_j$ in $\mathbb{R}^n$. The latter states that there are smooth functions $H_i$ defined on $M \times M \setminus \text{Cut}(M)$ such that

$$p_t(x,y) \sim (2\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(x,y)}{2t}} \sum_{i=0}^{\infty} H_i(x,y)t^i$$

with $H_0(x,y) = J_x^{-\frac{1}{2}}(y)$, as $t \to 0$. Both converge uniformly on compact subsets of $M \times M \setminus \text{Cut}(M)$, where $\text{Cut}(M)$ denotes the cut locus of $M$. See also [AH05].

### 3.2 First Order Formula

We return to the $h$-Brownian motion whose generator is $\frac{1}{2}\Delta + \nabla h$. Set $\beta_t^h = \exp\left(\int_0^t (\Phi^h - V)(\tilde{x}_s) ds\right)$.

**Theorem 3.4** *Assume $y_0$ is a pole, $V \in C^{1,\alpha} \cap BC^1$, $\Phi^h - V$ is bounded above, $f \in L_\infty$ and $\underline{\text{Ric}} - 2\,\text{Hess}\, h \geq K$. Then*

$$dp_T^{h,V}(\cdot, y_0) = \frac{1}{T} e^{h(y_0)-h(x_0)} k_T(x_0,y_0) \mathbf{E}\left[\beta_T^h \left(\int_0^T \langle \tilde{W}_r(\cdot), \tilde{u}_r d\tilde{B}_r\rangle - \int_0^T (T-r) dV(\tilde{W}_r(v)) dr\right)\right]$$

*where $\tilde{W}$ is the solution to (2.4) and for $r \in [0,T)$*

$$d\tilde{B}_r = dB_r + \tilde{u}_r^{-1} \nabla(\log(k_{T-r}e^{-h}))(\tilde{x}_r)\, dr.$$

**Proof** For all $0 < t \leq T$ and $v \in T_{x_0}M$ we have, by Theorem 2.5 and Fubini's theorem, that

$$\begin{aligned} dP_t^{h,V} f(v) &= \frac{1}{t}\mathbf{E}\left[\mathbb{V}_t f(x_t) \int_0^t \langle W_s(v), u_s dB_s\rangle - (t-s) dV(W_s(v))\, ds\right] \\ &= \frac{1}{t}\mathbf{E}\left[\mathbb{V}_t f(x_t) \int_0^t \langle W_s(v), u_s dB_s - (t-s)\nabla V ds\rangle\right]. \end{aligned}$$

Therefore, it follows that, for all $0 < t < T$,

$$dP_t^{h,V}(\phi k_{T-t})(v) = \frac{1}{t}\mathbf{E}\left[\mathbb{V}_t(\phi k_{T-t})(x_t) \int_0^t \langle W_r(v), u_r dB_r - (t-r)\nabla V dr\rangle\right],$$

where $\phi$ is a smooth function with compact support and $\phi(y_0) = 1$, as in the proof of Theorem 3.3. By Lemma 3.2 this yields

$$dP_t^{h,V}(\phi k_{T-t})(v) = \frac{1}{t}e^{-h(x_0)}k_T(x_0, y_0)\mathbf{E}\left[\phi(\tilde{x}_t)\beta_t^h e^{h(\tilde{x}_t)}\int_0^t \langle \tilde{W}_r(v), \tilde{u}_r d\tilde{B}_r - (t-r)\nabla V dr\rangle\right].$$

We now take limits. For the left-hand side of the previous equation we see that

$$\begin{aligned}\lim_{t\uparrow T} dP_t^{h,V}(\phi k_{T-t})(v) &= \lim_{t\uparrow T} d\left(\int_M P_t^{h,V}(\cdot, y)\phi(y)k_{T-t}(y)\,dy\right)(v)\\ &= \lim_{t\uparrow T}\int_M dP_t^{h,V}(\cdot, y)(v)\phi(y)k_{T-t}(y)\,dy = dP_T^{h,V}(\cdot, y)(v)\end{aligned}$$

where the final equality follows from the compactness of the support of $\phi$ and the fact that $J(y_0) = 1$. For the right-hand side, we use $\lim_{t\to T}\tilde{x}_t = y_0$, apply the dominated convergence theorem, using upper and lower bounds on $\Phi^h$ and $\mathrm{Ric} - 2\,\mathrm{Hess}\, h$, respectively, obtaining the result as claimed. □

Immediate applications of Theorems 3.3 and 3.4 are:

**Corollary 3.5** *Under the assumptions of Theorem 3.4 we have,*

$$d\log p_T^{h,V}(\cdot, y_0)(v) = \frac{1}{T}\mathbf{E}\left[Z^h\int_0^T \langle \tilde{W}_r(v), \tilde{u}_r d\tilde{B}_r - (T-r)\nabla V dr\rangle\right],$$

*where* $Z_T^h = \frac{\beta_T^h}{\mathbf{E}(\beta_T^h)} = \frac{\exp\left(\int_0^t(\Phi^h - V)(\tilde{x}_s)ds\right)}{\mathbf{E}\exp\left(\int_0^t(\Phi^h - V)(\tilde{x}_s)ds\right)}$.

For $V$ bounded Hölder continuous, the above argument and Lemma 2.3 lead to:

**Corollary 3.6** *Assume that $y_0$ is a pole for $M$, $\underline{\mathrm{Ric}} - 2\,\mathrm{Hess}(h)$ is bounded below and $\Phi^h$ is bounded above with $V$ Hölder continuous and bounded. Then*

$$\begin{aligned}d\log p_T^{h,V}(\cdot, y_0)_{x_0} =& \frac{1}{T}\mathbf{E}\left[Z_T^h\int_0^T \langle \tilde{W}_s(\cdot), \tilde{u}_s d\tilde{B}_s\rangle\right]\\ &+ \mathbf{E}\left[Z_T^h\int_0^T V(\tilde{x}_{T-s})\, e^{-\int_{T-s}^T V(\tilde{x}_u)du}\frac{1}{T-s}\int_0^{T-s}\langle \tilde{W}_r(\cdot), \tilde{u}_r d\tilde{B}_r\rangle\right].\end{aligned}$$

In the paper [CLY81, Thm.6] by S.-Y. Cheng, P. Li and S.-T. Yau, an estimate of the following form

$$\nabla p_t(x_0, y_0) \le C\delta^{-\alpha(n)}(x_0)t^{-(n+\frac{1}{2})}e^{-\frac{\alpha(n)d^2(x_0,y_0)}{t}}$$

is given for Riemannian manifold of bounded curvature. We have the following corresponding estimates.

**Corollary 3.7** *Assume $y_0$ is a pole, $\Phi^h - V$ is bounded above, $\underline{\mathrm{Ric}} - 2\,\mathrm{Hess}(h) \geq K$, and $\nabla h$, and $\nabla \log J$ are bounded. Suppose that $V \in C^{1,\alpha} \cap BC^1$. Then for an explicit constant $C$ depending only on $n$, $K$ and $|\nabla \log J|_\infty$, the following estimate holds,*

$$\frac{|\nabla p_T^{h,V}(\cdot, y_0)|_{x_0}}{k_T(x_0,y_0)} \leq C e^{h(y_0)-h(x_0)} |\beta_T^h|_\infty \left( \frac{d(x_0,y_0)}{T} + |\nabla h|_\infty + 1 + \frac{1}{\sqrt{T}} + T|dV|_\infty \right).$$

**Proof** Since $\tilde{W}$ is defined by (2.4), $|\tilde{W}_t(v)| \leq e^{-\frac{1}{2}Kt}|v|$,

$$\begin{aligned}
&\left| \int_0^T \langle \tilde{W}_r(v), \tilde{u}_r d\tilde{B}_r \rangle - \int_0^T (t-r) dV(\tilde{W}_r(v)) dr \right| \\
&= \left| \int_0^T \langle \tilde{W}_r(v), \tilde{u}_r dB_r \rangle + \int_0^T \left\langle \tilde{u}_r \tilde{W}_r(v), \nabla \log(e^{-h} k_{T-r})(\tilde{x}_r) \right| dr - \int_0^T (t-r) dV(\tilde{W}_r(v)) dr \rangle \right| \\
&\leq \left| \int_0^T \langle \tilde{W}_r(v), \tilde{u}_r dB_r \rangle \right| + |v| \int_0^T e^{-Kr/2} \left| \nabla \log k_{T-r}(\tilde{x}_r) - \nabla h)(\tilde{x}_r) \right| dr \\
&\quad + |v|\; |dV|_\infty \int_0^T e^{-Kr/2} (t-r) dr.
\end{aligned}$$

The expectation of the first term can be easily estimated: $\mathbf{E} \left| \int_0^T \langle \tilde{W}_r(v), \tilde{u}_r dB_r \rangle \right| \leq \sqrt{\mathbf{E} \int_0^T |\tilde{W}_r(v)|^2 dr} \leq \left( \int_0^T e^{-Kr} |v|^2 dr \right)^{\frac{1}{2}}$. We apply Theorem 3.4 to see that

$$\begin{aligned}
&\left| e^{h(x_0)-h(y_0)} \frac{\nabla p_T^{h,V}(\cdot, y_0)(v)}{k_T(x_0,y_0)} \right| \\
&\leq \frac{1}{T} |v|\; |\beta_T^h|_\infty \left( \int_0^T e^{-Kr} dr \right)^{\frac{1}{2}} + \frac{1}{T} |v|\; |\beta_T^h|_\infty |dV|_\infty \int_0^T e^{-Kr/2} (T-r) dr \\
&+ \frac{1}{T} |v|\; |\beta_T^h|_\infty \mathbf{E} \left[ \int_0^T e^{-Kr/2} (|\nabla h|_\infty + |\nabla \log k_{T-r}(\tilde{x}_r)|) dr \right].
\end{aligned}$$

The sum of the first two terms on the right hand side are nicely bounded by

$$|v|\; |\beta_T^h|_\infty \frac{1}{T} (\sqrt{C_1(K,T)} + |dV|_\infty C_2(K,T))$$

where $C_1$ and $C_2$ are the obvious integrals of order $T$, and so is bounded for small $T$. Since $\nabla \log k_t(\cdot, y_0) = -\frac{\nabla r^2}{2t} - \frac{1}{2} \nabla \log J$, the last term can be estimated using the Euclidean bridge. Then

$$\begin{aligned}
&\mathbf{E}\left[\int_0^T e^{-Kr/2}|\nabla \log k_{T-r}(\tilde{x}_r)|dr\right] \\
&\le \frac{1}{2}\int_0^T e^{-Kr/2}|\nabla \log J|_\infty dr + \mathbf{E}\left[\int_0^T e^{-Kr/2}\frac{d(\tilde{x}_r, y_0)}{(T-r)}dr\right] \\
&\le \frac{1}{2}|\nabla \log J|_\infty \int_0^T e^{-Kr/2}dr + \int_0^T e^{-Kr/2}\frac{\sqrt{\mathbf{E}d^2(\tilde{x}_r, y_0)}}{(T-r)}dr.
\end{aligned}$$

Since $r_t = d(\tilde{x}_t, y_0)$ is the $n$-dimensional Bessel bridge,

$$\mathbf{E}d^2(\tilde{x}_r, y_0) = \left(\frac{T-r}{T}\right)^2 d^2(x_0, y_0) + \frac{r}{T}(T-r),$$

and consequently,

$$\int_0^t e^{-Kr/2}\frac{\sqrt{\mathbf{E}d^2((\tilde{x}_r), y_0)}}{(T-r)}dr \le \int_0^t e^{-Kr/2}\left(\frac{1}{T}d(x_0, y_0) + \sqrt{\frac{r}{T}}\frac{1}{\sqrt{T-r}}\right)dr,$$

which is bounded by $C(d(x_0, y_0) + \sqrt{T})$. This completes the proof. □

Together with Theorem 3.3 we easily have an estimate for the gradient of the logarithmic Feynman-Kac kernel, which follows from a similar estimate as above:

$$\left|\nabla \log p_T^{h,V}(\cdot, y_0)\right|_{x_0} \le C|Z_T|_{L_2} C\left(\frac{d(x_0, y_0)}{T} + 1 + |\nabla h|_\infty + \frac{1}{\sqrt{T}} + |dV|_\infty T\right),$$

where $C$ is a constant depending on $|\nabla \log J|_\infty$ and on $K$.

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## References


[AH05] Teppei Ariyoshi and Masanori Hino. Small-time asymptotic estimates in local Dirichlet spaces. *Electron. J. Probab.*, 10:no. 37, 1236–1259 (electronic), 2005.

[Aid01] Shigeki Aida. Precise Gaussian lower bounds on heat kernels. In *Stochastics in finite and infinite dimensions*, Trends Math., pages 1–28. Birkhäuser Boston, Boston, MA, 2001.

[AKR12] Sergio Albeverio, Hiroshi Kawabi, and Michael Röckner. Strong uniqueness for both Dirichlet operators and stochastic dynamics to Gibbs measures on a path space with exponential interactions. *J. Funct. Anal.*, 262(2):602–638, 2012.

[ATW06] Marc Arnaudon, Anton Thalmaier, and Feng-Yu Wang. Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below. *Bull. Sci. Math.*, 130(3):223–233, 2006.

[Aub76] Thierry Aubin. Espaces de Sobolev sur les variétés riemanniennes. *Bull. Sci. Math. (2)*, 100(2):149–173, 1976.

[Aze74] Robert Azencott. Behavior of diffusion semi-groups at infinity. *Bull. Soc. Math. France*, 102:193–240, 1974.

[Bis84] Jean-Michel Bismut. *Large deviations and the Malliavin calculus*, volume 45 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1984.

[BQ97] Dominique Bakry and Zhongmin Qian. On Harnack estimates for positive solutions of the heat equation on a complete manifold. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(9):1037–1042, 1997.

[Bra98] Maxim Braverman. On self-adjointness of a Schrödinger operator on differential forms. *Proc. Amer. Math. Soc.*, 126(2):617–623, 1998.

[CGT82] Jeff Cheeger, Mikhail Gromov, and Michael Taylor. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.*, 17(1):15–53, 1982.

[CLY81] Siu Yuen Cheng, Peter Li, and Shing Tung Yau. On the upper estimate of the heat kernel of a complete Riemannian manifold. *Amer. J. Math.*, 103(5):1021–1063, 1981.

[Dav88] E. B. Davies. Gaussian upper bounds for the heat kernels of some second-order operators on Riemannian manifolds. *J. Funct. Anal.*, 80(1):16–32, 1988.

[DP15] G. Da Prato. An integral inequality for the invariant measure of some finite dimensional stochastic differential equation. Preprint, 2015.

[DPZ97] Giuseppe Da Prato and Jerzy Zabczyk. Differentiability of the Feynman-Kac semigroup and a control application. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 8(3):183–188, 1997.

[DT01] Bruce K. Driver and Anton Thalmaier. Heat equation derivative formulas for vector bundles. *J. Funct. Anal.*, 183(1):42–108, 2001.

[DvC00] Michael Demuth and Jan A. van Casteren. *Stochastic spectral theory for self-adjoint Feller operators*. Probability and its Applications. Birkhäuser Verlag, Basel, 2000. A functional integration approach.

[EL94a] K. D. Elworthy and X.-M. Li. Differentiation of heat semigroups and applications. In *Probability theory and mathematical statistics (Vilnius, 1993)*, pages 239–251. TEV, Vilnius, 1994.

[EL94b] K. D. Elworthy and X.-M. Li. Formulae for the derivatives of heat semigroups. *J. Funct. Anal.*, 125(1):252–286, 1994.

[Elw82] K. D. Elworthy. *Stochastic differential equations on manifolds*, volume 70 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1982.

[Eng06] A. Engoulatov. A universal bound on the gradient of logarithm of the heat kernel for manifolds with bounded Ricci curvature. *J. Funct. Anal.*, 238(2):518–529, 2006.

[ET81] David Elworthy and Aubrey Truman. Classical mechanics, the diffusion (heat) equation and the Schrödinger equation on a Riemannian manifold. *J. Math. Phys.*, 22(10):2144–2166, 1981.

[ETW85] David Elworthy, Aubrey Truman, and Keith Watling. The semiclassical expansion for a charged particle on a curved space background. *J. Math. Phys.*, 26(5):984–990, 1985.

[Fit89] P. J. Fitzsimmons. Time changes of symmetric Markov processes and a Feynman-Kac formula. *J. Theoret. Probab.*, 2(4):487–501, 1989.

[Fre85] Mark Freidlin. *Functional integration and partial differential equations*, volume 109 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[Gaf59] Matthew P. Gaffney. The conservation property of the heat equation on Riemannian manifolds. *Comm. Pure Appl. Math.*, 12:1–11, 1959.

[GHL08] Alexander Grigor'yan, Jiaxin Hu, and Ka-Sing Lau. Obtaining upper bounds of heat kernels from lower bounds. *Comm. Pure Appl. Math.*, 61(5):639–660, 2008.

[Gri91] A. A. Grigor′yan. The heat equation on noncompact Riemannian manifolds. *Mat. Sb.*, 182(1):55–87, 1991.

[Gri99] Alexander Grigor′yan. Estimates of heat kernels on Riemannian manifolds. In *Spectral theory and geometry (Edinburgh, 1998)*, volume 273 of *London Math. Soc. Lecture Note Ser.*, pages 140–225. Cambridge Univ. Press, Cambridge, 1999.

[Gro93] Leonard Gross. Uniqueness of ground states for Schrödinger operators over loop groups. *J. Funct. Anal.*, 112(2):373–441, 1993.

[Gün12] Batu Güneysu. On generalized Schrödinger semigroups. *J. Funct. Anal.*, 262(11):4639–4674, 2012.

[Heb99] Emmanuel Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, volume 5 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[Hsu99] Elton P. Hsu. Estimates of derivatives of the heat kernel on a compact Riemannian manifold. *Proc. Amer. Math. Soc.*, 127(12):3739–3744, 1999.

[LHB11] József Lőrinczi, Fumio Hiroshima, and Volker Betz. *Feynman-Kac-type theorems and Gibbs measures on path space*, volume 34 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2011. With applications to rigorous quantum field theory.

[Li92] Xue-Mei Li. Stochastic differential equations on noncompact manifolds. University of Warwick Thesis, url=http://www.xuemei.org/Xue-Mei.Li-thesis.pdf, 1992.

[Li95] Xue-Mei Li. On extensions of Myers' theorem. *Bull. London Math. Soc.*, 27(4):392–396, 1995.

[Li16] Xue-Mei Li. On the semi-classical brownian bridge measure. arxiv:1607.06498, 2016.

[LY86] Peter Li and Shing-Tung Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156(3-4):153–201, 1986.

[LZ96] Xue-Mei Li and H. Z. Zhao. Gradient estimates and the smooth convergence of approximate travelling waves for reaction-diffusion equations. *Nonlinearity*, 9(2):459–477, 1996.

[Mol68] S. A. Molčanov. The strong Feller property of diffusion processes on smooth manifolds. *Teor. Verojatnost. i Primenen.*, 13:493–498, 1968.

[MP49] S. Minakshisundaram and Å. Pleijel. Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds. *Canadian J. Math.*, 1:242–256, 1949.

[MS96] Paul Malliavin and Daniel W. Stroock. Short time behavior of the heat kernel and its logarithmic derivatives. *J. Differential Geom.*, 44(3):550–570, 1996.

[Ndu09] Martin N. Ndumu. Heat kernel expansions in vector bundles. *Nonlinear Anal.*, 71(12):e445–e473, 2009.

[Nor97] J. R. Norris. Long-time behaviour of heat flow: global estimates and exact asymptotics. *Arch. Rational Mech. Anal.*, 140(2):161–195, 1997.

[Ole93] I. M. Oleuinik. On the essential selfadjointness of the Schrödinger operator on complete Riemannian manifolds. *Mat. Zametki*, 54(3):89–97, 159, 1993.

[SC10] Laurent Saloff-Coste. The heat kernel and its estimates. In *Probabilistic approach to geometry*, volume 57 of *Adv. Stud. Pure Math.*, pages 405–436. Math. Soc. Japan, Tokyo, 2010.

[She91] Shuenn Jyi Sheu. Some estimates of the transition density of a nondegenerate diffusion Markov process. *Ann. Probab.*, 19(2):538–561, 1991.

[Shu01] Mikhail Shubin. Essential self-adjointness for semi-bounded magnetic Schrödinger operators on non-compact manifolds. *J. Funct. Anal.*, 186(1):92–116, 2001.

[Sim82] Barry Simon. Schrödinger semigroups. *Bull. Amer. Math. Soc. (N.S.)*, 7(3):447–526, 1982.

[ST98] Daniel W. Stroock and James Turetsky. Upper bounds on derivatives of the logarithm of the heat kernel. *Comm. Anal. Geom.*, 6(4):669–685, 1998.

[Str00] Daniel W. Stroock. *An introduction to the analysis of paths on a Riemannian manifold*, volume 74 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.

[Stu93] Karl-Theodor Sturm. Schrödinger semigroups on manifolds. *J. Funct. Anal.*, 118(2):309–350, 1993.

[Tha97] Anton Thalmaier. On the differentiation of heat semigroups and Poisson integrals. *Stochastics Stochastics Rep.*, 61(3-4):297–321, 1997.

[Var67] S. R. S. Varadhan. Diffusion processes in a small time interval. *Comm. Pure Appl. Math.*, 20:659–685, 1967.

[Wal42] A. G. Walker. Note on a distance invariant and the calculation of Ruse's invariant. *Proc. Edinburgh Math. Soc. (2)*, 7:16–26, 1942.

[Wat88] Keith D. Watling. Formulæ for the heat kernel of an elliptic operator exhibiting small-time asymptotics. In *Stochastic mechanics and stochastic processes (Swansea, 1986)*, volume 1325 of *Lecture Notes in Math.*, pages 167–180. Springer, Berlin, 1988.