Stochastic Homogenization on Homogeneous Spaces

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Abstract

Motivated by inhomogeneous scaling of left invariant Riemannian metrics on a Lie group G, we consider a family of stochastic differential equations (SDEs) on G with Markov operators $\mathcal{L}^\epsilon = \frac{1}{\epsilon}\sum_k (A_k)^2 + \frac{1}{\epsilon}A_0 + Y_0$, where $\epsilon > 0$, $\{A_k\}$ generates a sub Lie-algebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$, and $Y_0 \in \mathfrak{g}$. Assuming that G/H is a reductive homogeneous space, in the sense of Nomizu, we take $\epsilon \to 0$ and study the asymptotics of the SDEs through the stochastic homoganization problem, and study the effective limit operators on G and on G/H.

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Contents

1 Introduction

By deforming the fibres of the Hopf fibration, the canonical round metric on the Lie group S^3 gives rise to a family of left invariant Berger's metrics which we denot by m_ϵ. As ϵ approaches zero, the Riemannian manifolds (S^3, m_ϵ) converge to the lower dimensional two sphere $S^2(\frac{1}{2})$ of radius $\frac{1}{2}$, while keeping the sectional curvatures bounded. See [13, J. Cheeger, M. Gromov], [25, K. Fukaya], and [37, A. Kasue, H. Kumura] for other types of convergences of Riemannian manifolds. Let us denote by $\Delta^\epsilon_{S^3}$ and Δ_{S^1} the Laplacians on (S^3, m_ϵ) and on S^1 respectively, also denote by Δ^h the horizontal Laplacian identified with the Laplacian on $S^2(\frac{1}{2}) = S^3/S^1$. These operators commute and $\Delta^\epsilon_{S^3} = \frac{1}{\epsilon}\Delta_{S^1} + \Delta^h$. If $\{X_1, X_2, X_3\}$ are the Pauli matrices, identified with left invariant vector fields, then $\Delta_{S^1} = (X_1)^2$, $\Delta^h = (X_2)^2 + (X_3)^2$. As ϵ approaches 0, any eigenvalues of the Laplacian $\Delta^\epsilon_{S^3}$ coming from a non-zero eigenvalue of $\frac{1}{\epsilon}\Delta_{S^1}$ is pushed to the back of the spectrum and an eigenfunction of $\Delta^\epsilon_{S^3}$, not constant in the fibre, flies away. In other words the spectrums of S^3 converge to that of S^2, see [65, S. Tanno], [3, L. Bérard-Bergery and J. -P. Bourghignon] [66, H. Urakawa] for discussions on the spectrum of Laplacians on spheres, on homogeneous Riemannian manifolds and on Riemannian submersions with totally geodesic fibres.

Another interesting family of operators is $\{\frac{1}{\epsilon}\mathcal{L}^\epsilon = \frac{1}{\epsilon^2}\Delta_{S^1} + \frac{1}{\epsilon}Y_0, \epsilon > 0\}$, where $Y_0 = aX_2 + bX_3$ is a non-zero unit length left invariant vector field. The summands no longer commute and the operators do not converge, but their orbits, the orbits of the solutions to the corresponding initial value parabolic equations, exhibit interesting asymptotics. We use the Hopf fibration to construct a converging family of first order random linear differential operators $\frac{1}{\epsilon}\tilde{\mathcal{L}}^\epsilon$ sharing the same orbits, who converge with effective limit $\lambda\Delta^h$ where λ is an explicit constant, see Example 11.1. A related family of operators were studied in [44, Li]. We will generalising this in a different direction, proving this type of convergence for homogeneous manifolds which are not necessarily compact.

Let G be a smooth connected real Lie group with a non-trivial closed proper subgroup H. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ their respective Lie algebras. We assume that H is connected, and compact for certain type of results and in this introduction. If H is compact, there exists an Ad_H-invariant inner product on $\mathfrak{g}$, descending to T_oM and induing a G-invariant Riemmanian metric.

For a positive real number ϵ, let us consider the sum of squares of operator arising from non-homogeneous scaling of Riemannian metrics,

$$\frac{1}{\epsilon}\mathcal{L}^\epsilon = \frac{1}{\epsilon^2}\mathcal{L}_0 + \frac{1}{\epsilon}Y_0, \quad \text{where} \quad \mathcal{L}_0 = \sum (A_k)^2 + A_0.$$

where A_k and Y_0 are left invariant vector fields and $\{A_k \in \mathfrak{h}\}$ is bracket generating. and $\mathcal{L}_0$ is not necessarily symmetric. We study the asymptotics of $\mathcal{L}^\epsilon$ and that of $\frac{1}{\epsilon}\mathcal{L}^\epsilon$.

The operators $\mathcal{L}^\epsilon$ are not necessarily hypoelliptic in G, and they will certainly not converge in any reasonable sense. We first consider $\mathcal{L}^\epsilon$ as a perturbation to the hypoelliptic diffusions induced by $\frac{1}{\epsilon}\mathcal{L}_0$ who would stay in the orbits/fibres of their initial values, the perturbation in Y_0 direction induces motions transversal to the fibres. Our first task is to understand the nature of the perturbation and to extract from them a family of first order random differential operators, $\tilde{\mathcal{L}}^\epsilon$, called the 'slow motions'. These slow motions would converge and their effective limit is either a one parameter subgroups of G in which case our study terminates, or a fixed point in which case we study the dynamics on the time scale $[0, \frac{1}{\epsilon}]$. On the Riemmannian homogeneous manifold, the first limit is a geodesic and the second a fixed point. On the scale of $[0, \frac{1}{\epsilon}]$ we would however consider $\frac{1}{\epsilon}\mathcal{L}_0$ as perturbation. It is counter intuitive to consider the dominate part as the perturbation. But the perturbation, although very large in magnitude, is fast oscillating. The large oscillating motion is averaged out, leaving an effective stochastic motion corresponding to a second order differential operator on G.

We will treat this as a stochastic homoganization problem for the following family of stochastic differential equations (SDEs) on G,

$$dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^N A_k(g_t^\epsilon) \circ db_t^k + \frac{1}{\epsilon} A_0(g_0^\epsilon) dt + Y_0(g_t^\epsilon) dt, \quad g_0^\epsilon = g_0, \tag{1.1}$$

where $\circ$ denotes Stratonovich integrals. These SDEs belong to a family of equations, see §3, which interpolate between translates of one parameter subgroups of G and hypoelliptic diffusion on H. Scaled by $1/\epsilon$, the Markov generator of (g_t^ϵ) is precisely $\frac{1}{\epsilon}\mathcal{L}^\epsilon$.

We use a *reductive structure*, which exists if H is compact, see §2 for further discussion including a non-Riemannian example. This is the reductive property in the sense of Nomizu [55], i.e. an Ad_H-invariant sub-space of $\mathfrak{g}$ complementing $\mathfrak{h}$ which we denote by $\mathfrak{m}$, not in the sense of having a completely irreducible adjoint representation. We further assume that $Y_0 \in \mathfrak{m}$ and describe the scale at which the limit is taken (§4-§6.2) by irreducible invariant subspaces of $\mathfrak{m}$, denoted by $\mathfrak{m}_l$, and to classify the effective limit (§7-§10).

The convergence will be in terms of the probability distributions induced by $\frac{1}{\epsilon}\tilde{\mathcal{L}}_\epsilon$ in the weak topology and Wasserstein distance on the space $C([0,1];G)$ of continuous paths over G and over G/H, see §6, where the rate of convergence for their evaluations at specific times in the Wasserstein distance is also given. The proof is a based on a theorem in [46], we only need to take care of the non-compactness of G.

A heuristic argument based on multi-scale analysis, which we give shortly, appears to suggest the following centring condition: $\int_H Y_0 F_0 dh = 0$ where F_0 is a solution to the effective parabolic equation and where dh is the right invariant Haar measure on H. This is not quite the right assumption, we will assume instead that $Y_0 \neq 0$, $\bar{Y}_0 \equiv \int_H \mathrm{Ad}(h)(Y_0) dh = 0$.

Indeed, we show that $\frac{1}{\epsilon}\tilde{\mathcal{L}}_\epsilon = \mathrm{Ad}(h_{\frac{t}{\epsilon}}(\omega))(Y_0)$ where (h_t) is a stochastic process in H, and $\bar{Y}_0 = 0$ precisely when Y_0 is orthogonal to $\mathfrak{h} \oplus \mathfrak{m}_0$, where $\mathfrak{m}_0$ is the set of Ad_H-invariant vectors in $\mathfrak{m}$ (§6.2).

The heuristic Argument. Let us give the heuristics argument, to be made rigorous in §4-§6. For simplicity take $A_0 = 0$ and denote by dh the normalised Haar measure on a compact H. Let $\mathcal{L}_0 = \frac{1}{2}\sum_{k=1}^p (A_k)^2 + A_0$ and let $F : \mathbb{R}_+ \times G \to \mathbb{R}$ be a solution to the evolution equation $\frac{\partial F}{\partial t} = \mathcal{L}^\epsilon F$. Expand F in ϵ, $F(t) = F_0(t) + \epsilon F_1(t) + \epsilon^2 F_2(t) + o(\epsilon^2)$, and plug this into the parabolic equation. Equating coefficients of ϵ^0 and ϵ^{-1} we obtain the following equations: $\mathcal{L}_0 F_1 + Y_0 F_0 = 0$, $\frac{\partial F_0}{\partial t} = Y_0 F_1 + \mathcal{L}_0 F_2$. If $Y_0 F_0$ averages to zero, the formal solution to the first equation is $F_1 = -\mathcal{L}_0^{-1}(Y_0 F_0)$. We should interpret this as an equation on H. The second equation reduces to $\frac{\partial F_0}{\partial t} = -Y_0(\mathcal{L}_0^{-1}(Y_0 F_0)) + \mathcal{L}_0 F_2$ which we integrate with respect to dh. Since $\mathcal{L}_0$ is symmetric by Lemma 5.1 and $\mathcal{L}_0 F_2$ averages to zero. We obtain the equation for the effective motion:

$$\frac{d}{dt}\int_H F_0 dh = -\int_H Y_0 \mathcal{L}_0^{-1}(Y_0 F_0) dh.$$

The computations are formal. Firstly we neglected higher powers of $\frac{1}{\epsilon}$, which are very large for ϵ small. Secondly we assumed that the Poisson equation $\mathcal{L}_0 = Y_0 F_0$ is solvable. Finally we assumed that $\int_H \mathcal{L}_0 F_2 dh = 0$. We should remark that $\mathcal{L}_0$ is not hypoelliptic on G and we must reduce it to a space where it is, to justify these assertions.

Main result. We assume that $\mathfrak{m}$ is a reductive structure in the sense of Nomizu, a list of useful reductive decompositions are given in §2. Denote by $\mathrm{Ad}_H : H \to \mathbb{L}(\mathfrak{g}; \mathfrak{g})$ the restriction of the adjoint of G to H and also the restricted representation

$\mathrm{Ad}_H : H \to \mathbb{L}(\mathfrak{m};\mathfrak{m})$. Denote by $\mathfrak{m}_0 \subset \mathfrak{m}$ the space of invariant vectors of Ad_H. Let $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_l$ be an invariant decomposition for Ad_H where, for each $l \neq 0$, $\mathfrak{m}_l$ is an irreducible invariant space. If H is compact we may and will assume that the decomposition is orthogonal. The main results are in three parts.

(1) *Separation of slow and fast variables and Reduction.* Use the Ehreshman connection on the principal bundle $\pi : G \to G/H$, determined by $\mathfrak{m}$, and horizontal lifts of curves from G/H to G we deduce stochastic equations for the horizontal lifts $\tilde{x}_t^\epsilon$ of $x_t^\epsilon = \pi(g_t^\epsilon)$, the slow motions, and for the fast motions h_t^ϵ on H.

(2) *Convergence of the slow components.* If Y_0 is orthogonal to $\mathfrak{m}_0$ and $\{A_0, A_1, \ldots, A_N\}$ generates $\mathfrak{h}$, the stochastic processes $(\tilde{x}^\epsilon_{\frac{t}{\epsilon}}, t \in [0,T])$ and $(x^\epsilon_{\frac{t}{\epsilon}}, t \in [0,T])$ converge weakly, as $\epsilon \to 0$, to $\bar{u}_t$ and $\bar{x}_t$ respectively, in the weak topology on the path space over G and in Wasserstein distance. A rate of convergence is given.

(3) *Effective Process.* Assume $A_0 = 0$ and $Y_0 \in \mathfrak{m}_l$. The effective process on G is a Markov process with generator $c\Delta_{m_l}$. If furthermore M is isotropy irreducible and if M is naturally reductive (and more generally) then $\bar{x}_t$ is a scaled Brownian motion whose scale is computable.

We indicate the problems pertinent to part (3). The reduced first oder random differential operators give rise to second order differential operators by the action of the Lie brackets. If the Lie bracket between A_k and Y_0 is not trivial, randomness is generated in the $[A_k, Y_0]$ direction and transmitted from the vertical to the horizontal directions. We ask the question weather the limiting operators are scaled horizontal Laplacians. This is so for the Hopf fibration. However it only takes a moment to figure out this cannot be always true. The noise cannot be transmitted to directions in an irreducible invariant subspace of $\mathfrak{m}$ not containing Y_0. If $\mathfrak{m}_l$ is the irreducible invariant subspace containing Y_0, the action of Ad_H generates directions in $\mathfrak{m}_l$, not any direction in a component complementary to $\mathfrak{m}_l$ and so the rank of $\bar{\mathcal{L}}$ is at most $\dim(\mathfrak{m}_l)$. Within an irreducible Ad_H-invariant subspace $\mathfrak{m}_l$, the transmission of the noise should be 'homogeneous' and we might expect that there is essentially only one, up to scalar multiplication, candidate effective second order differential operator: the generalised Laplacian' operator' $\Delta_{\mathfrak{m}_l} := \sum L_{Y_i} L_{Y_i}$ where $\{Y_i\}$ is an orthonormal basis of $\mathfrak{m}_l$. The generalised Laplacian $\Delta_{\mathfrak{m}_l}$ is independent of the choice of the basis.

If $\{A_1, \ldots, A_p\}$ is a basis of $\mathfrak{h}$, we solve a Poisson equation and prove that $\bar{\mathcal{L}} = \sum_{i,j} a_{i,j}(Y_0) L_{Y_i} L_{Y_j}$, where $\{Y_i\}$ is a basis of the irreducible Ad_H invariant space $\mathfrak{m}_l$ containing Y_0 and $a_{i,j}(Y_0)$ trigonometric functions of the adjoint sub-representation in this basis. It can be written in terms of eigenvalues of $\mathcal{L}_0$, computable from the 'Casimir', and has an Ad_H-invariant form. If $Y_0 \in \mathfrak{m}_l$ then by the real Peter-Weyl theorem, a proof of which by Dmitriy Rumynin is appended at the end of the paper, we see that $\bar{\mathcal{L}} = \frac{1}{\dim(\mathfrak{m}_l)\lambda_l} |Y_0|^2 \Delta_{\mathfrak{m}_l}$. In case G is uni-modular and $\mathfrak{m}$ is irreducible this is the 'horizontal Laplacian'. If $\{A_k\}$ is only a set of generator of the Lie algebra $\mathfrak{h}$, a similar formula holds with the constant λ_l replaced by a constant $\lambda(Y_0)$, depending on Y_0 in general.

The above descriptions are algebraic, in §10 we discuss their differential geometric counterpart. Let G be endowed with a left invariant and Ad_H-invariant Riemannian metric and let $M = G/H$ be given the induced G-invariant Riemannian metric. The translates of the one parameter subgroups of G are not necessarily geodesics for the Levi-Civita connection on G. Their projections to the Riemannian homogeneous manifold are not necessarily geodesics either. In general $\bar{\mathcal{L}}$ needs not be the Laplace-Beltrami operator even if it is elliptic for it may have a nontrivial drift. The more symmetries there are, the closer is the effective diffusion to a Brownian motion. A maximally symmetric and non-degenerate effective process ought to be a scaled Brownian motion. In other words we like to see the convergence of the random smooth curves in the orbit manifold to a scaled Brownian motion, which will be studied under the condition on the trace of ad(Z), under which $\text{trace}_{\mathfrak{m}_l} \nabla^L d = \text{trace}_{\mathfrak{m}_l} \nabla d$ and $\text{trace}_{\mathfrak{m}_l} \nabla^c d = \text{trace}_{\mathfrak{m}_l} \nabla d$ on M, where ∇ denotes the Levi-Civita connections on G and on M and ∇^c denotes the canonical connection on M.

The context. The study of parabolic differential equation of the type $\frac{\partial}{\partial t} = Y_0 + \frac{1}{\epsilon}\mathcal{L}_0$ where $\mathcal{L}_0$ is an elliptic operator and Y_0 a vector field, in the non-geometric settings goes back to S. Smoluchowski (1916) and to H. Kramers (1940) [40] and are known as Smoluchowski-Kramers limits. This was taken up in [54, E. Nelson] for a dynamical description of Brownian motion. The further scaling by $1/\epsilon$, leading to the asymptotic problems for $\frac{1}{\epsilon}Y_0 + \frac{1}{\epsilon^2}\mathcal{L}_0$, are known by the a number of terminologies in a great many subjects: averaging principle, stochastic homogenization, multi-scale analysis, singular perturbation problem, or taking the adiabatic limit. They also appear in the study of interacting particle systems. We treat our problem as a perturbation to a non conventional conservation law and uses techniques from multi-scale analysis and stochastic homogenization. We consider our SDEs as a perturbations to a generalised 'Hamiltonian' systems. For perturbation to Hamiltonian systems see M. Freidlin [22], see also [23, M. Freidlin and M. Weber]. The use of 'Hamiltonian' is quite liberal, by which we merely mean that they are conserved quantities, here they are manifold valued functions or orbits. Some ideas in this paper were developed from [44, X.-M. Li]. Convergences of Riemannian metrics from the probabilistic point of view, have been studied by N. Ikeda and Y. Ogura [35] and Y. Ogura and S. Taniguchi [57].

A collection of limit theorems, which we use, together with a set of more complete references can be found in [46, X.-M. Li]. For stochastic averaging with geometric structures, we refer to [43, X.-M. Li] for a stochastic integrable Hamiltonian systems in symplectic manifolds, to [26, P. Gargate and P. Ruffino], [31, M. Hogele and P. Ruffino] for studies of stochastic flows on foliated manifolds.

We use the multi-scale analysis to separate the slow and fast variables and draw from the work in [20, K. D. Elworthy, Y. LeJan and X.-M. Li]. Our reduction in complexity bears resemblance to aspects of geometric mechanics and Hamiltonian reduction, see [42, L-C. Joan-Andreu and O. Juan-Pablo], also [32, D. Holm, J. Marsden, T. Ratiu, A. Weinstein], and [49, J. Marsden, G. Misiolek, J.-P. Ortega, M. Perlmutter and T. Ratiu].

Our reduced equations are ODEs on manifolds with random coefficients, the study of random ODEs on $\mathbb{R}^n$ goes back to [64, R.L. Stratonovich], [29, R.Z. Khasminski], [58, W. Kohler and G. C. Papanicolaou] and [59, G. C. Papanicolaou and S. R. S. Varadhan], [11, A. N. Borodin and M. Freidlin]. The convergence from operator semigroup point of view is studied in [41, T. Kurtz].

The asymptotics of linear operators of the form $\frac{1}{\epsilon^2}\mathcal{L}_0 + \frac{1}{\epsilon}\mathcal{L}_1 + \mathcal{L}_0$ are often referred as taking the adiabatic limits in some contexts, see for example [50, R.R. Mazzeo-R. B. Melrose] and [5, N. Berline, E. Getzler, and M. Vergne]. These operators act not just on scalar functions and their objectives are motivated by problems in topology. Motivated by analysis of loop spaces J.-M. Bismut [8, 9] studied the differential operators $\frac{1}{2\epsilon^2}(\Delta + |Y|^2 - n) - \frac{U}{\epsilon}$ where U is a vector field on the tangent bundle generating a geodesic flow and Y an element of the tangent bundle or cotangent bundle of a compact

Riemannian manifold M including special studied on symmetric spaces and reductive Lie groups, see also [10, J.-M. Bismut and G. Lebeau] for an earlier work. In another direction, E. Nelson [54], prove that the physical particles whose velocity are one dimensional Ornstein-Uhlenbeck processes approximates, in high temperature, one dimensional Brownian motions. The was generalised to manifolds in [15, R. M. Dowell]. This study bears much resemblance to the above mentioned articles; however the model and the objectives in [45, Li] are much closer: SDEs on the orthonormal frame bundle are used to show Brownian motions are generated by geodesics with rapidly changing directions.

The problem of identifying limit operators is reminiscent of that in M. Freidlin and A. D. Wentzell [24] where an explicit Markov process on a graph was obtained from the level sets of a Hamiltonian function in $\mathbb{R}^2$, but they are obviously very different in nature.

2 Notation, Preliminaries, and examples

If $g \in G$ we denote by L_g and R_g the left and right multiplications on G. Denote by dL_g and dR_g their differentials. The differentials are also denoted by TL_g and TR_g where these are traditionally used. Denote by Ad the adjoint representation of G with Ad_H its restriction to H and ad the adjoint representation on $\mathfrak{g}$ with restriction $\mathrm{ad}_{\mathfrak{h}}$. Both the notation $\mathrm{ad}_X Y$ and $[X, Y]$ are used. If $\mathfrak{m}$ is a subspace of $\mathfrak{g}$, $g\mathfrak{m}$ denotes the set of left translates of elements of $\mathfrak{m}$ to T_gG. We identify the set of left invariant vector fields with elements of the Lie algebra, denote them by the same letters or with the superscript $*$. If V is a vector field and f function, both $L_V f$ and Vf are use to denote Lie differentiation in the direction V, the latter for simplicity, the first to avoid confusing with operators denoted by $\mathcal{L}$. Also, Δ_H denotes the Laplacian for a bi-invariant metric on the Lie group H.

A manifold M is homogeneous if there is a transitive action by a Lie group G. It can be represented as the coset space G/H where H is the isotropy group at a point o. The coset space is given the unique manifold structure such that $gH \in G/H \mapsto \bar{L}_g o \in M$ is a diffeomorphism where $\bar{L}_g$ denotes the action of g. If $\pi : G \to M$ denotes the projection taking g to the coset gH and $1 \in G$ the identity, the sub-Lie algebra $\mathfrak{h}$ is the kernel of $(d\pi)_1$ and T_oM is isomorphic to a complement of $\mathfrak{h}$ in $\mathfrak{g}$.

The homogeneous space G/H is reductive if in the Lie algebra $\mathfrak{g}$ there exists a subspace $\mathfrak{m}$ such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a vector space direct sum. By $\mathrm{Ad}_H(\mathfrak{m})$ we mean the image of H under the adjoint map, treated as linear maps restricted to $\mathfrak{m}$. We say that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a *reductive decomposition* and $(\mathfrak{g}, h)$ a reductive pair. This implies that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and vice versa if H is connected. The homogeneous space is also called reductive, a reductive property in the sense of Nomizu, a concept different from a Lie group being reductive. In particular, the Lie group G is not necessarily reductive, by which we mean its adjoint representation is completely reducible.

We discuss briefly the connectedness of the Lie group. Firstly if H and G/H are connected, so is G. This follows from the fact that a topological space X is connected if and only if every continuous function from X to $\{0, 1\}$ is constant. The identity component G^0 of G is a normal subgroup of G, and any other component of G is a coset of G^0. The component group G/G^0 is discrete. If a Lie group G acts transitively on a connected smooth manifold M, so does G^0. See [27]. Our stochastic processes are continuous in time, and hence we assume that both G and H are connected.

The existence of an Ad_Hinvariant inner product is much easier than requesting an Ad-invariant inner product on $\mathfrak{g}$ which is equivalent to G is of compact type. If H is compact, by the unitary trick, there exists an Ad_H-invariant inner product on $\mathfrak{g}$ and a reductive structure by setting $\mathfrak{m} = \mathfrak{h}^\perp$. The compactness of H is not a restriction for a Riemannian homogeneous manifold. if G/H is a connected Riemannian homogeneous space and G is connected then by a theorem of van Danzig and van der Waerden (1928), the isotropy isometry groups at every point is compact, [38, Kobayashi]. If H is connected and its Lie algebra is reductive in $\mathfrak{g}$, in the sense that ad($\mathfrak{h}$) in $\mathfrak{g}$ is completely reducible, then G/H is reductive. The Euclidean space example below will cover the averaging model used in [47, M. Liao and L. Wang]. Let $G = E(n)$ be the space of rigid motions on $\mathbb{R}^n$,

$$E(n) = \left\{ \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix} : R \in SO(n), v \in \mathbb{R}^n \right\}$$

and H its subgroup of rotations. Elements of H fix the point $o = (0,1)^T$, $E(n)/H = \{(x,1)^T, x \in \mathbb{R}^n\}$ and a matrix in $E(n)$ projects to its last column. We may take

$$\mathfrak{h} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, A \in \mathfrak{so}(n) \right\}, \ \mathfrak{m} = \left\{ \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}, v \in \mathbb{R}^n \right\}.$$

A reductive structure may not be unique. For example, let G be a connected compact Lie group, $H = \{(g,g) : g \in G\}$ and $\mathfrak{h} = \{(X,X) : X \in \mathfrak{g}\}$. Then $G = (G \times G)/H$ is a reductive homogeneous space in three ways: $\mathfrak{m}^0 = \{(X,-X), X \in \mathfrak{g}\}$,

$$\mathfrak{m}^+ = \{(0,X), X \in \mathfrak{g}\}, \qquad \mathfrak{m}^- = \{(X,0), X \in \mathfrak{g}\}.$$

The first one, $\mathfrak{h} \oplus m^0$, is the symmetric decomposition.

A definite metric is not necessary either. If $\mathfrak{g}$ admits an Ad_H invariant non-degenerate bilinear form such that its restriction to $\mathfrak{h}$ is non-degenerate, let $\mathfrak{m} = \{Y \in \mathfrak{g} : B(X,Y) = 0, \forall X \in \mathfrak{h}\}$. In [39, Chap. 10], Kobayashi and Nomizu considered the case where B is Ad(G) invariant. Their proof can be modified to work here.

2.1 The Motivating Example

As a motivating example, we take $G = SU(2)$, $H = U(1)$ and the bi-invariant Riemannian metric such that $\langle A, B\rangle = \frac{1}{2}$ trace AB^* where $A, B \in \mathfrak{g}$. Let

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{2.1}$$

be the Pauli matrices. They form an orthonormal basis of the Lie algebra $\mathfrak{g}$ with respect to the bi-invariant metric and G is the canonical three sphere in $\mathbb{R}^4$. The Lie algebra $\mathfrak{h}$ is generated by X_1 and we take $\mathfrak{m}$ to be the vector space generated by the remaining two Pauli matrices and obtain the family of Berger's metric, which we denote as before by $\{m^\epsilon, \epsilon > 0\}$.

Let us first take the Brownian motions on Bergers' spheres. A Brownian motion on (S^3, m^ϵ) is determined by the Hörmander type operator $\Delta^\epsilon = \frac{1}{\epsilon}(X_1)^2 + \sum_{i=2}^3 (X_i)^2$. Although no longer associated with the round metric, there are many symmetries in the following SDEs,

$$dg_t = \frac{1}{\sqrt{\epsilon}} X_1(g_t) \circ db_t^1 + X_2(g_t) \circ db_t^2 + X_3(g_t) \circ db_t^3. \tag{2.2}$$

In particular the probability distributions of its slow components at time t are independent of ϵ, see Example 4.3 in Section 4. Breaking up the symmetry we may consider the equation,

$$dg_t = \frac{1}{\sqrt{\epsilon}} X_1(g_t) \circ db_t^1 + X_2(g_t) \circ db_t^2,$$

in which the incoming noise is 2-dimensional and its Markov generator satisfies the strong Hörmander's conditions. We will go one step further and use a one dimensional noise. Let (b_t) be a real valued Brownian motion. If $Y_0 = c_2 X_2 + c_3 X_3$, where c_2, c_3 are numbers not simultaneously zero, the infinitesimal generator of the equation

$$dg_t = \frac{1}{\sqrt{\epsilon}} X_1(g_t) \circ db_t + (Y_0)(g_t) dt, \tag{2.3}$$

satisfies weak Hörmander's conditions. Indeed by the structural equations, $[Y_0, X_1] = 2c_2 X_3 - 2c_3 X_2$, and the matrix

$$\begin{pmatrix} \frac{1}{\sqrt{\epsilon}} & 0 & 0 \\ 0 & c_2 & -2c_3 \\ 0 & c_3 & 2c_2 \end{pmatrix}.$$

is not degenerate. In general, equation (1.1) need not satisfy Hörmander's condition. This example is concluded in Example 11.1, using Corollary 9.2.

3 The Interpolation Equations

Given a left invariant Riemannian metric on G, we consider a family of non-homogeneously scaled Riemannian metrics. To define these let $\{X_1, \dots, X_n\}$ be an orthonormal basis of $\mathfrak{g}$ extending an orthonormal basis $\{X_1, \dots, X_p\}$ of $\mathfrak{h}$ and let

$$E^\epsilon = \left\{ \frac{1}{\sqrt{\epsilon}} X_1^*, \dots, \frac{1}{\sqrt{\epsilon}} X_p^*, X_{p+1}^*, \dots X_n^* \right\}.$$

The superscript $*$ above a letter denotes the corresponding left invariant vector field which we from time to time will omit in favour of simplicity. By declaring E^ϵ an orthonormal frame, we have a family of left invariant Riemannian metrics on G, which we denote by m^ϵ. In this article we are not concerned with the problem of keeping the sectional curvatures bounded, and G needs not be compact.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space satisfying the usual assumptions and $(b_t^k, k = 1, \dots, N,)$ a family of independent real valued Brownian motions. Let γ, δ be positive real numbers, $X_k \in \mathfrak{h}$ as above, and $Y_0 \in \mathfrak{g}$. We study the stochastic dynamics associated with the above inhomogeneous scalings of Riemannian metrics and propose the following interpolation equation that, in the limit of $\epsilon \to 0$, describes the 'effective motion' across the 'orbits'.

$$dg_t = \sum_{k=1}^{p} \gamma X_k(g_t) \circ db_t^k + \delta Y_0(g_t) dt, \tag{3.1}$$

with a given initial value g_0. Here $\circ$ denotes Stratonovich integration. Their solutions are Markov processes whose probability laws are determined by the fundamental solutions to the following parabolic equation $\frac{\partial}{\partial t} = \frac{1}{2}\gamma^2 \sum_{k=1}^{p} (X_k)^2 + \delta Y_0$.

For $\gamma = 1$ and $\delta = \frac{1}{|Y_0|}$, these equations are driven by unit length vector fields on the Riemannian manifold (G, m^ϵ). If $\delta = 0$ the solution with its initial value the identity of the group is a scaled Brownian motion on the subgroup H. If $\gamma = 0$, the solutions are translates of the one parameter family subgroup generated by Y_0. We denote the vector space generated by $\{X_{p+1}, \dots, X_n\}$ by $\mathfrak{m}$, assumed to be Ad_H-invariant. If $Y_0 \in \mathfrak{m}$, these one parameter families, $\exp(\delta t Y_0)$, are horizontal curves for the horizontal distribution determined by $\mathfrak{m}$. The solutions of (3.1) interpolate between translates of the one parameter group, generated by δY_0, on G and Brownian motion on H. If we take $\gamma \to \infty$ while keeping δ fixed, what can we say about the solutions of these equations?

We will work on more general operators, allowing $\{A_k\}$ to be a Lie algebra generating subset of $\mathfrak{h}$ in place of ellipticity. Take $Y_0 \in \mathfrak{m}$. We study the family of SDEs, where $\epsilon > 0$ is a small parameter,

$$dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^N A_k(g_t^\epsilon) \circ db_t^k + \frac{1}{\epsilon} A_0(g_0^\epsilon) dt + Y_0(g_t^\epsilon) dt, \quad g_0^\epsilon = g_0.$$

The condition $g_0^\epsilon = g_0$, independent of ϵ, is assumed only for the simplicity of the statements. Let $\mathcal{L}_0 = \sum (A_k)^2 + \frac{1}{\epsilon} A_0$. The corresponding parabolic problems are:

$$\frac{\partial}{\partial t} = \frac{1}{\epsilon}\mathcal{L}_0 + Y_0, \qquad \frac{\partial}{\partial t} = \frac{1}{\epsilon^2}\mathcal{L}_0 + \frac{1}{\epsilon} Y_0.$$

For intuition, let us review the theory for randomly perturbed Hamiltonian dynamics. If H is a function on R^{2n}, it is a first integral of the Hamiltonian system: $\dot{q} = -\frac{\partial H}{\partial p}, \dot{p} = \frac{\partial H}{\partial q}$. Let $x^\epsilon(t) := (q^\epsilon(t), p^\epsilon(t))$ denote solutions to a perturbed Hamiltonian system, which we do not specify, then $H^\epsilon(t) := H(p^\epsilon(t), q^\epsilon(t))$ varies slowly with t on $[0, 1]$. Under suitable mixing conditions on the perturbation the stochastic processes $H^\epsilon(t/\epsilon)$ converge, see e.g. [24, M. Freidlin and A. D. Wentzel] and [2, V. I. Arnold]. Although we do not have a Hamiltonian system, the projection π is a conservation law for an 'unperturbed' system. If (y_t^ϵ) are solutions to the equations $dy_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^N A_k(y_t^\epsilon) \circ db_t^k + \frac{1}{\epsilon} A_0(y_t^\epsilon)$, where $A_k \in \mathfrak{h}$, then $\pi(y_t^\epsilon) = \pi(y_0^\epsilon)$ for all t. The stochastic dynamics can be considered as perturbations to the orbits of the vertical motions whose 'constant of motion' taking its value in the homogeneous manifold, the orbit manifold. There will be of course extra difficulties, due to the fact that our slow motions are not local functions of (g_t^ϵ), they depend not simply on (g_t^ϵ) but on its whole trajectory.

For ϵ small, we would expect $\pi(g_t^\epsilon)$ measures its deviation from the 'orbit' containing g_0^ϵ. The inherited non-linearity from π causes some technical problems, for example the homogeneous manifold is in general not prarallelizable, so it is not easy to work directly with $x_t^\epsilon = \pi(g_t^\epsilon)$. The operator $\frac{1}{\epsilon}\mathcal{L}_0 + Y_0$ does not satisfy Hörmander's conditions, so we do not wish to work directly with (g_t^ϵ) either. To overcome these difficulties we assume that $\mathfrak{g}$ has a reductive decomposition, with which we construct a stochastic process (u_t^ϵ) on G, having the same projection as (g_t^ϵ). Since $\mathfrak{m}$ is Ad_H invariant, G is a principal bundle over M with structure group H. We lift (x_t^ϵ) to G to obtain a 'horizontal' stochastic process $(\tilde{x}_t^\epsilon)$ covering (x_t^ϵ). The horizontality is with respect to the Ehreshmann connection determined by $\mathfrak{m}$. Then (u_t^ϵ) is the horizontal lift process. As 'perturbations' to the random motions on the fibres, the horizontal lifts $(\tilde{x}_t^\epsilon)$ describe transverse motions across the fibres.

This consideration has a bonus: in addition to asymptotic analysis of the x processes, we also obtain information on the asymptotic properties of their horizontal lifts. This is more striking if we get out of the picture of homogenization for a moment, and consider instead the three dimensional Heisenberg group as a fibre bundle over $\mathbb{R}^2$. The horizontal lift of an $\mathbb{R}^2$-valued Brownian motion to the Heisenberg group is their stochastic Lévy area. Horizontal lifts of stochastic processes are standard tools in the study of Malliavin calculus in association with the study of the space of continuous paths over a Riemannian manifold. In §4 below we deduce an explicit equation for the horizontal lift in terms of the vertical component of (g_t^ϵ), making further analysis possible.

4 Reduction

The sub-group H acts on G on the right by group multiplication and we have a principal bundle structure $P(G, H, \pi)$ with base space M and structure group H. Each fibres $\pi^{-1}(x)$ is diffeomorphic to H and the kernel of $d\pi_g$, the differential at g, is $g\mathfrak{h} = \{TL_g(X) : X \in \mathfrak{h}\}$ For $a \in G$, denote by $\bar{L}_a$ the left action on M: $\bar{L}_a(gH) = agH$. Then $\pi \circ L_a = \bar{L}_a \circ \pi$ and $(d\pi)_a dL_a|_{T_oM} = (d\bar{L}_a)(d\pi)_1$.

In this section we assume that H contains no normal subgroup of G, which is equivalent to G acts on G/H effectively, i.e. an element of G acting as the identity transformation on G/H is the identity element of g. This is not a restriction on the homogeneous space. If G does not act transitively on $M = G/H$, then there exists a normal subgroup H^0 of H such that G/H^0 acts transitively on $G/H = (G/H^0)/(H/H^0)$. We identify $\mathfrak{m}$ with T_oM as below:

$$X \mapsto (d\pi)_1(X) = \frac{d}{dt}|_{t=0}\, d\bar{L}_{\exp(tX)}o.$$

An Ehreshmann connection is a choice of a set of complements of $g\mathfrak{h}$ that is right invariant by the action of H. Our basic assumption is that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a *reductive decomposition*, in which case $(gh)\mathfrak{m} = TR_h(g\mathfrak{m})$ and $T_gG = g\mathfrak{h} \oplus g\mathfrak{m}$ is an Ehreshmann connection. See [39, S. Kobayashi and K. Nomizu].

An Ehreshmann connection determines and is uniquely determined by horizontal lifting maps $\mathfrak{h}_u : T_{gH}M \to T_uG$ where $u \in \pi^{-1}(gH)$. Furthermore, to every piecewise C^1 curve c on M and every initial value $\tilde{c}(0) \in \pi^{-1}(c(0))$, there is a unique curve $\tilde{c}$ covering c with the property that $\frac{d}{dt}\tilde{c}(t) \in \tilde{c}(t)\mathfrak{m}$. See Besse [7]. We can also horizontally lift a sample continuous semi-martingale. The case of the linear frame bundle is specially well known, see J. Eells and D. Elworthy [17], P. Malliavin [48]. See also M. Arnaudon [1], M. Emery [21] and D. Elworthy [18], and N. Ikeda, S. Watanabe [36]. This study has been taken further in D. Elworthy, Y. LeJan, X.-M. Li [20], in connection with horizontal lifts of intertwined diffusions. A continuous time Markov process, whose infinitesimal generator is in the form of the sum of squares of vector fields, is said to be horizontal if the vector fields are horizontal vector fields.

Let $\{b_t^l, w_t^k, k = 1, \dots, N_1, l = 1, \dots, N_2\}$ be real valued, not necessarily independent, Brownian motions. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive structure . Let $\{A_i, 1 \le i \le p\}$ be a basis of $\mathfrak{h}$, $\{X_j, p+1 \le j \le n\}$ a basis of $\mathfrak{m}$, and $\{c_k^i, c_l^j\}$ is a family of real valued smooth functions on G. Let ϖ be the canonical *connection 1-form* on the principal bundle $P(G, H, \pi)$, determined by $\varpi(A_k^*) = A_k$ whenever $A_k \in \mathfrak{h}$ and $\varpi(X_j^*) = 0$ whenever $X_j \in \mathfrak{m}$. A left invariant vector field corresponding to a Lie algebra element is denoted by an upper script $*$ for emphasizing.

Definition 4.1 A semi-martingale $(\tilde{x}_t)$ in G is horizontal if $\varpi(\circ d\tilde{x}_t) = 0$. If (x_t) is a semi-martingale on M, we denote by $(\tilde{x}_t)$ its horizontal lift.

Let $Y_k^h(g) = \sum_{i=p+1}^n c_k^i(g)X_i^*(g)$ and $Y_l^v(g) = \sum_{j=1}^p c_l^j(g)A_j^*(g)$. Denote by $(g_t, t < \zeta)$ the maximal solution to the following system of equations,

$$dg_t = Y_0^h(g_t)dt + \sum_{k=1}^{N_1} Y_k^h(g_t) \circ dw_t^k + Y_0^v(g_t)dt + \sum_{l=1}^{N_2} Y_l^v(g_t) \circ db_t^l, \tag{4.1}$$

with initial value g_0 and $x_t = \pi(g_t)$. For simplicity let $b_t^0 = t$ and $w_t^0 = t$.

In the lemma below, we split (g_t) into its 'horizontal' and 'vertical part' and describe the horizontal lift of the projection of (g_t) by an explicit stochastic differential equation where the role played by the vertical part is transparent.

Lemma 4.1 *Let $x_0 = \pi(g_0)$. Take $u_0 \in \pi^{-1}(x_0)$ and define $a_0 = u_0^{-1}g_0$. Let $(u_t, a_t, t < \eta)$ be the maximal solution to the following system of equations*

$$du_t = \sum_{k=0}^{N_1} \sum_{i=p+1}^{n} c_k^i(u_t a_t) \left(\mathrm{Ad}(a_t)X_i\right)^* (u_t) \circ dw_t^k \tag{4.2}$$

$$da_t = \sum_{l=0}^{N_2} \sum_{j=1}^{p} c_l^j(u_t a_t) A_j^*(a_t) \circ db_t^l. \tag{4.3}$$

Then the following statements hold.

(1) $(u_t a_t, t < \eta)$ solves (4.1). Furthermore $\eta \le \zeta$ where ζ is the life time of (g_t).

(2) $(u_t, t < \eta)$ is a horizontal lift of $(x_t, t < \zeta)$. Consequently $\zeta = \eta$ a.s.

(3) If $(\tilde{x}_t, t < \zeta)$ is an horizontal lift of $(x_t, t < \zeta)$, it is a solution of (4.2) with $u_0 = \tilde{x}_0$.

Proof (1) Define $\tilde{g}_t := u_t a_t$. On $\{t < \eta\}$, we have

$$d\tilde{g}_t = dR_{a_t} \circ du_t + (a_t^{-1} \circ da_t)^*(\tilde{g}_t).$$

Here dR_g denotes the differential of the right translation R_g;, a_t^{-1} in the last term denotes the action of the differential, $dL_{(a_t)^{-1}}$, of the left multiplication. See page 66 of S. Kobayashi, K. Nomizu [39]. The stochastic differential d on both the left and right hand side denotes Stratonovich integration. Then $(\tilde{g}_t, t < \eta)$ is a solution of (4.1), which follows from the computations below.

$$\begin{aligned} d\tilde{g}_t =& \sum_{k=0}^{N_1} \sum_{i=p+1}^{n} c_k^i(u_t a_t) dR_{a_t} \left(\mathrm{Ad}(a_t)X_i\right)^* (u_t) \circ dw_t^k \\ &+ \sum_{l=0}^{N_2} \sum_{j=1}^{p} c_l^j(u_t a_t) A_j^*(\tilde{g}_t) \circ db_t^l. \end{aligned}$$

Since $dR_{a_t} \left(\mathrm{Ad}(a_t)(X_j)\right)^* (u_t) = X_j^*(\tilde{g}_t)$,

$$d\tilde{g}_t = \sum_{k=0}^{N_1} \sum_{j=p+1}^{n} c_k^j(\tilde{g}_t) \left(X_k\right)^* (u_t) \circ dw_t^k + \sum_{l=0}^{N_2} Y_l^v(\tilde{g}_t) \circ db_t^l,$$

which is equation (4.1). Since the coefficients of (4.1) are smooth, pathwise uniqueness holds. In particular $g_t = \tilde{u}_t a_t$ and the life time ζ of (4.1) must be greater or equal to η.

(2) It is clear that $a_0 \in H$ and $\varpi(\circ du_t) = 0$. Let $y_t = \pi(u_t)$. Then

$$dy_t = \sum_{k=0}^{N_1} \sum_{i=p+1}^{n} c_k^i(u_t a_t) d\bar{L}_{u_t a_t} \left(d\pi(X_i)\right) \circ dw_t^k,$$

following from the identity $d\pi\left((\mathrm{Ad}(a)X_i)^*(u)\right) = d\bar{L}_u d\bar{L}_a \left(d\pi(X_i)\right)$. By the same reasoning (x_t) satisfies the equation

$$dx_t = \sum_{k=0}^{N_1} d\pi(Y_k^h(g_t)) \circ dw_t^k.$$

By the definition, $d\pi(Y_k^h(g_t)) = \sum_{i=p+1}^n c_k^i(g_t) d\pi(X_i^*(g_t))$ and $d\pi(X_i^*(g_t)) = T\bar{L}_{g_t} d\pi(X_i)$. Using part (1), $g_t = u_t a_t$, we conclude that the two equations above are the same and $\pi(u_t) = x_t$. This concludes that (u_t) is a horizontal lift of (x_t) up to time η.

It is well known that through each u_0 there is a unique horizontal lift $(\tilde{x}_t)$ and the life time of $(\tilde{x}_t)$ is the same as the life time of (x_t). See I. Shigekawa [62] and R. Darling [14]. The life time of (x_t) is ζ. Let $\tilde{a}_t$ be the process such that $g_t = u_t \tilde{a}_t$ for $t < \eta$. On $\{t < \eta\}$, $\tilde{a}_t = a_t$ and $u_t = \tilde{x}_t$. If $\eta < \zeta$, as $t \to \eta$, $\lim_{t\to\eta} u_t$ leaves every compact set. This is impossible as it agrees with $\tilde{x}_t$. Similarly (a_t) cannot explode before ζ.

(3) Let $(g_t, t < \zeta)$ be a solution of (4.1) and set $x_t = \pi(g_t)$. For each t, g_t and $\tilde{x}_t$ belong to the same fibre. Define $k_t = \tilde{x}_t^{-1} g_t$, which takes values in H and is defined for all $t < \zeta$. Then,

$$d\tilde{x}_t = dR_{(k_t)^{-1}} \circ dg_t + \left(k_t \circ d(k_t)^{-1}\right)^* (\tilde{x}_t).$$

From this and equation (4.1) we obtain the following,

$$\begin{aligned} d\tilde{x}_t = dR_{(k_t)^{-1}} &\left(\sum_{k=0}^{N_1} \sum_{i=p+1}^{n} c_k^i(g_t) X_i^*(g_t) \circ dw_t^k \right) \\ + dR_{(k_t)^{-1}} &\left(\sum_{l=0}^{N_2} \sum_{j=1}^{p} c_l^j(g_t) A_j^*(g_t) \circ db_t^l \right) + ((k_t) \circ d(k_t)^{-1})^*(\tilde{x}_t). \end{aligned} \tag{4.4}$$

We apply the connection 1-form ϖ to equation (4.4), observing $\omega(\circ d\tilde{x}_t) = 0$ and $\mathrm{Ad}(k_t)(X_i) \in \mathfrak{m}$,

$$\begin{aligned} 0 &= \sum_{l=0}^{N_2} \sum_{j=1}^{p} c_l^j(g_t) \varpi_{\tilde{x}_t} \left(dR_{(k_t)^{-1}} A_j^*(g_t)\right) \circ db_t^l + (k_t) \circ d(k_t)^{-1} \\ &= \sum_{l=0}^{N_2} \sum_{j=1}^{p} c_l^j(g_t) \mathrm{Ad}(k_t) A_j \circ db_t^l + (k_t) \circ d(k_t)^{-1}. \end{aligned}$$

We have used the fact that X_j^* are horizontal, $dR_{(k_t)^{-1}}\left((X_j)^*(g_t)\right) = \left(\mathrm{Ad}(a_t)(X_j)\right)^*(\tilde{x}_t)$, and $\varpi_{ga^{-1}}(R_{(a^{-1})_*} w) = \mathrm{Ad}(a)\varpi_g(w)$ for any $w \in T_g G$. It follows that $d(k_t)^{-1} =$

$-\sum_{l=0}^{N_2}\sum_{j=1}^{p} c_l^j(g_t)R_{(k_t)^{-1}}A_j \circ db_t^l$. By the product rule,

$$dk_t = \sum_{l=0}^{N_2}\sum_{j=1}^{p} c_l^j(g_t)A_j^*(k_t) \circ db_t^l.$$

Thus (k_t) solves equation (4.3) and we take this back to (4.4). Since the vertical vector field associated to $k_t \circ dk_t^{-1}$ evaluated at $\tilde{x}_t$ is given by the formula

$$(k_t \circ d(k_t)^{-1})^*(\tilde{x}_t) = -\sum_{l=0}^{N_2}\sum_{j=1}^{p} c_l^j(g_t)(\mathrm{Ad}(k_t)A_j)^*(\tilde{x}_t) \circ db_t^l,$$

the second term and the third term on the right hand side of (4.4) cancel. Using the same computation given earlier, we see that

$$\begin{aligned} d\tilde{x}_t &= dR_{(k_t)^{-1}}\left(\sum_{k=0}^{N_1}\sum_{i=p+1}^{n} c_k^i(g_t)X_i^*(g_t) \circ dw_t^k\right) \\ &= \sum_{k=0}^{N_1}\sum_{i=p+1}^{n} c_k^i(\tilde{x}_t k_t)\left(\mathrm{Ad}(k_t)X_i\right)^*(\tilde{x}_t) \circ dw_t^k, \end{aligned}$$

proving that $(\tilde{x}_t, t < \zeta)$ is a solution of (4.2) and concludes the proof. In particular $\zeta \geq \tau$. □

We observe that the Ehreshmann connection induced by the reductive decomposition is independent of the scaling of the Riemannian metric.

Corollary 4.2 *Let $\epsilon_l > 0$ and $Y_0 \in \mathfrak{m}$. Let (g_t^ϵ) be a solution to the equation*

$$dg_t^\epsilon = Y_0^h(g_t^\epsilon)dt + \sum_{k=1}^{N_1} Y_k^h(g_t^\epsilon) \circ dw_t^k + \frac{1}{\epsilon_0}Y_0^v(g_t^\epsilon)dt + \sum_{l=1}^{N_2}\frac{1}{\epsilon_l}Y_k^v(g_t^\epsilon) \circ db_t^l, \quad g_0^\epsilon = g_0. \tag{4.5}$$

Then the horizontal lift of $\pi(g_t^\epsilon)$ satisfies the following system of equations

$$\begin{aligned} d\tilde{x}_t^\epsilon =& \sum_{k=1}^{N_1}\sum_{i=p+1}^{n} c_k^i(\tilde{x}_t^\epsilon a_t^\epsilon)\left(\mathrm{Ad}(a_t^\epsilon)X_i\right)^*(\tilde{x}_t^\epsilon) \circ dw_t^k \\ &+ \sum_{i=p+1}^{n} c_0^i(\tilde{x}_t^\epsilon a_t^\epsilon)\left(\mathrm{Ad}(a_t^\epsilon)X_i\right)^*(\tilde{x}_t^\epsilon)\, dt, \qquad (4.6) \\ da_t^\epsilon =& \sum_{l=1}^{N_2}\frac{1}{\epsilon_l}\sum_{j=1}^{p} c_l^j(\tilde{x}_t^\epsilon a_t^\epsilon)A_j^*(a_t^\epsilon) \circ db_t^l + \sum_{j=1}^{p}\frac{1}{\epsilon_0}c_0^j(\tilde{x}_t^\epsilon a_t^\epsilon)A_j^*(a_t^\epsilon)\, dt, \end{aligned}$$

up to an explosion time. Here $\tilde{x}_0^\epsilon = g_0$ and a_0^ϵ is the identity.

Example 4.3 Let us take the Hopf fibration $\pi : SU(2) \to S^2(\frac{1}{2})$, given the bi-invariant metric. If we represent $SU(2)$ by the unit sphere in $\mathbb{C}^2$ and $S^2(\frac{1}{2})$ as a subset in $\mathbb{R} \oplus \mathbb{C}$, the Hopf map is given by the formula $\pi(z, w) = (\frac{1}{2}(|w|^2 - |z|^2), z\bar{w})$. It is a Riemannian submersion. Let $\{X_1, X_2, X_3\}$ be Pauli matrices defined by (2.1) and let $\mathfrak{m} = \langle X_2, X_3\rangle$. Then $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$. This is easily seen from the structure of the Lie bracket: $[X_1, X_2] = -2X_3, \quad [X_2, X_3] = -2X_1, \quad [X_3, X_1] = -2X_2$. Let (g_t^ϵ) be a solution to the equation (2.2) and $x_t^\epsilon = \pi(g_t^\epsilon)$. Denote by (u_t^ϵ) the horizontal lift of (x_t^ϵ). By Corollary 4.2, $(u_t^\epsilon, a_t^\epsilon)$ satisfies

$$du_t^\epsilon = (\mathrm{Ad}(a_t^\epsilon)X_2)(u_t^\epsilon) \circ db_t^2 + (\mathrm{Ad}(a_t^\epsilon)X_3)(u_t^\epsilon) \circ db_t^3,$$
$$da_t^\epsilon = \frac{1}{\sqrt{\epsilon}} X_1(a_t^\epsilon) \circ db_t^1.$$

Since the metric is invariant by Ad_H, $(u_t^\epsilon, t \geq 0)$ is a horizontal Brownian motion, its Markov generator is the horizontal Laplacian $\Delta^h = \mathrm{trace}\,\nabla_{\mathfrak{m}} d$. Furthermore $\{\pi_*((\mathrm{Ad}(a_t^\epsilon)X_2)^*), \pi_*((\mathrm{Ad}(a_t^\epsilon)X_3)^*)\}$ is an orthonormal frame in $S^2(\frac{1}{2})$ and then for each $\epsilon > 0$, $(x_t^\epsilon, t \geq 0)$ is a Brownian motion on $S^2(\frac{1}{2})$.

Corollary 4.4 *Let $Y_0 \in \mathfrak{m}$, $A_0 \in \mathfrak{h}$ and $\{A_k, 1 \leq k \leq N\} \subset \mathfrak{h}$. If there is an Ad_H invariant inner product on $\mathfrak{g}$, the following SDEs are conservative for every ϵ.*

$$\dot{u}_t^\epsilon = \left(\mathrm{Ad}(h_{\frac{t}{\epsilon}})Y_0\right)^*(u_t^\epsilon), \tag{4.7}$$

$$dh_t = \sum_{k=1}^N A_k^*(h_t) \circ db_t^k + A_0^*(h_t)dt \tag{4.8}$$

$$dg_t^\epsilon = Y_0^*(g_t^\epsilon)dt + \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^N A_k^*(g_t^\epsilon) \circ db_t^k + \frac{1}{\epsilon} A_0^*(g_t^\epsilon)dt. \tag{4.9}$$

Furthermore $\pi(g_t^\epsilon) = \pi(u_t^\epsilon)$ and $g_t^\epsilon = u_t^\epsilon h_{\frac{t}{\epsilon}}$ a.s..

Proof With respect to the left invariant Riemannian metric on G generated by the Ad_H-invariant inner product, the random vector fields $(\mathrm{Ad}(h_{\frac{t}{\epsilon}}^\epsilon)Y_0)^*$ are bounded, so (4.7) is conservative for almost surely all ω. That (4.8) does not explode is clear, c.f. Lemma 5.1 below. Since $\pi(g_t^\epsilon) = \pi(u_t^\epsilon)$, by Lemma 4.1, (4.9) is also conservative. □

5 Elementary Lemmas

Let $\{X_k, k = 1, \ldots, m\}$ be smooth vector fields on a smooth manifold N. Denote by $\mathcal{L}$ the Hörmander type operator $\frac{1}{2}\sum_{k=1}^m (X_k)^2 + X_0$. If at each point, X_1, $\ldots$, X_m and their Lie brackets generate the tangent space, we say that $\mathcal{L}$ satisfies strong Hörmander's condition. It satisfies Hörmander's condition if X_0 is allowed. A Hörmander type operator on a compact manifold satisfying strong Hörmander's condition has a unique invariant probability measure π; furthermore for $f \in C^\infty(N; \mathbb{R})$, $\mathcal{L}F = f$ is solvable if and only if $\int f d\pi = 0$. We denote by $\mathcal{L}^{-1}f$ a solution to the Poisson equation $\mathcal{L}F = f$ whenever it exists. If a Markov operator $\mathcal{L}$ has a unique invariant probability measure π and $f \in L^1(N; \pi)$ we write $\bar{f} = \int_N f d\pi$.

Lemma 5.1 *Let G be a Lie group with left invariant Riemannian metric. Then an SDE driven by left invariant vector fields is conservative. If μ is a right invariant measure on G then left invariant vector fields are divergence free and the linear operator $\mathcal{B} = \frac{1}{2}\sum_{i=1}^m (X_i)^2$, where $X_i \in \mathfrak{g}$, is symmetric on $L^2(G; d\mu)$.*

Proof The SDE is conservative follows from the fact that a Lie group with left invariant metric is geodesically complete and has positive injectivity radius. The left invariant vector fields and their covariant derivatives are bounded, so by localisation or the uniform cover criterion in D. Elworthy [18, Chapt vii], solutions of equation (4.8) from any initial point exist for all time. If $f \in BC^1(G;\mathbb{R})$, using the right invariance of the measure μ, $\int_G (Xf)(g)\mu(dg) = \int_G \frac{d}{dt} f(g\exp(tX))\Big|_{t=0}\mu(dg) = 0$. Consequently X having vanishing divergence with respect to μ and for $f_1, f_2 \in BC^1(G.;\mathbb{R})$, $\int_G f_2(Xf_1)d\mu = -\int_G f_1(Xf_2)d\mu$. In particular $\mathcal{B}$ is symmetric on $L^2(G,\mu)$. $\square$

We say a family of vectors $\{A_1,\dots,A_N\}$ in $\mathfrak{h} \subset \mathfrak{g}$ is *Lie algebra generating* if $\{A_1,\dots,A_N\}$ and their iterated brackets generate $\mathfrak{h}$. Define $\mathcal{L}_0 = \frac{1}{2}\sum_{k=1}^N (A_k)^2 + A_0$. We restrict A_k to the compact manifold H and treat $\mathcal{L}_0$ as an operator on H. If $\mathcal{L}_0$ is symmetric and satisfies Hörmander's condition, the maximal principle states that $\mathcal{L}_0^* u = 0$ has only constant solutions. Otherwise, is $\mathcal{L}_0^*$ satisfies also Hörmander condition, $\mathcal{L}_0 = 0$ has a unique solution.

Lemma 5.2 *If H is compact and $\{A_0, A_1, \dots, A_N\} \subset \mathfrak{h}$ is Lie algebra generating, the following statements hold.*

1. *The normalised Haar measure dh is the unique invariant probability measure for $\mathcal{L}_0$, and $\mathcal{L}_0$ is a Fredholm operator with Fredholm index* 0.
2. *If $\int_H \langle \mathrm{Ad}(h)(Y_0), Y\rangle dh = 0$, where $Y \in \mathfrak{g}$, there is a unique function $F \in C^\infty(H;\mathbb{R})$ solving the Poisson equation $\mathcal{L}_0 F = \langle \mathrm{Ad}(\cdot)(Y_0), Y\rangle$.*

Proof Denote by $\mathcal{L}_0^*$ the dual of $\mathcal{L}_0$ on $L^2(H;\mathbb{R})$ which, by Lemma 5.1, is $\mathcal{L}_0^* = \sum_k (A_k)^2 - A_0$. Both $\mathcal{L}_0$ and $\mathcal{L}_0^*$ satisfies Hörmander's condition. We have seen that $\int_H \mathcal{L}_0 f dh$ vanishes for all $f \in C^\infty$ and dh is an invariant measure, with full topoligical support. Distinct ergodic invariant measures have disjoint supports, and since every invariant measure is a convex combination of ergodic invariant measures, the Haar measure is therefore the only invariant measure, up to a scaling. Also $\mathcal{L}_0$ satisfies a sub-elliptic estimate: $\|u\|_s \le \|\mathcal{L}_0 u\|_{L^2} + c\|u\|_{L^2}$, [33, L. Hörmander], which implies that $\mathcal{L}_0$ has compact resolvent, and $\mathcal{L}_0$ is a Fredholm operator. In particular $\mathcal{L}_0$ has closed range, [34, L. Hörmander] and $\mathcal{L}_0 u = \langle \mathrm{Ad}(h)(Y_0), Y\rangle$ is solvable if and only if $\langle \mathrm{Ad}(h)(Y_0), Y\rangle$ annihilates the kernel of $\mathcal{L}_0^*$, i.e. $\int_H \langle \mathrm{Ad}(h)(Y_0), Y\rangle dh$ vanishes. By the earlier argument the dimension of the kernels of $\mathcal{L}_0$ and $\mathcal{L}_0^*$ agree and $\mathcal{L}_0$ has Fredholm index 0. $\square$

6 Convergence and Effective Limits

Let (h_t) be a Markov process on a compact manifold H with generator $\mathcal{L}_0 = \sum_k (A_k)^2 + A_0$ where A_k are smooth vector fields satisfying Hörmander's condition, and a invariant probability measure μ. Let $\Phi_t^\epsilon(y)$ be the solution to a family of conservative random differential equations $\dot{y}_t^\epsilon = \sum_{k=1}^m \alpha_k(h_{\frac{t}{\epsilon}})Y_k(y_t^\epsilon)$ with $y_0^\epsilon = y_0$ and Y_k smooth vector fields.

Lemma 6.1 *Suppose that N is compact ; or satisfies the following conditions.*

- *The injectivity radius of N is greater than a positive number* $2a$.
- $C_1(p) := \sup_{s,t\le 1}\sup_{x\in N} \mathbf{E}\left(|Y_j(y_{\frac{t}{\epsilon}}^\epsilon)|^p \mathbf{1}_{\rho(y_{\frac{t}{\epsilon}}^\epsilon, x)\le 2a}\right) < \infty$, *for all p; also* $C_2(p) := \sup_{s,t\le 1}\sup_{x\in N} \mathbf{E}\left(|\nabla d\rho^2(y_{\frac{t}{\epsilon}}^\epsilon, x)|^p \mathbf{1}_{\rho(y_{\frac{t}{\epsilon}}^\epsilon, x)\le 2a}\right) < \infty$.

Then $(y^\epsilon_{\frac{t}{\epsilon}})$ *converge weakly to a Markov process with generator* $-\sum_{i,j}\overline{\alpha_i\mathcal{L}_0^{-1}\alpha_j}L_{Y_i}L_{Y_j}$.

Proof Let $\beta_j = \mathcal{L}_0^{-1}\alpha_j$ and we first prove the tightness of the family of stochastic processes$\{y^\epsilon_{\frac{t}{\epsilon}}, \epsilon > 0\}$ for which we take a smooth function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(r) = r$ if $r < a$ and $\phi(r) = 1$ for all $r > 2a$ and define $\tilde\rho = \phi\circ\rho$. We apply Itô's formula, see [46, Lemma 3.1], to $\tilde\rho^2$ to give for $s, t \le 1$,

$$\begin{aligned}
&\tilde\rho^2\left(y^\epsilon_{\frac{s}{\epsilon}}, y^\epsilon_{\frac{t}{\epsilon}}\right)\\
=&\epsilon\sum_{j=1}^m \left(L_{Y_j}\tilde\rho^2(y^\epsilon_{\frac{s}{\epsilon}}, y^\epsilon_{\frac{t}{\epsilon}})\right)\ \beta_j(h_{\frac{t}{\epsilon^2}}) - \epsilon\sum_{i,j=1}^m\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}\left(L_{Y_i}L_{Y_j}\tilde\rho^2(y^\epsilon_{\frac{s}{\epsilon}}, y^\epsilon_r)\right)\ \alpha_i(h_{\frac{r}{\epsilon^2}})\ \beta_j(h_{\frac{r}{\epsilon}})\ dr\\
&-\sqrt{\epsilon}\sum_{j=1}^m\sum_{k=1}^{m'}\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}(L_{Y_j}\tilde\rho^2)(y^\epsilon_r))\ L_{A_k}\beta_j)(h_{\frac{r}{\epsilon}})\ db^k_r.
\end{aligned}$$

We raise both sides to the power p where $p > 2$ to see for a constant c_p depending on $|\beta_j|_\infty$, $|\alpha_j|_\infty$, $|A_k|_\infty$, m, and p, C_1 and C_2, which may represent a different number in a different line,

$$\begin{aligned}
&\mathbf{E}\left[\tilde\rho^{2p}\left(y^\epsilon_{\frac{s}{\epsilon}}, y^\epsilon_{\frac{t}{\epsilon}}\right)\right]\\
\le& c_p\epsilon^p\sum_{j=1}^m\mathbf{E}\left|L_{Y_j}\tilde\rho^2(y^\epsilon_{\frac{s}{\epsilon}}, y^\epsilon_{\frac{t}{\epsilon}})\right|^p + c_p\epsilon^p\sum_{i,j=1}^m\mathbf{E}\left(\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}\left|L_{Y_i}L_{Y_j}\tilde\rho^2(y^\epsilon_{\frac{s}{\epsilon}}, y^\epsilon_r)\right|\ dr\right)^p\\
&+c_p\epsilon^{\frac{p}{2}}\sum_{j=1}^m\sum_{k=1}^{m'}\mathbf{E}\left(\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}|L_{Y_j}\tilde\rho^2(y^\epsilon_r))|^2dr\right)^{\frac{p}{2}}\\
&\le c_pC_1(p)\epsilon^p + c_p(t-s)^p\sqrt{C_1(4p)}\sqrt{C_2(2p)} + c_p(t-s)^{\frac{p}{2}}\sqrt{C_1(2p)}\sqrt{C_2(2p)}.
\end{aligned}$$

Applying $\tilde\rho$ directly to $y^\epsilon_{\frac{t}{\epsilon}}$ giving another estmate:

$$\mathbf{E}\left|\tilde\rho\left(y^\epsilon_{\frac{s}{\epsilon}}, y^\epsilon_{\frac{t}{\epsilon}}\right)\right|^p \le c_p\mathbf{E}\sum_j\left|\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}L_{Y_j}\tilde\rho^2(y^\epsilon_{\frac{r}{\epsilon}})\alpha_j(h_{\frac{r}{\epsilon}})dr\right|^p \le c_p\left(\frac{t-s}{\epsilon}\right)^p\sqrt{C_1(2p)}\sqrt{C_2(2p)}.$$

Interpolate the estimates for $\epsilon^p \le (t-s)^{\frac{p}{2}}$ and for $\epsilon^p \ge (t-s)^{\frac{p}{2}}$, to see

$$\mathbf{E}\left|\tilde\rho\left(y^\epsilon_{\frac{s}{\epsilon}}, y^\epsilon_{\frac{t}{\epsilon}}\right)\right|^{2p} \le c_p|t-s|^{\frac{p}{2}}.$$

Take $p > 4$ and apply Kolmogrov's theorem to obtain the required tightness. The weak convergence follows just as for the proof of [46, Theorem 5.4], using a law of large numbers with rate of convergence the square root of time [46, Lemma 5.2]. The limit is identified as following for a Fredholm operator $\mathcal{L}_0$ of index zero. Let $\{u_i, i = 1, \dots, n_0\}$ be a basis in ker$(\mathcal{L}_0)$ and $\{\pi_i\ i = 1, \dots, n_0\}$ the dual basis for the null space of $\mathcal{L}_0^*$. Then $\bar{\mathcal{L}} = -\sum_{i,j}\sum_{b=1}^{n_0}u_b\langle\alpha_i\beta_j, \pi_b\rangle L_{Y_i}L_{Y_j}$, where the bracket denotes the dual pairing between L^2 and $(L^2)^*$. In our case we know that there is only one invariant probability measure for $\mathcal{L}_0$ from which we conclude that $\bar{\mathcal{L}} = -\sum_{i,j}\overline{\alpha_i\beta_j}L_{Y_i}L_{Y_j}$. $\square$

Let use return to our equations on the product space $G\times H$,

$$\dot u^\epsilon_t = \left(\mathrm{Ad}(h_{\frac{t}{\epsilon}})Y_0\right)^*(u^\epsilon_t), \quad u^\epsilon_0 = u_0 \tag{6.1}$$

$$dh_t = \sum_{k=1}^{N} A_k^*(h_t) \circ db_t^k + A_0^*(h_t)dt, \quad \mathfrak{h}_0 = 1. \tag{6.2}$$

Let $x^\epsilon_{\frac{\cdot}{\epsilon}} = \pi(u^\epsilon_{\frac{\cdot}{\epsilon}})$ and $x_0 = \pi(u_0)$. Let $\langle , \rangle$ denote a left invariant and Ad_H invariant scalar product on $\mathfrak{g}$, $\{Y_j\}$ an orthonormal basis of $\mathfrak{m}$ and define

$$\alpha(Y_0, Y_j)(h) = \langle \mathrm{Ad}(h)(Y_0), Y_j \rangle \,. \tag{6.3}$$

Proposition 6.2 *Suppose the subgroup H is compact, $\{A_0, A_1, \ldots, A_N\}$ a Lie algebra generating subset of $\mathfrak{h}$, and $\bar{Y}_0 = 0$. Then as $\epsilon \to 0$, $(u^\epsilon_{\frac{s}{\epsilon}}, s \le T)$ converge weakly to a Markov process $(\bar{u}_s, s \le T)$ with Markov operator*

$$\bar{\mathcal{L}} = -\sum_{i,j=1}^{m} \overline{\alpha(Y_0, Y_i)\,(\mathcal{L}_0^{-1}\alpha(Y_0, Y_j))}\; L_{Y_i^*} L_{Y_j^*}.$$

Also, $(x^\epsilon_{\frac{t}{\epsilon}}, t \le T)$ converges weakly to a stochastic process $(\bar{x}_t, t \le T)$.

Proof By the left invariance of the Riemannian metric, Equation (6.1) is equivalent to $u_t^\epsilon = \sum_{j=1}^N \alpha(Y_0, Y_j)(h_{\frac{t}{\epsilon}}) Y_j^*(u_t^\epsilon)$ and

$$(\mathrm{Ad}(h)(Y_0))^*(g) = \sum_{j=1}^{N} \left\langle (\mathrm{Ad}(h)(Y_0))^*, Y_j^* \right\rangle Y_j^*(g) = \sum_{j=1}^{N} \alpha(Y_0, Y_j)(h) Y_j^*(g).$$

We may rewrite (6.1) as $\dot{u}_t^\epsilon = \sum_{j=1}^N Y_j(u_t^\epsilon)\alpha(Y_0, Y_j)(h_{\frac{t}{\epsilon}})$. Since $\bar{Y}_0$ vanishes apply Lemma 5.2, so $\mathcal{L}_0^{-1}\alpha_j$ exists and is smooth for each j. Furthermore by Lemma 5.1, the $\bar{\mathcal{L}}$ diffusions exist for all time. Since the Riemannian metric on G is left invariant, its Riemannian distance function ρ is also left invariant, $\rho(gg_1, gg_2) = \rho(g_1, g_2)$ for any $g, g_1, g_2 \in G$. Furthermore, $\sup_{p:\rho(p,y)<\delta} \nabla^2 \rho(y,p)$ is finite, for sufficiently small $\delta > 0$, and is independent of y, where ∇ is the Levi-Civita connection. Thus $C_1(p)$ and $C_2(p)$, from Lemma 6.1, are both finite. We observe that G has positive injectivity radius and bounded geometry, and we then apply Lemma 6.1 to conclude that $(u^\epsilon_{\frac{s}{\epsilon}}, s \le t)$ converges weakly as ϵ approaches 0. As any continuous real valued function f on M lifts to a continuous function on G, the weak convergence passes, trivially, to the processes $(x^\epsilon_{\frac{t}{\epsilon}})$. □

For $p \ge 1$, denote by W_p or $W_p(N)$ the Wasserstein distance on probability measures over a metric space N. For T a positive number, let $C_x([0,T]; N)$ denote the space of continuous curves defined on the interval $[0,T]$ starting from x. Given two probability measures μ_1, μ_2 and $C_x([0,T]; N)$,

$$W_p(\mu_1, \mu_2) := \left(\inf \int \sup_{0 \le s \le T} \rho^p(\sigma_1(s), \sigma_2(s)) d\nu(\sigma_1, \sigma_2) \right)^{\frac{1}{p}}.$$

Here σ_1 and σ_2 take values in $C_x([0,T]; N)$ and the infimum is taken over all probability measures ν whose marginals are μ_1, μ_2.

Remark 6.3 Weak convergence implies convergence in the Wasserstein p distance if the limit belongs to W_p and the $\mathbf{E} \sup_{s \le t} \rho(0, y^\epsilon_{\frac{s}{\epsilon}})^p$ is uniformly bounded in ϵ. If G is compact or is diffeomorphic to an Euclidean space, our processes converge in the

Wasserstein distance. To the latter just note that in the proof of the lemma, $\tilde{\rho}^2$ can be replaced by ρ^2, the left invariant vector fields are bounded. Since the sectional curvature are bounded, $|\nabla d\rho^2|$ are bounded by the Hessian comparison theorem which states that $\nabla d\rho$ is bounded by the curvature of the Lie group, the curvature is bounded as they are preserved by isometries. Thus $\rho(0, y^\epsilon_{\frac{s}{\epsilon}})^2$ has finite moments, the assertion follows.

6.1 Rate of convergence.

The definition of BC^r functions on a manifold depend on the linear connection on TM, in general. We use the flat connection ∇^L for BC^r functions on the Lie group G. The canonical connection ∇^c is a convenient connection for defining BC^r functions on $M = G/H$, as the parallel transports for ∇^c are differentials of the left actions of G on M. See §10 below for more discussions on the canonical connection. In the theorem below, M is given the induced G-invariant Riemannian metric. Set

$$BC^r(M;\mathbb{R}) = \{f \in C^r(M;\mathbb{R}) : |f|_\infty + |\nabla f|_\infty + \sum_{k=1}^r |(\nabla^c)^{(k)} df|_\infty < \infty\}$$

$$BC^r(G;\mathbb{R}) = \{f \in C^r(G;\mathbb{R}) : |f|_\infty + |\nabla f|_\infty + \sum_{k=1}^r |(\nabla^L)^{(k)} df|_\infty < \infty\}.$$

Denote $|f|_{r,\infty} = \sum_{k=0}^r |\nabla^{(k)} df|_\infty$ where ∇ is one of the connections above.

Remark 6.4 A function f belongs to $BC^r(G)$ if and only if for an orthonormal basis $\{Y_i\}$ of $\mathfrak{g}$, the following functions are bounded for any set of indices: f, $Y^*_{i_1} f$, $\dots$, $Y^*_{i_1} \dots Y^*_{i_r} f$. In fact $\nabla f = \sum_i df(Y_i^*) Y_i^*$ and $\nabla^L_\cdot \nabla f = \sum_i Y_i^* L_\cdot (Y_i^* f)$, and so on. Here $L_\cdot$ denotes Lie differentiation so $L_\cdot f = df(\cdot)$.

Denote by $\bar{\mu}$ the probability measure of $(\bar{u}_\cdot)$ and $\bar{P}_t$ the associated probability semigroup. We give the rate of convergence, which essentially follows from [46, Theorem 7.2], to follow which we would assume one of the following conditions. (1) G is compact; (2) For some point $o \in G$, ρ_o^2 is smooth and $|\nabla^L d\rho_o^2| \le C + K\rho_o^2$. (3) There exist $V \in C^2(V;\mathbb{R}_+)$, $c > 0$, $K > 0$ and $q \ge 1$ such that

$$|\nabla V| \le C + KV, \quad |\nabla dV| \le c + KV, \quad |\nabla d\rho_o^2| \le c + KV^q.$$

However it is better to prove the rate of convergence directly.

Theorem 6.5 *Suppose that H is compact and $\{A_1, \dots, A_N\} \subset$ is Lie algebra generating and $\bar{Y}_0 = 0$. Then, (1) Both $(u^\epsilon_{\frac{t}{\epsilon}}, t \le T)$ and $(x^\epsilon_{\frac{t}{\epsilon}}, t \le T)$ converge in W_p for any $p > 1$.*
(2) There are numbers c such that for all $f \in BC^4(G;\mathbb{R})$,

$$\sup_{0 \le t \le T} \left| \mathbf{E} f\left(u^\epsilon_{\frac{t}{\epsilon}}\right) - \bar{P}_t f(u_0) \right| \le C\,\epsilon\,\sqrt{|\log \epsilon|}\,(1 + |f|_{4,\infty}), \quad u_0 \in G.$$

(3) For any $r \in (0, \frac{1}{4})$, $\sup_{0 \le t \le T} W_1(\mathrm{Law}(u^\epsilon_{\frac{t}{\epsilon}}), \bar{\mu}_t) \le C\epsilon^r$. An analogous statement holds for $x^\epsilon_{\frac{t}{\epsilon}}$.

Proof If ρ^2 is bounded or smooth, $L_{Y_i^*}L_{Y_j^*}\rho = \nabla^L d\rho(Y_i^*, Y_j^*) + \nabla^L_{Y_i^*} Y_j^*$ is bounded. Denote $\mathcal{T}^L$ the torsion of ∇^L and ∇ the Levi-Civita connection. The derivative flow for (6.1) satisfying the following equation

$$\nabla v_t^\epsilon = -\frac{1}{2}\sum_j \alpha(Y_0, Y_j)(h_t^\epsilon)\,\mathcal{T}^L(v_t^\epsilon, Y_j(u_t^\epsilon)).$$

Indeed this follows from linearising the equation $\dot{u}_t^\epsilon = \sum_{j=1}^N Y_j^*(u_t^\epsilon)\alpha(Y_0, Y_j)(h_{\frac{t}{\epsilon}})$ and the relation $\nabla^L = \nabla + \frac{1}{2}\mathcal{T}^L$. Since $\mathcal{T}^L(u,v) = -[u,v]$, the torsion tensor and their covariant derivatives are bounded. Thus all covariant derivatives of Y_j^* with respect to the Levi-Civita connection are bounded. We use Lemma 6.2 and follow the proof of [46, Theorem 7.2], to conclude the first assertion for the u process. These are proved by discretising time and writing the differences as telescopic sums. The main ingredients are: (1) (h_t) has an exponential mixing rate which follows from Hörmander's conditions, (2) estimates for $|\bar{P}_t f - \bar{P}_s f - (t-s)\bar{P}_s \bar{\mathcal{L}} f| \le C(t-s)^2$ which follows from the fact that the vector fields Y_j have bounded derivatives of all order.

For the process on the homogeneous manifold, take $f \in BC^4(M;\mathbb{R})$ and let $\tilde{f} = f\circ\pi$. Then $Y_iY_j\tilde{f} = \nabla^c df(\pi_* Y_i^*, \pi_* Y_j^*) + df\left(\nabla^c_{Y_i^*} d\pi(Y_j^*)\right)$. The last term vanishes, as $(d\pi)_u(Y_i^*) = T\bar{L}_u((d\pi)_1 Y_i))$, c.f. Lemma 10.2 below. Since π_* is a Riemannian isometry, $Y_i^*Y_j^*\tilde{f}$ is bounded if $\nabla^c df$ is. The same assertion holds for higher order derivatives. Thus $f\circ\pi \in BC^4(G;\mathbb{R})$ and

$$\sup_{0\le t\le T}\left|\mathbf{E} f\left(x^\epsilon_{\frac{t}{\epsilon}}\right) - \pi_*\bar{P}_t f(x_0)\right| \le C\,\epsilon\,\sqrt{|\log\epsilon|}\;\gamma(x_0)\,(1+|f|_{4,\infty}),$$

where γ is a function in $B_{\rho,o}$.

For the convergence in the Wasserstein distance, denote by $\tilde{\rho}$ and ρ respectively the Riemannian distance function on G and on M. For $i=1,2$, let $x_i \in M$ and $u_i \in \pi^{-1}(x_i)$. If u_1 and u_2 are the end points of a horizontal lift of the unit speed geodesic connecting x_1 and x_2, then $\tilde{\rho}(u_1,u_2) = \rho(x_1,x_2)$. Otherwise $\tilde{\rho}(u_1,u_2) \ge \rho(x_1,x_2)$. If c_1 and c_2 are C^1 curves on M with c^1 curves $\tilde{c}_1$ and $\tilde{c}_2$ on G covering c_1 and c_2 respectively, then $\rho(c_1,c_2) = \sup_{t\in[0,T]}\rho(c_1(t),c_2(t)) \le \tilde{\rho}(\tilde{c}_1,\tilde{c}_2)$.

Since $C_{u_0}([0,T];G)$ is a Polish space, there is an optimal coupling of the probability law of $u^\epsilon_{\frac{\cdot}{\epsilon}}$ and $\bar{\mu}$ which we denote by μ. Then $\pi_*\bar{\mu}$ is a coupling of $\mathrm{Law}(x^\epsilon_{\frac{\cdot}{\epsilon}})$ and $\pi_*\bar{\mu}$ and

$$\begin{aligned}
W_p(\mathrm{Law}(u^\epsilon_{\frac{\cdot}{\epsilon}}), \bar{\mu}) &= \left(\int_{C_{u_0}([0,T];G)} \tilde{\rho}^p(\gamma_1,\gamma_2)\,d\mu(\gamma_1,\gamma_2)\right)^{\frac{1}{p}}\\
&\left(\int_{C_{x_0}([0,T];M)} \rho^p(\pi(\gamma_1),\pi(\gamma_2))\,d\mu(\gamma_1,\gamma_2)\right)^{\frac{1}{p}}\\
&= \left(\int_{C_{x_0}([0,T];M)} \sup_{0\le t\le T}\rho^p(\sigma_1(t),\sigma_2(t))\,d\pi_*\bar{\mu}(\sigma_1,\sigma_2)\right)^{\frac{1}{p}}\\
&\ge W_p(\mathrm{Law}(x^\epsilon_{\frac{\cdot}{\epsilon}}), \pi_*\bar{\mu}).
\end{aligned}$$

Consequently $x^\epsilon_{\frac{\cdot}{\epsilon}}$ converges in W_p. Similarly the rate of convergence passes from the u process to the x process. For $r<\frac{1}{4}$ there is a number C such that, $W_1(\mathrm{Law}(x^\epsilon_{\frac{t}{\epsilon}}), \pi_*\bar{\mu}_t) \le W_p(\mathrm{Law}(u^\epsilon_{\frac{t}{\epsilon}}), \bar{\mu}) \le C\gamma(x_0)\epsilon^r$. □

6.2 The Center Condition

We identify those vectors Y_0 satisfying $\bar{Y}_0 = 0$. Given a reductive structure, Ad_H is the direct sum of sub representations $\mathrm{Ad}_H = \mathrm{Ad}_H|_{\mathfrak{h}} \oplus \mathrm{Ad}_H|_{\mathfrak{m}}$. The condition $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$ implies that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $\mathrm{Ad}(H)(\mathfrak{m}) = \mathfrak{m}$. Let

$$\mathfrak{m}_0 = \{X \in \mathfrak{m} : \mathrm{Ad}(h)(X) = X \text{ for all } h \in H\},$$

be the subspace on which Ad_H acts trivially. We consider an Ad_H-invariant subspace space $\tilde{\mathfrak{m}}$ of $\mathfrak{m}$, transversal to $\mathfrak{m}_0$, i.e. $\tilde{\mathfrak{m}} \cap \mathfrak{m}_0 = \{0\}$. It is more intuitive to study the action of Ad_H through its isotropy representation τ of H on T_oM which is defined by the formula $\tau_h(\pi_* X) := (\bar{L}_h)_*(\pi_* X)$. Since left translation on T_oM corresponds to adjoint action on $\mathfrak{m}$, then $\pi_*\mathrm{Ad}(h)(X)$ agree with $(\bar{L}_h)_*(\pi_* X)$, the linear representation Ad_H is equivalent to the isotropy representation. A representation ρ of H is said to acts transitively on the unit sphere, of the representation space V, if for any two unit vectors in V there is a $\rho(h)$ taking one to the other. If Ad_H acts transitively, its representation space is irreducible and $\mathfrak{m}_0 = \{0\}$.

Recall that H is unimodular is equivalent to that the Haar measure dh is bi-invariant.

Lemma 6.6 *Suppose that H is uni-modular.*

(1) If $Y_0 \in \mathfrak{m}_0$ is non-trivial, then $\bar{Y}$ does not vanish.

(2) $\bar{Y} = 0$ for every $Y \in \tilde{\mathfrak{m}}$. In particular, if $\mathrm{Ad}_H|_{\mathfrak{m}}$ has no non-trivial invariant vectors, then $\bar{Y} = 0$ for all $Y \in \mathfrak{m}$.

Proof Part (1) is clear by the definition. For any $Y \in \tilde{\mathfrak{m}}$, the integral $\bar{Y} := \int_H \mathrm{Ad}(h)(Y)dh$ is an invariant vector of Ad_H, using the bi-invariance of the measure dh. Since $\bar{Y} \in \tilde{\mathfrak{m}}$ and $\tilde{\mathfrak{m}} \cap \{\mathfrak{m}_0\} = \{0\}$ the conclusion follows. □

Example 6.7 Let M be the Stiefel manifold $S(k, n)$ of oriented k frames in $\mathbb{R}^n$. The orthogonal group takes a k-frame to a k frame and acts transitively. The isotropy group of the k-frame $o = (e_1, \dots, e_k)$, the first k vectors from the standard basis of $\mathbb{R}^n$, contains rotation matrices that keep the first k frames fixed and rotates the rest. Hence, $S(k, n) = SO(n)/SO(n-k)$. Then $\mathfrak{h} = \left\{\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}\right\}$ where $A \in \mathfrak{so}(n-k)$. Denote by $M_{n-k,k}$ the set of $(n-k)\times k$ matrices and let $\mathfrak{m} = \left\{\begin{pmatrix} S & -C^T \\ C & 0 \end{pmatrix}\right\}$ where $S \in \mathfrak{so}(k)$ and $C \in M_{n-k,k}$. Since $\mathrm{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}\right)\left(\begin{pmatrix} S & -C^T \\ C & 0 \end{pmatrix}\right) = \begin{pmatrix} S & -C^TR^T \\ RC & 0 \end{pmatrix}$ for $R \in SO(n-k)$, we see $\mathfrak{m}_0 = \left\{\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}\right\}$. Define $\tilde{\mathfrak{m}} = \left\{\begin{pmatrix} 0 & -C^T \\ C & 0 \end{pmatrix}\right\}$. Let us identify $\tilde{\mathfrak{m}}$ with $(n-k)\times k$ matrices. For $C \in M_{n-k,k}$ denote by Y_0 the corresponding skew symmetric matrix in $\tilde{m}$. It is clear that $\int_H RCdR$ and $\int_H \mathrm{Ad}(R)(Y_0)dR$ vanish.

The symmetry group of a Riemmannian homogeneous space of dimension d has at most dimension $d(d+1)/2$. If G/H is a connected d-dimensional manifold and if G admits an Ad_H-invariant inner product with $\dim(G) = \frac{1}{2}d(d+1)$, then $\bar{Y}$ vanishes for all $Y \in \mathfrak{m}$. Such Riemannian homogenous manifolds are of constant curvature, isometric to one of the following spaces: an Euclidean space, a sphere, a real projective space, and a simply connected hyperbolic space.

Example 6.8 Suppose that $M = G/H$ is a symmetric space, it is in particular a reductive homogeneous space. Then $L_h\pi = \pi \mathrm{Ad}(h)$ and $(dL_h)_o d\pi = d\pi \mathrm{Ad}_h$, where o is the identity coset. Since $d\pi$ restricts to an isomorphism on $\mathfrak{m}$, left actions correspond to the Ad action on G. For every Y_0 there is an h with the property that $\mathrm{Ad}(h)(Y_0) = -Y_0$. A compact Lie group with the bi-invariant metric is a symmetric space.

Example 6.9 A more general class of manifolds, for which $\bar{Y}_0 = 0$ for every $Y_0 \in \mathfrak{m}$, are weakly symmetric spaces among the Riemannian homogeneous spaces. A Riemannian manifold is weakly symmetric if there exists some closed subgroup K of the Isometry group $I(M)$ and some $\mu \in I(M)$ such that $\mu^2 \in K$ and for all $p, q \in M$ there exists $I \in K$ such that $I(p) = \mu(q)$ and $I(q) = \mu(p)$. This concept was introduced in [61, A. Selberg] and studied by Szabó who characterises them as ray symmetric space. Also, a Riemannian manifold is weakly symmetric if and only if any two points in it can be interchanged by some isometry, see [6, J. Berndt, L. Vanhecke]. The following property characterizes weakly symmetric spaces among Riemannian homogeneous spaces: let H be the isotropy subgroup of $I(M)$ and $\tau : H \to GL(T_oM)$ its isotropy representation. Then for each $X \in \mathfrak{h}$ there exists an element of $h \in H$ such that $\rho(h)(X) = -X$. Some of the weakly symmetric spaces are given in the form of G/H where G is a connected semi simple Lie group, K a closed subgroup of G, and H a closed subgroup K. Then there is a principal fibration $G/H \to G/K$ with fibre K/H.

The three spheres with the left invariant Bergers metrics, $\epsilon \neq 1$, are not locally Riemannian symmetric spaces, they are reductive homogeneous spaces. We remark also that irreducible symmetric spaces are in fact strongly isotropy irreducible. A connected homogeneous manifold G/H is strongly isotropy irreducible if H is compact and its identity component acting irreducibly on the tangent space. Such spaces admit left invariant Einstein metric and are completely classified, for more study see [7] and see also [12]. If the symmetric space has rank 1, i.e. it is the quotient space of a semi-simple Lie group G whose maximal torus group H is of dimension 1, they are two point homogeneous spaces, see [30, Prop. 5.1].

Theorem 6.10 *Suppose that H is compact and $\{A_0, A_1, \dots, A_N\}$ is Lie algebra generating. Let $Y_0 \in \tilde{\mathfrak{m}}$, $x_t^\epsilon = \pi(g_t^\epsilon)$, where (g_t^ϵ) is the solution to (1.1), and (u_t^ϵ) its horizontal lift through g_0. Then $(u_{\frac{t}{\epsilon}}^\epsilon)$ and $(x_{\frac{t}{\epsilon}}^\epsilon)$ converge weakly with respective limit diffusion process $(\bar{u}_t)$ and $(\bar{x}_t)$. Let $\{Y_j, j = 1, \dots, N\}$ be an orthonormal basis of $\tilde{\mathfrak{m}}$. Then the Markov generator of $(\bar{u}_t)$ is*

$$\bar{\mathcal{L}} = -\sum_{i,j=1}^{N} a_{i,j}(Y_0)\, L_{Y_i^*} L_{Y_j^*}, \quad a_{i,j}(Y_0) = \int_H \alpha(Y_0, Y_i)\, \mathcal{L}_0^{-1}(\alpha(Y_0, Y_j)) dh.$$

Proof By Corollary 4.4 the horizontal lift process of (x_t^ϵ) solves equations (6.1-6.2). By Lemma 6.6, $\bar{Y}_0 = 0$. Since $\tilde{\mathfrak{m}}$ is an invariant space of Ad_H and $Y_0 \in \tilde{\mathfrak{m}}$, $\mathrm{Ad}(h)(Y_0) \in \tilde{\mathfrak{m}}$. In Proposition 6.2 it is sufficient to take $\{Y_j\}$ to be an orthonormal basis of the invariant subspace $\tilde{\mathfrak{m}}$. $\square$

In the rest of the paper we study the effective limits $\bar{\mathcal{L}}$ given in Proposition 6.10. Although $\bar{u}_t$ is always a Markov process, that its projection $(\bar{x}_t)$ (describing the motion of the effective orbits) is a Markov process on its own is not automatic. In the next two sections we study this and identify the stochastic processes $(\bar{u}_t)$ and $(\bar{x}_t)$ by computing the Markov generator $\bar{\mathcal{L}}$ and its projection.

7 Effective Limits, Casimir, and The Markov Property

In this section H is a compact connected proper subgroup of G, $\mathcal{L}_0 = \frac{1}{2}\sum_{k=1}^N (A_k)^2$, $\langle , \rangle$ an Ad_H-invariant inner product on $\mathfrak{g}$ and $\mathfrak{m} = \mathfrak{h}^\perp$. Let $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$ be the orthogonal sum of Ad_H-invariant subspaces, where each $\mathfrak{m}_j$ is irreducible for $j \neq 0$. This is also an invariant decomposition for $\mathrm{ad}_\mathfrak{h}$.

Let $\mathcal{B}_{\mathrm{ad}_\mathfrak{h}, \mathfrak{m}_l}(X, Y)$ be the symmetric associative bilinear form of the adjoint subrepresentation of $\mathfrak{h}$ on $\mathfrak{m}_l$. The symmetric bilinear form for a representation ρ of a Lie algebra $\mathfrak{g}$ on a finite dimensional vector space is given by the formula $\mathcal{B}_\rho(X, Y) = \mathrm{trace}\, \rho(X)\rho(Y)$ for $X, Y \in \mathfrak{h}$. A bilinear form $\mathcal{B}$ on a Lie algebra is associative if $\mathcal{B}([X, Y], Z]) = \mathcal{B}(X, [Y, Z])$. Let $\mathrm{Id}_{\mathfrak{m}_l}$ be the identity map on $\mathfrak{m}_l$. The following lemma allows us to compute the coefficients of the generator.

Lemma 7.1 *Let $\{A_j\}$ be an orthonormal basis of $\mathfrak{h}$. The following statements hold.*

1. *There exists λ_l such that $\frac{1}{2}\sum_{k=1}^p \mathrm{ad}^2(A_k)|_{\mathfrak{m}_l} = -\lambda_l \,\mathrm{Id}_{\mathfrak{m}_l}$. Furthermore $\lambda_l = 0$ if and only if $l = 0$.*
2. *Suppose that $\mathrm{ad}_\mathfrak{h} : \mathfrak{h} \to \mathbb{L}(\mathfrak{m}_l; \mathfrak{m}_l)$ is faithful. Then $\mathcal{B}_{\mathrm{ad}_\mathfrak{h}, \mathfrak{m}_l}$ is non-degenerate, negative definite, and Ad_H-invariant. If $l \neq 0$,*
$$\lambda_l = -\mathcal{B}_{\mathrm{ad}_h, \mathfrak{m}_l}(X, X)\frac{\dim(\mathfrak{h})}{2\dim(\mathfrak{m}_l)},$$
where X is any unit vector in $\mathfrak{h}$. If the inner product on $\mathfrak{h}$ agrees with $-\mathcal{B}_{\mathrm{ad}_h, \mathfrak{m}_l}$ then $\lambda_l = \frac{\dim(\mathfrak{h})}{2\dim(\mathfrak{m}_l)}$.

Proof Part (1). For $Z \in \mathfrak{h}$ and $X, Y \in \mathfrak{g}$, we differentiate the identity
$$\langle \mathrm{Ad}(\exp(tZ))(X), \mathrm{Ad}(\exp(tZ))(Y)\rangle = \langle X, Y\rangle$$
at $t = 0$ to see that
$$\langle \mathrm{ad}(Z)(X), Y\rangle + \langle X, \mathrm{ad}(Z)(Y)\rangle = 0. \tag{7.1}$$
Let $\{A_1, \dots, A_p\}$ be an orthonormal basis of $\mathfrak{h}$, for the Ad_H invariant metric, then $\sum_{k=1}^p (\mathrm{ad}(A_k))^2$ commutes with every element of $\mathfrak{h}$. Indeed if $X \in \mathfrak{h}$ let $[X, A_k] = \sum_{l'} a_{kl'} A_{l'}$, then
$$a_{kl} = \sum_{l'} a_{k,l'}\langle A_{l'}, A_l\rangle = \langle [X, A_k], A_l\rangle = -\langle ([X, A_l], A_k\rangle = -a_{lk},$$
$$[\mathrm{ad}(X), \sum_{k=1}^p \mathrm{ad}^2(A_k)] = \sum_k [\mathrm{ad}(X), \mathrm{ad}(A_k)]\mathrm{ad}(A_k) + \sum_k \mathrm{ad}(A_k)[\mathrm{ad}(X), \mathrm{ad}(A_k)]$$
$$= \sum_k \mathrm{ad}([X, A_k])\mathrm{ad}(A_k) + \sum_k \mathrm{ad}(A_k)\mathrm{ad}([X, A_k]) = 0.$$
If $Y_0 \in \mathfrak{m}$, $\mathrm{ad}(A_k)(Y_0) \in \mathfrak{m}$ and by the skew symmetry, (7.1),
$$\langle \sum_{k=1}^p (\mathrm{ad}(A_k))^2(Y_0), Y_0\rangle = -\sum_{k=1}^p \langle \mathrm{ad}(A_k)(Y_0), \mathrm{ad}(A_k)(Y_0)\rangle.$$
Thus $\sum_{k=1}^p (\mathrm{ad}(A_k))^2(Y_0) = 0$, where $p = \dim(\mathfrak{h})$, implies that $\mathrm{ad}(A_k)(Y_0) = 0$ for all k which in turn implies that $Y_0 \in \mathfrak{m}_0$. Conversely if $Y_0 \in \mathfrak{m}_0$ it is clear that $\sum_{k=1}^p \mathrm{ad}^2(A_k) = 0$.

(2) Firstly, $\mathcal{B}_{\mathrm{ad}_{\mathfrak{h}},\mathfrak{m}_l}$ is non-degenerate. If $\{Y_j\}$ is a basis of $\mathfrak{m}_l$ with respect to the ad_H invariant inner product and $X \in \mathfrak{h}$, then

$$\mathcal{B}_{\mathrm{ad}_{\mathfrak{h}},\mathfrak{m}_l}(X,X) = -\sum_j |[X,Y_j]|^2,$$

which vanishes only if $[X,Y_j] = 0$ for all j. By the skew symmetry of $\mathrm{ad}_{\mathfrak{h}}$, for any $Y \in \mathfrak{m}_l$, $0 = \langle [X,Y_j], Y\rangle = -\langle Y_j, [X,Y]\rangle$ for all j. Since $\mathrm{ad}(X)(\mathfrak{m}_l) \subset \mathfrak{m}_l$, $\mathcal{B}_{\mathrm{ad}_{\mathfrak{h}},\mathfrak{m}_l}(X,X) = 0$ implies that $[X,Y] = 0$ for all Y which implies X vanishes from the assumption that $\mathrm{ad}_{\mathfrak{h}}$ is faithful. It is clear that $\mathcal{B}_{\mathrm{ad}_h,\mathfrak{m}_l}$ is Ad_H-invariant. Let $X \in \mathfrak{h}$ and $h \in H$, then

$$\begin{aligned}
&\mathcal{B}_{\mathrm{ad}_{\mathfrak{h}},\mathfrak{m}_l}(\mathrm{Ad}(h)(X), \mathrm{Ad}(h)(X)) = \sum_j \langle \mathrm{ad}(\mathrm{Ad}(h)(X))\mathrm{ad}(\mathrm{Ad}(h)(X))Y_j, Y_j\rangle\\
&= -\sum_j \left\langle [\mathrm{Ad}(h)(X), Y_j], [\mathrm{Ad}(h)(X), Y_j]\right\rangle\\
&= -\sum_j \left\langle \mathrm{Ad}(h)[X, \mathrm{Ad}(h^{-1})Y_j], \mathrm{Ad}(h)[X, \mathrm{Ad}(h^{-1})Y_j]\right\rangle\\
&= -\sum_j \left\langle [X, \mathrm{Ad}(h^{-1})Y_j], [X, \mathrm{Ad}(h^{-1})Y_j]\right\rangle = \mathcal{B}_{\mathrm{ad}_{\mathfrak{h}},\mathfrak{m}_l}(X,X).
\end{aligned}$$

Since there is a unique, up to a scalar multiple, Ad_H-invariant inner product on a compact manifold, $\mathcal{B}_{\mathrm{ad}_{\mathfrak{h}},\mathfrak{m}_l}$ is essentially the inner product on H. There is a positive number a_l such that $\mathcal{B}_{\mathrm{ad}_{\mathfrak{h}},\mathfrak{m}_l} = -a_l\langle,\rangle$. It is clear that $a_l > 0$. For the orthonormal basis $\{A_k\}$, $\mathcal{B}_{\mathrm{ad}_{\mathfrak{h}},\mathfrak{m}_l}(A_i,A_j) = -a_l\delta_{i,j}$. We remark that $\{\frac{1}{a_l}A_k\}$ is a dual basis of $\{A_k\}$ with respect to $\mathcal{B}_{\mathrm{ad}_{\mathfrak{h}},\mathfrak{m}_l}$ and $\frac{1}{a_l}\sum_k \mathrm{ad}^2(A_k)$ is the Casimir element.

By part (1), there is a number λ_l such that $\frac{1}{2}\sum_{k=1}^{\dim(\mathfrak{h})} \mathrm{ad}^2(A_k) = -\lambda_l \,\mathrm{Id}_{\mathfrak{m}_l}$. The ration between the symmetric form $\mathcal{B}_{\mathrm{ad}_{\mathfrak{h}},\mathfrak{m}_l}$ and the inner products on $\mathfrak{m}_l$ can be determined by any unit length vector in $\mathfrak{h}$. It follows that

$$\mathrm{trace}_{\mathfrak{m}_l}\sum_{k=1}^{\dim(\mathfrak{h})} \mathrm{ad}^2(A_k) = \mathrm{trace}_{\mathfrak{m}_l}(-2\lambda_l \mathrm{Id}_{\mathfrak{m}_l}) = -2\lambda_l \dim(\mathfrak{m}_l).$$

On the other hand,

$$\mathrm{trace}_{\mathfrak{m}_l}(\sum_{k=1}^{\dim(\mathfrak{h})} \mathrm{ad}^2(A_k)) = \sum_{k=1}^{\dim(\mathfrak{h})} \mathrm{trace}_{\mathfrak{m}_l}(\mathrm{ad}^2(A_k)) = \sum_{k=1}^{\dim(\mathfrak{h})} \mathcal{B}_{\mathrm{ad}_h,\mathfrak{m}_l}(A_k,A_k).$$

Consequently $\lambda_l = -\mathcal{B}_{\mathrm{ad}_h,\mathfrak{m}_l}(A_1,A_1)\frac{\dim(\mathfrak{h})}{2\dim(\mathfrak{m}_l)}$. We completed part (2). □

If $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ s.t. $\mathfrak{h}_0$ acts trivially on $\mathfrak{m}_l$ and $\mathfrak{h}_1$ a sub Lie-algebra acts faithfully, we take $\mathfrak{h}_1$ in place of $\mathfrak{h}$, λ_l can be computed using the formula in (2) with $\{A_k\}$ taken to be an orthonormal basis of $\mathfrak{h}_1$.

If $\mathcal{A} = \frac{1}{2}\sum_{k=1}^N (X_k)^2 + X_0$ where $X_k \in \mathfrak{g}$, we denote by $c(\mathcal{A}) = \frac{1}{2}\sum_{k=1}^N \mathrm{ad}^2(X_k) + \mathrm{ad}(X_0)$ to be the linear map on $\mathfrak{g}$.

Lemma 7.2 *For any $Y_0 \in \mathfrak{m}$, $\mathcal{L}_0(\mathrm{Ad}(\cdot)Y_0) = \mathrm{Ad}(\cdot)(c(\mathcal{L}_0)(Y_0))$. If Y_0 is an eigenvector of $c(\mathcal{L}_0)$ corresponding to an eigenvalue $-\lambda(Y_0)$ then for any $Y \in \mathfrak{m}$, $\langle Y, \mathrm{Ad}(h)(Y_0)\rangle$ is an eigenfunction of $\mathcal{L}_0$ corresponding to $-\lambda(Y_0)$. The converse also holds.*

Proof Just note that

$$L_{A_k(h)}(\mathrm{Ad}(\cdot)(Y_0)) = \frac{d}{dt}|_{t=0}\mathrm{Ad}(h\exp(tA_k))(Y_0) = \mathrm{Ad}(h)([A_k, Y_0]),$$

which by iteration leads to $L_{A_k(h)}L_{A_k(h)}\mathrm{Ad}(\cdot)(Y_0) = \mathrm{Ad}(h)([A_k,[A_k,Y_0]])$ and to the required identity $\mathcal{L}_0\left(\mathrm{Ad}(\cdot)Y_0\right) = \mathrm{Ad}(\cdot)c(\mathcal{L}_0)(Y_0)$. Furthermore, for every $Y \in \mathfrak{m}$,

$$\mathcal{L}_0\left(\langle \mathrm{Ad}(\cdot)Y_0, Y\rangle\right) = \langle \mathrm{Ad}(\cdot)(c(\mathcal{L}_0)Y_0), Y\rangle,$$

from which follows the statement about the eigenfunctions. □

We take an orthonormal basis $\{Y_j\}$ of $\mathfrak{m}_l$ and for Y_0 fixed set $\alpha_j(Y_0) = \langle \mathrm{Ad}(\cdot)(Y_0), Y_j\rangle$. If f is a real valued function, set $\overline{f} = \int_H f dh$ and by $\mathcal{L}_0^{-1} f$ we denote a solution to the Poisson equation $\mathcal{L}_0 u = f$. Also set $\bar{\mathcal{L}} = -\sum_{i,j=1}^{m_l} \overline{\alpha_j(Y_0)\,\mathcal{L}_0^{-1}(\alpha_i(Y_0))}\; L_{Y_i}L_{Y_j}$. Below we give an invariant formula for the limiting operator on G.

Theorem 7.3 *Suppose that $\{A_1,\dots,A_p\}$ is a basis of $\mathfrak{h}$ and $Y_0 \in \mathfrak{m}_l$ where $l \neq 0$. Then, for any orthonormal basis $\{Y_1,\dots,Y_d\}$ of $\mathfrak{m}_l$,*

$$\bar{\mathcal{L}} f = \sum_{i,j=1}^{d} a_{i,j}(Y_0)\nabla^L df(Y_i, Y_j), \quad f \in C^2(G;\mathbb{R}),$$

where $a_{i,j}(Y_0) = \frac{1}{\lambda_l}\int_H \langle Y_i, \mathrm{Ad}(h)Y_0\rangle\langle Y_j, \mathrm{Ad}(h)Y_0\rangle\, dh$, λ_l is an eigenvalue of $\mathcal{L}_0$, and $-\frac{1}{2}\sum_{k=1}^{N}\mathrm{ad}^2(A_k) = \lambda_l\,\mathrm{Id}_{\mathfrak{m}_l}$. Equivalently,

$$\bar{\mathcal{L}} f = \frac{1}{\lambda_l}\int_H \nabla^L df\left((\mathrm{Ad}(h)(Y_0))^*, (\mathrm{Ad}(h)(Y_0))\right) dh. \tag{7.2}$$

Furthermore, $\bar{x}_t$ is a Markov process with Markov generator

$$\bar{\mathcal{L}}(F\circ\pi)(u) = \frac{1}{\lambda_l}\int_H (\nabla^c dF)_{\pi(u)}\left(d\bar{L}_{uh}Y_0, d\bar{L}_{uh}Y_0\right) dh, \quad F \in C^2(M;\mathbb{R}),$$

where ∇^c be the canonical connection on the Riemannian homogeneous manifold.

Proof For the Left invariant connection, $\nabla^L_{Y_i}Y_j^* = 0$ for any i,j, and so

$$-\sum_{i,j=1}^{m}\overline{\alpha_i(Y_0)\,(\mathcal{L}_0^{-1}\alpha_j(Y_0))}Y_i^*Y_j^* f = -\sum_{i,j=1}^{m}\overline{\alpha_i(Y_0)\,(\mathcal{L}_0^{-1}\alpha_j(Y_0))}\nabla^L df(Y_i^*, Y_j^*).$$

We can always take the Ad_H invariant inner product on $\mathfrak{h}$ w.r.t. which $\{A_1,\dots,A_p\}$ is an orthonormal basis of $\mathfrak{h}$, and so Lemma 7.1 applies. By Lemma 7.1, $\frac{1}{2}c(\mathcal{L}_0) = -\lambda_l\,\mathrm{Id}_{\mathfrak{m}}$. For any $Y_0, Y \in \mathfrak{m}_l$, $\beta(Y, Y_0)(\cdot) := \langle Y, \mathrm{Ad}(\cdot)Y_0\rangle$ is an eigenfunction of $\mathcal{L}_0$ with eigenvalue $-2\lambda_l$, using Lemma7.2. Note also that $\lambda \neq 0$. Consequently,

$$\overline{\alpha_i(Y_0)\,(\mathcal{L}_0^{-1}\alpha_j(Y_0))} = -\frac{1}{\lambda_l}\int_H \langle Y_i, \mathrm{Ad}(h)Y_0\rangle\langle Y_j, \mathrm{Ad}(h)(Y_0)\rangle\, dh. \tag{7.3}$$

The equivalent formula is obtained from summing over the basis of $\mathfrak{m}_l$:

$$\sum_{i,j}\alpha_{i,j}(Y_0)\nabla^L df(Y_i^*(u), Y_j^*(u))$$
$$= \frac{1}{\lambda_l}\int_H (\nabla^L df)_u\left((\mathrm{Ad}(h)(Y_0))^*(u), (\mathrm{Ad}(h)(Y_0))^*(u)\right) dh.$$

Finally we prove the Markov property of the projection of the effective process. For $g \in G$,

$$\begin{aligned}&\nabla^L d(F \circ \pi)\left((\mathrm{Ad}(h)(Y_0))^*(g), (\mathrm{Ad}(h)(Y_0))^*(g)\right)\\ &= L_{\mathrm{Ad}(h)(Y_0)^*}\left(dF\Big(d\pi\big((\mathrm{Ad}(h)Y_0)^*\big)\Big)\right)(g) = \nabla^c dF\left(d\pi(\mathrm{Ad}(h)(Y_0)^*), d\pi(\mathrm{Ad}(h)(Y_0)^*)\right).\end{aligned}$$

The last step is due to part (4) of Lemma 10.2 in §10, from which we conclude that $dF\Big(\nabla^c_{\pi_*\mathrm{Ad}(h)(Y_0)^*} d\pi\big((\mathrm{Ad}(h)Y_0)^*\big)\Big)$ vanishes. It is clear that,

$$\begin{aligned}&\int_H (\nabla^c dF)_{\pi(g)}\left(d\pi(\mathrm{Ad}(h)(Y_0)^*(g)), d\pi(\mathrm{Ad}(h)(Y_0)(g))\right) dh\\ &= \int_H (\nabla^c dF)_{\pi(g)}\left(d\bar{L}_{gh} d\pi(Y_0), d\bar{L}_{gh} d\pi(Y_0)\right) dh.\end{aligned}$$

Since dh is right invariant, the above formulation is independent of the choice of g in $\pi^{-1}(x)$. That the stochastic process $\pi(u_t)$ is a Markov process follows from Dynkin's criterion which states that if (u_t) is a Markov process with semigroup P_t and if $\pi : G \to M$ is a map such that $P_t(f \circ \pi)(y)$ depends only on $\pi(y)$ then $\pi(u_t)$ is a Markov process. See e.g. [16, E. Dynkin] and [60, M. Rosenblatt]. □

Remarks. If we take the Ad_H invariant product on $\mathfrak{h}$ to be the one for which $\{A_k\}$ is an o.n.b. then $\mathcal{L}_0 = \Delta_H = \sum_{k=1}^p (A_k)^2$ which follows from H being compact, c.f. Lemma 10.1. The operator $\bar{\mathcal{L}}$ in the above Lemma provides an example of a cohesive operator, as defined in [19, D. Elworthy, Y. LeJan, X.-M. Li].

8 Symmetries and Special Functions

We fix an Ad_H invariant inner product on $\mathfrak{g}$, and as usual, set $\mathfrak{m} = \mathfrak{h}^\perp$ w.r.t an Ad_H invariant inner product on $\mathfrak{g}$. Let $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_k$ be an orthogonal decomposition. By $\tilde{\mathfrak{m}}$ we denote an Ad_H invariant subspace of $\mathfrak{m}$ not containing any non-trivial Ad_H-invariant vectors. In this section we explore the symmetries of the manifold M to study the functions of the form $\langle \mathrm{Ad}(\cdot)Y_0, Y_j\rangle$ where $\{Y_i\}$ is an orthonormal basis of $\tilde{\mathfrak{m}}$.

Definition 8.1 We say that Ad_H acts *quasi doubly transitively* (on the unit sphere) of $\tilde{\mathfrak{m}}$ if for any orthonormal basis $\{Y_i\}$ of $\tilde{\mathfrak{m}}$ and for any pair of numbers $i \neq j$ there is $h^{i,j} \in H$ such that $\mathrm{Ad}(h^{i,j})(Y_i) = Y_i$ and $\mathrm{Ad}(h^{i,j})(Y_j) = -Y_j$.

The family of Riemannian manifold, with a lot of symmetry, are two-point homogeneous spaces by which mean for any two points x_1, x_2 and y_1, y_2 with $d(x_1, x_2) = d(y_1, y_2)$ sufficiently small, there exist an isometry taking (x_1, x_2) to (y_1, y_2). Such spaces were classified by H.-C. Wang (1952, Annals) to be isometric to a symmetric Riemannian space $\mathrm{Iso}^o(M)/K$ where K is compact. They have constant sectional curvatures in odd dimensions and the non-compact spaces are all simply connected and homeomorphic to an Euclidean space. On a locally two point homogeneous space Ad_H acts transitively on the unit sphere, and acts quasi doubly transitively. If (e, f) or orthogonal unit vectors at T_oM. Let $x = \exp_o(\delta e)$, $x' = \exp_o(-\delta e)$, and $y = \exp_o(\delta f)$ where δ is a number sufficiently small for them to be defined. There is an isometry ϕ taking x to y leaving o fixed. This isometry taking the geodesic $0x$ to the geodesic oy. It would take y to x' which implies that $\rho(x, y) = \rho(y, x')$.

Lemma 8.1 *Let* $\{Y_1, \ldots, Y_d\}$ *be an orthonormal basis of* $\tilde{\mathfrak{m}}$*, an* Ad_H *invariant subspace of* $\mathfrak{m}$ *with* $\tilde{\mathfrak{m}} \cap \mathfrak{m}_0 = \{0\}$*. Let* $Y_0 \in \tilde{\mathfrak{m}}$*.*

1. *If* Ad_H *acts transitively on the unit sphere of* $\tilde{\mathfrak{m}}$*.*
$$\int_H \langle Y_i, \mathrm{Ad}(h)Y_0\rangle^2\, dh = \frac{|Y_0|^2}{\dim(\tilde{\mathfrak{m}})}, \quad i = 1, \ldots, \dim(\tilde{\mathfrak{m}}).$$

2. *If* Ad_H *acts quasi doubly transitively on* $\tilde{\mathfrak{m}}$ *then for* $i \neq j$*,*
$$\int_H \langle Y_i, \mathrm{Ad}(h)Y_0\rangle \langle Y_j, \mathrm{Ad}(h)(Y_0)\rangle\, dh = 0.$$

3. $\dim(\tilde{\mathfrak{m}}) > 1$.

4. *If* $\dim(\tilde{\mathfrak{m}}) = 2$ *then* $\int_H \langle Y_i, \mathrm{Ad}(h)Y_0\rangle \langle Y_j, \mathrm{Ad}(h)(Y_0)\rangle\, dh = \frac{|Y_0|^2}{2}\delta_{i,j}$ *for all* i, j*.*

Proof Suppose that $\mathrm{Ad}(h_0)$ take Y_j to Y_1. Then

$$\int_H \langle Y_j, \mathrm{Ad}(h)Y_0\rangle^2 dh = \int_H \langle Y_j, \mathrm{Ad}(h_0^{-1})\mathrm{Ad}(h)(Y_0)\rangle^2 dh = -\int_H \langle Y_1, \mathrm{Ad}(h)(Y_0)\rangle^2 dh,$$

using the Ad_H-invariance of the Haar measure and the inner product. Set $d = \dim(\tilde{\mathfrak{m}})$. Since $\tilde{\mathfrak{m}}$ is an invariant space of Ad_H we sum over the basis vectors to obtain

$$\sum_{j=1}^d \int_H \langle Y_j, \mathrm{Ad}(h)Y_0\rangle^2 dh = \int_H \sum_{j=1}^d \langle \mathrm{Ad}(h^{-1})Y_j, Y_0\rangle^2 dh = |Y_0|^2.$$

It follows that $\int_H \langle Y_j, \mathrm{Ad}(h)Y_0\rangle^2 dh = \frac{|Y_0|^2}{\dim(\tilde{\mathfrak{m}})}$. For $i \neq j$, there exists $h^{i,j} \in H$ such that $\mathrm{Ad}(h^{i,j})(Y_i) = Y_i$ and $\mathrm{Ad}(h^{i,j})(Y_j) = -Y_j$. By the bi-invariance of the Haar measure and the Ad_H -invariance of the inner product again we obtain,

$$\begin{aligned}
&\int_H \langle Y_i, \mathrm{Ad}(h)Y_0\rangle \langle Y_j, \mathrm{Ad}(h)Y_0\rangle dh \\
&= \int_H \langle Y_i, \mathrm{Ad}(h^{i,j})\mathrm{Ad}(h)(Y_0)\rangle \langle Y_j, \mathrm{Ad}(h^{i,j})\mathrm{Ad}(h)Y_0\rangle dh \\
&= -\int_H \langle Y_i, \mathrm{Ad}(h)(Y_0)\rangle \langle Y_j, \mathrm{Ad}(h)Y_0\rangle dh.
\end{aligned}$$

and part (2) follows.

We observe that no Ad_H-invariant subspace of $\mathfrak{m}$, containing no trivial Ad_H-invariant vectors, can be one dimensional, for otherwise, every orthogonal transformation $\mathrm{Ad}(h)$ takes a $Y_0 \in \tilde{\mathfrak{m}}$ to itself or to $-Y_0$, and one of which takes Y_0 to $-Y_0$, violating connectedness. The connected component of the compact matrix group $\mathrm{Ad}(H)$ is its normal subgroup of the same dimension to which we apply the following facts to conclude the lemma.

Suppose that $d = 2$, then $\tilde{\mathfrak{m}}$ is irreducible. If we identify $\tilde{\mathfrak{m}}$ with $\mathbb{R}^2$ then H can be identified with $SO(2)$. Let $\{Y_1, Y_2\}$ be an orthonormal basis of $\mathfrak{m}$. Then

$$\int_H \langle Y_1, \mathrm{Ad}(h)Y_0\rangle \langle Y_2, \mathrm{Ad}(h)Y_0\rangle dh = \int_{SO(2)} \langle Y_1, gY_0\rangle \langle Y_2, gY_0\rangle dg$$

where dg is the pushed forward measure $\mathrm{Ad}^*(dh)$. Since dg is bi-invariant it is the standard measure on $SO(2)$, normalised to have volume 1. Thus the above integral vanishes. This can be computed explicitly. The same proof as in Proposition 8.2 shows that $\int_H \langle Y, \mathrm{Ad}(h)Y_0\rangle^2 dh = \frac{|Y_0|^2}{2}$. □

Proposition 8.2 *Let* $\tilde{\mathfrak{m}}$ *be an* Ad_H *invariant subspace of* $\mathfrak{m}$ *not containing any non-trivial* Ad_H*-invariant vectors. Suppose* $Y_0 \in \tilde{\mathfrak{m}}$ *and let*

$$\bar{\mathcal{L}}f = \frac{1}{\lambda(Y_0)} \int_H \nabla^L df\left((\mathrm{Ad}(h)(Y_0))^*, (\mathrm{Ad}(h)(Y_0))^*\right) dh.$$

Then $\bar{\mathcal{L}} = \frac{|Y_0|^2}{\lambda(Y_0)\dim(\tilde{\mathfrak{m}})}\Delta_{\tilde{\mathfrak{m}}}$ *under one of the following conditions:*

(a) $\dim(\tilde{\mathfrak{m}}) = 2$*;*

(b) Ad_H *acts transitively and quasi doubly transitively on the unit sphere of* $\tilde{\mathfrak{m}}$*.*

Proof Expanding $(\mathrm{Ad}(h)(Y_0))^*$ in $\{Y_j^*\}$ we see that $\bar{\mathcal{L}}f = \sum_{i,j=1}^d a_{i,j}(Y_0)\nabla^L df(Y_i^*, Y_j^*)$ where $a_{i,j}(Y_0) = \frac{1}{\lambda(Y_0)} \int_H \langle Y_i, \mathrm{Ad}(h)Y_0)\rangle \langle Y_j, \mathrm{Ad}(h)(Y_0)\rangle\, dh$. If $\dim(\tilde{\mathfrak{m}}) = 2$ then $\mathrm{Ad}(H)$ can be identified with $SO(2)$. In both cases, by Lemma 8.1, $a_{11}(Y_0) = \cdots = a_{dd}(Y_0)$ and the cross terms disappear. □

Since ad($\mathfrak{h}$) consists of skew symmetric matrices, with respect to the invariant Riemannian metric on the homogeneous manifold G/H, H acts as a group of isometries. So do the left actions by elements of H on $\pi_*(\tilde{\mathfrak{m}})$. Since π takes the identity to the coset H, we may rule out translations and every element of $\mathrm{Ad}(h)$ is rotation on $\tilde{\mathfrak{m}}$. We identify $\tilde{\mathfrak{m}}$ with $\mathbb{R}^d$, where d is the dimension of $\tilde{\mathfrak{m}}$. Since H is connected, they are orientation preserving rotations. Since H is compact, so is $\mathrm{Ad}(H)$ and $\mathrm{Ad}(H)$ can be identified with a compact subgroup of $SO(d)$.

It would be nice to classify all subgroups of $SO(d)$ that acts transitively and quasi doubly transitively on the spheres. Sub-groups of $SO(d)$ acting transitively on the spheres are completely classified and coincide with the list of possible holonomy groups of simply connected, see [4], , non-symmetric irreducible complete Riemannian manifolds [63, J. Simons]. See also [28, E. Heintze, W. Ziller]. The large subgroups of $SO(d)$ are reasonably well understood by a theorem of Montgomery and Samelson [52] also a theorem of M. Obata [56]. There are two exceptions: the non-simple group $SO(4)$ and also $SO(8)$ which has two interesting subgroups: the 21 dimensional $spin(7)$ and the exceptional 14 dimensional compact simple Lie group G_2, automorphism of the Octonians. We end this discussion with the following remark.

Remark 8.3 Let $d = \dim(\tilde{\mathfrak{m}})$ where $\tilde{\mathfrak{m}}$ is Ad_H-invariant. Suppose that $\dim(\mathrm{Ad}(H)|_{\tilde{\mathfrak{m}}}) \geq \dim(SO(d-1))$; or suppose that $\tilde{\mathfrak{m}}$ has no two dimensional invariant subspace with $d \geq 13$ and suppose that $\dim(\mathrm{Ad}(H)|_{\tilde{\mathfrak{m}}}) \geq \dim(O(d-3)) + \dim(O(3)) + 1$. Then the identity component of $\mathrm{Ad}(H)_{\tilde{\mathfrak{m}}}$ is

1. $SO(d)$ if $d \neq 4, 8$;
2. $SO(4)$ or S^3 if $d = 4$;
3. $SO(8)$ or $spin(7)$, if $d = 8$.

Furthermore, $\mathrm{Ad}_H : H \to \mathbb{L}(\tilde{\mathfrak{m}}; \tilde{\mathfrak{m}})$ acts transitively on the unit sphere of $\tilde{\mathfrak{m}}$, and acts quasi doubly transitively on $\tilde{\mathfrak{m}}$ if $d > 2$.

Our aim is to prove that Ad_H acts transitively and quasi doubly transitively, using properties of connected closed subgroups of the rotation groups, we are not concerned with weather the image of H under the representation, considered as subgroups of $SO(d)$, $d = \dim(\mathfrak{m}_l)$, is connected. It is sufficient prove its identity component has the required property. The connected component of the compact matrix group Ad(H) is its normal subgroup of the same dimension to which we apply the following facts to conclude the lemma. For our purpose, the image of Ad_H acts faithfully on $\mathfrak{m}_l$ in the sense that if H_0 is the group of identity transformations, acting on $\mathfrak{m}_l$, then H/H_0 has the same orbit as H_0 and both acts transitively.

We have already seen that no Ad_H-invariant subspace of $\mathfrak{m}$ can be one dimensional. By a theorem of D. Montgomery and H. Samelson[52, Lemma 3, 4], there is no proper closed subgroup H' of $SO(d)$ of dimension greater than the dimension of $SO(d-1)$. If H' is a connected closed sub-group of $SO(d)$ of the dimension of $SO(d-1)$, then H is continuously isomorphic to $SO(d-1)$ or to the double cover of $SO(d-1)$. If $d \neq 4, 8$, then H is conjugate with $Q(d-1)$, the sub-matrix of $SO(d)$ leaving invariant the first axis. Note that the identity component of Ad(H) has the dimension of Ad(H). Since $\mathfrak{m}$ has no one dimensional invariant subspace we conclude that Ad$(H) = SO(d)$ for $d = 4, 8$.

In dimension 4, the subgroup S^3 is not conjugate with $Q(d-1)$. The subgroup S^3 acts on itself transiently, freely, and leaves no one dimensional sub-space invariant. It is also doubly transitive.

If $d = 8$, the 21-dimensional sub-group $spin(7)$ is embedded into $SO(8)$ by the spin representation, it acts transitively on S^7 and its isotropy subgroup at a point is G_2. We learnt from Dmitriy Rumynin that $G_2 \sim SU(4) \subset spin(7)$ and G_2 acting transitively on unit spheres of S^7 and transitive on any two pairs of orthogonal unit vectors. Finally we use a theorem from [56, M. Obata] : if K is a Lie group of orthogonal $d\times d$ matrices where $d \geq 13$ and if $\dim(O(d)) > \dim(K) \geq \dim(O(d-3)) + \dim(d(3)) + 1$, then K is reducible in the real vector space. Since $\mathfrak{m}_l$ is irreducible, the group Ad(H) must be $SO(d)$ where $d = \dim(m_l)$. We may then apply the same analysis as above.

9 Laplacian Like Operators as Effective Limits

Let $\{Y_j\}$ be an orthonormal basis of $\tilde{\mathfrak{m}}$. Denote $\Delta_{\tilde{\mathfrak{m}}} = \sum_{j=1}^m L_{Y_j^*} L_{Y_j^*}$, this is the 'round' operator on $\tilde{\mathfrak{m}}$. For the left invariant connection ∇^L, $\Delta_{\tilde{\mathfrak{m}}} = \mathrm{trace}_{\tilde{\mathfrak{m}}} \nabla^L d$ is a 'generalised' horizontal Laplacian and is independent of the choice of the basis. In the special case where G is isomorphic to the Cartesian product of a compact group and an additive vector group, there is a bi-invariant metric, ∇ is the Levi-Civita connection for a bi-invariant metric, then $\nabla_X X^* = 0$ and $\Delta_{\tilde{\mathfrak{m}}} = \mathrm{trace}_{\tilde{\mathfrak{m}}} \nabla d$. In the irreducible case, this operator $\Delta_{\mathfrak{m}}$ is the *horizontal Laplacian* and we denote the operator by Δ^{hor}. Its corresponding diffusion is a *horizontal Brownian motion*. In the reducible case, we abuse the notation and define a similar concept. Since the limit operator in Theorem 6.10 is given by averaging the action of Ad_H, we expect that the size of the isotropy group H is correlated with the 'homogeneneity' of the diffusion operator, which we explore in the remaining of the section.

Definition 9.1 A sample continuous Markov process is a (generalised) horizontal Brownian motion if its Markov generator is $\frac{1}{2}\Delta_{\tilde{\mathfrak{m}}}$; it is a (generalised) scaled horizontal Brownian motion with scale c if its Markov generator is $\frac{1}{2}c\Delta_{\tilde{\mathfrak{m}}}$ for some constant $c \neq 0$.

We use the notation in Theorem 6.10. Let , $\alpha(Y_j, Y_0) = \langle Y_j, \mathrm{Ad}(h)(Y_0)\rangle$ and

$$\bar{\mathcal{L}} = \sum_{i,j=1}^{d} a_{i,j}(Y_0)\, L_{Y_i^*} L_{Y_j^*}, \quad a_{i,j}(Y_0) = -\int_H \alpha(Y_0, Y_i)\, \mathcal{L}_0^{-1}(\alpha(Y_0, Y_j))dh.$$

If the representation space of Ad_H were complex, then $\langle Y, \mathrm{Ad}(h)Y_0)\rangle$ where $Y, Y_0 \in \mathfrak{m}_l$ are known as trigonometric functions.

Theorem 9.1 *Suppose that* $\{A_1, \dots, A_N\}$ *generates* $\mathfrak{h}$, $\mathcal{L}_0 = \frac{1}{2}\sum_{k=1}^N (A_k)^2$, *and* $Y_0 \in \mathfrak{m}_l$ *where* $l \neq 0$. *If* $Y_0 = \sum_{m=1}^d c_m Y_m$, *then*

$$\bar{\mathcal{L}} = \sum_{m=1}^{d} \frac{(c_m)^2}{\lambda(Y_m)} \frac{1}{\dim(\mathfrak{m}_l)} \Delta_{\mathfrak{m}_l}.$$

If furthermore $\{A_1, \dots, A_N\}$ *is an o.n.b. of* $\mathfrak{h}$, *then* $\bar{\mathcal{L}} = \frac{|Y_0|^2}{\dim(\mathfrak{m}_l)\lambda_l} \Delta_{\mathfrak{m}_l}$.

Proof Set $d = \dim(\mathfrak{m}_l)$. With respect to the Ad_H-invariant inner product on $\mathfrak{g}$, $\mathrm{ad}^2(A_k)$ is a self-adjoint linear map on $\mathfrak{m}_l$. For an orthonormal basis $\{Y_1, \dots, Y_d\}$ of $\mathfrak{m}_l$ consisting of eigenvectors of $-\frac{1}{2}\sum_{k=1}^N \mathrm{ad}^2(A_k)$ with $\lambda(Y_j)$ the corresponding eigenvalues. Then $\alpha(Y_j, Y_0)$ is an eigenfunction of $\mathcal{L}_0$ corresponding to the eigenvalue $-\lambda(Y_0)$, and $-\frac{\alpha(Y_j, Y_0)}{\lambda(Y_0)}$ solves the Poisson equation with right hand side $\alpha(Y_j, Y_0)$, see Lemma 7.2. Evidently $\lambda(Y_j) \neq 0$, for otherwise $\langle \sum_k \mathrm{ad}^2(A_k)(Y_j), Y_j\rangle = -\sum_k |\mathrm{ad}(A_k)(Y_j)|^2 = 0$ which means Y_j is in the kernel of ad_H which is possible only if $l = 0$.

Consequently,

$$\begin{aligned} a_{i,j}(Y_0) &= -\sum_{m,m'} c_m c_{m'} \int_H \langle \mathrm{Ad}(h)(Y_{m'}), Y_i\rangle \mathcal{L}_0^{-1} \langle \mathrm{Ad}(h)(Y_m), Y_j\rangle dh \\ &= \sum_{m,m'} \frac{c_m c_{m'}}{\lambda(Y_m)} \int_H \langle \mathrm{Ad}(h)(Y_{m'}), Y_i\rangle \langle \mathrm{Ad}(h)(Y_m), Y_j\rangle dh. \end{aligned}$$

Peter-Weyl's theorem states that if V is an irreducible unitary representation of a compact Lie group H and $\{Y_i\}$ an o.n.b. of V, then the collection of functions $\{\langle Y_i, \rho(\cdot)Y_k\rangle\}$, where V ranges through all equivalent classes of irreducible unitary representations and where ρ is the representation, is orthogonal with norm $\sqrt{\dim(V)}$. In particular

$$\int_H \langle Y_i, \mathrm{Ad}(\{h)Y_k)\rangle \langle Y_j, \mathrm{Ad}(h)(Y_l)\rangle\, dh = \frac{1}{d}\delta_{ij}\delta_{kl}.$$

We are grateful to Dmitriy Rumynin for providing us with a version of this theorem, valid for orthogonal representations which is appended at the end of the paper. We do not know any other reference for this.

From which we see that $a_{i,j} = 0$ for $i \neq j$ and

$$a_{i,i}(Y_0) = \sum_{m=1}^{d} \frac{(c_m)^2}{\dim(\mathfrak{m}_l)\lambda(Y_m)}, \qquad \bar{\mathcal{L}} = \frac{1}{d}\sum_{m=1}^{d} \frac{(c_m)^2}{\lambda(Y_m)} \sum_{i=1}^{m} L_{Y_i} L_{Y_i}.$$

Then part (1) follows by seeting

$$\frac{1}{\lambda'(Y_0)} = \sum_{m=1}^{d} \frac{(c_m)^2}{\lambda(Y_m)}.$$

Note that $\sum_{i=1}^m L_{Y_i} L_{Y_i} = \text{trace}\, \nabla^L d$ is independent of the choice of the basis vectors $\{Y_j\}$, so is also $\bar{\mathcal{L}}$ as a Markov generator to (4.4) which means $\lambda'(Y_0)$ is independent of the choice of the basis vectors. For the case $\{A_1, \ldots, A_N\}$ is an orthonormal basis of $\mathfrak{h}$, we apply Lemma 7.1 to conclude. $\square$

If Y_0 belongs to a subspace of $\tilde{\mathfrak{m}}$, say $Y_0 \in m_l \oplus m_{l'}$, then an analogous claim holds if the representations Ad_H on $\mathfrak{m}_l$ and $m_{l'}$ are not equivalent, especially if $\mathfrak{m}_l$ and $m_{l'}$ have different dimensions.

Let $d = 2$. Then $\sum_k \mathrm{ad}^2(A_k)|_{\mathfrak{m}_l} = \lambda_l \,\mathrm{Id}\,|_{\mathfrak{m}_l}$, for some number $\lambda_l \neq 0$, if and only if $\bar{\mathcal{L}} = \frac{|Y_0|^2}{2\lambda_l} \Delta_{\mathfrak{m}_l}$. Indeed, H is essentially $SO(2)$. Let Y_1, Y_2 be a pair of orthogonal unit length eigenvectors of the linear map $\frac{1}{2} \sum_k \mathrm{ad}^2(A_k)$, restricted to $\mathfrak{m}_l$. Let $Y_0 = c_1 Y_1 + c_2 Y_2$. For $j', j, k', k = 1, 2$, let

$$b_{j',j}^{k,k'}(Y_0) = \frac{c_k c_{k'}}{\lambda(Y_{k'})} \int_{S^1} \langle \mathrm{Ad}(e^{i2\pi\theta})(Y_k), Y_{j'} \rangle \langle \mathrm{Ad}(e^{i2\pi\theta})(Y_{k'}), Y_j \rangle d\theta.$$

By direct computation, it is easy to see that $b_{j',j}^{k',k} = 0$ unless $j' = j$, $k = k'$ or $j' = k, j = k'$, In particular, we examine the cross term:

$$\begin{aligned} a_{1,2}(Y_0) &= \sum_{k,l=1}^{2} b_{1,2}^{k,l} = b_{1,2}^{2,1} + b_{1,2}^{1,2} \\ &= \frac{c_1 c_2}{\lambda(Y_2)} \int_{S^1} \langle \mathrm{Ad}(e^{i2\pi\theta})(Y_1), Y_1 \rangle \langle \mathrm{Ad}(e^{i2\pi\theta})(Y_2), Y_2 \rangle d\theta \\ &+ \frac{c_1 c_2}{\lambda(Y_1)} \int_{S^1} \langle \mathrm{Ad}(e^{i2\pi\theta})(Y_2), Y_1 \rangle \langle \mathrm{Ad}(e^{i2\pi\theta})(Y_1), Y_2 \rangle d\theta. \end{aligned}$$

Direct computation shows that

$$\begin{aligned} a_{1,2}(Y_0) &= \frac{c_1 c_2}{\lambda(Y_2)} \int \cos^2(i2\pi\theta) d\theta - \frac{c_1 c_2}{\lambda(Y_1)} \int \sin^2(i2\pi\theta) d\theta \\ &= \frac{1}{2} \left(\frac{c_1 c_2}{\lambda(Y_2)} - \frac{c_1 c_2}{\lambda(Y_1)}\right). \end{aligned}$$

Thus

$$\bar{\mathcal{L}} = \left(\frac{c_1 c_2}{\lambda(Y_2)} - \frac{c_1 c_2}{\lambda(Y_1)}\right) L_{Y_1} L_{Y_2} + \frac{1}{2} \left(\frac{c_1^2}{\lambda(Y_1)} + \frac{c_2^2}{\lambda(Y_2)} \right) \Delta_{\mathfrak{m}_l}.$$

Thus $a_{1,2}(Y_0)$ vanishes if and only if $\lambda_l := \lambda(Y_2) = \lambda(Y_1)$, the latter allows us to conclude that $a_{1,1}(Y_0) = a_{2,2}(Y_0) = \frac{1}{2} \frac{|Y_0|^2}{\lambda_l}$ and $\frac{1}{2} \sum_k \mathrm{ad}^2(A_k) = \lambda_l \,\mathrm{Id}$.

We next work with semi-simple Lie groups. A Lie algebra is simple if it is not one dimensional and if $\{0\}$ and $\mathfrak{g}$ are its only ideals; it is semi-simple if it is the direct sum of simple algebras. Cartans criterion for semi-simplicity states that $\mathfrak{g}$ is semi simple if and only if its killing form is non-degenerate. Another useful criterion is that a lie algebra is semi-simple if and if it has no solvable (i.e. Abelian) ideals. The special unitary group $SU(n)$ is semi-simple if $n \geq 2$; $SO(n)$ is semi-simple for $n \geq 3$; $SL(n, \mathbb{R})$ is a non-compact semi-simple Lie group for $n \geq 2$; $\mathfrak{so}(p, q)$ is semi-simple for $p + q \geq 3$.

Corollary 9.2 *Let H be a maximal torus group of a semi-simple group G. Let $Y_0 \in \mathfrak{m}_l$ and suppose that $A_0 = 0$ and $\{A_i\}$ generates $\mathfrak{h}$. Then $\bar{\mathcal{L}} = \frac{|Y_0|^2}{2\lambda_l} \Delta_{\mathfrak{m}_l}$ where λ_l is determined by $\frac{1}{2} \sum_{k=1}^p \mathrm{ad}^2(A_k) = -\lambda_l \,\mathrm{Id}_{\mathfrak{m}_l}$.*

Proof If $\mathfrak{h}$ is the Cartan sub-algebra of the semi-simple Lie algebra $\mathfrak{g}$ the dimension of $\mathfrak{m}_l$ is 2 and Proposition 8.2 applies. It is clear that $\dim(\mathfrak{m}_l) \geq 2$, for otherwise it consists of invariant vectors. It is well known that $\mathfrak{g}$ is semi-simple if and only if $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ is semi-simple. Denote the complexification of $\mathfrak{h}$ by $\mathfrak{t}$, it is a Cartan sub-algebra of $\mathfrak{g} \otimes \mathbb{C}$. Let $\alpha \in \mathfrak{t}^*$ be a weight for $\mathfrak{g}_{\mathbb{C}}$. Take $Y = Y_1 + iY_2$ from the root space $\mathfrak{g}_\alpha$ corresponding to α. Let $X \in \mathfrak{h}$, and write $\alpha = \alpha_1 + i\alpha_2$. Then

$$[X, Y] = [X, Y_1] + i[X, Y_2] = (\alpha_1(X) + i\alpha_2(X))(Y_1 + iY_2),$$

and Y_1 and Y_2 generate an invariant subspace for ad_H following from:

$$[X, Y_1] = \alpha_1(X)Y_1 - \alpha_2(X)Y_2; \quad [X, Y_2] = \alpha_1(X)Y_1 + \alpha_2(X)Y_2.$$

Restricted on each vector space $\mathfrak{g}_\alpha$ only one specific A_k in the sum $\sum_{k=1}^{p} \mathrm{ad}^2(A_k)$ makes non-trivial contribution to the corresponding linear operation. Thus we can restrict to the one dimensional torus sub-group of H, which acts faithfully on $\mathfrak{g}_\alpha$. By the earlier discussion, $\bar{\mathcal{L}} = \frac{|Y_0|^2}{\lambda_l \dim(\mathfrak{m}_l)} \Delta_{\mathfrak{m}_l} = \frac{1}{2\lambda_l} |Y_0|^2 \Delta_{\mathfrak{m}_l}$. □

Example 9.3 Let $G_0(k, n) = SO(n)/SO(k) \times SO(n-k)$ be the oriented Grassmannian manifold of k oriented planes in n dimensions. It is a connected manifold of dimension $k(n-k)$. The Lie group $SO(n)$ act on it transitively. Let o be k-planes spanned by the first k vectors of the standard basis in $\mathbb{R}^n$. Then $SO(k)\times O(n-k)$ keeps $\mathbb{R}^k$ fixed and as well as keeps the orientation of the first k-frame. Let $\pi : SO(n) \to G(k, n)$, then $\pi O = \{O_1, \dots, O_k\}$, the first k columns of the matrix O. Let $\sigma(A) = SAS^{-1}$, where S is the diagonal block matrix with $-I_k$ and I_{n-k} as entries, be the symmetry map on G. The Lie algebra has the symmetric decomposition $\mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{m}$:

$$\mathfrak{h} = \left\{ \begin{pmatrix} \mathfrak{so}(k) & 0 \\ 0 & \mathfrak{so}(n-k) \end{pmatrix} \right\}, \quad \mathfrak{m} = \left\{ Y_M := \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix}, M \in M_{k,n-k} \right\},$$

The adjoint action on $\mathfrak{m}$ is given by

$$\mathrm{Ad}(h)(Y_M) = \begin{pmatrix} 0 & RMQ^T \\ -(RMQ^T)^T & 0 \end{pmatrix},$$

for $h = \begin{pmatrix} R & 0 \\ 0 & Q \end{pmatrix}$, $R \in SO(k)$ and $Q \in SO(n-k)$. We identify $\mathfrak{m}$ with $T_oG_0(k, n)$. The isotropy action on $T_oG_0(k, n)$ is now identified with the map $(\begin{pmatrix} R & 0 \\ 0 & Q \end{pmatrix}, M) \to RMQ^T$. Let $M_1 = (e_1, 0 \dots, 0)$. Then $RM_1Q^T = R_1Q_1^T$ where R_1 and Q_1 are the first columns of R and Q respectively. The orbit of M_1 by the Adjoint action generates a basis of $M_{k,n-k}$ and M is isotropy irreducible. Let $M_2 = (0, e_2, 0, \dots, 0)$, dR and dQ the Haar measure on $SO(k)$ and $SO(n-k)$ respectively. Then if $R = (r_{ij})$ and $Q = (q_{ij})$. Then

$$\int_H \langle \mathrm{Ad}(h)M_1, M_1\rangle \langle \mathrm{Ad}(h)(M_1), M_2\rangle dRdQ = \int_H (r_{11})^2 q_{11}q_{12}dQdR = 0.$$

Hence $\bar{\mathcal{L}}$ is proportional to Δ^{hor}. It is easy to see that $\bar{\mathcal{L}}$ satisfies the one step Hörmander condition and is hypoelliptic on G. Given $M, N \in M_{k\times(n-k)}$ whose corresponding elements in $\mathfrak{m}$ denoted by $\tilde{M}, \tilde{N}$. Then $[\tilde{M}, \tilde{N}] = N^TM - MN^T$. There is a basis of $\mathfrak{h}$ in this form. If $\{e_i\}$ is the standard basis of $\mathbb{R}^k$, $E_{ij} = e_ie_j^T$, $\{E_{ij} - E_{ji}, i < j\}$ is a basis of $\mathfrak{so}(\mathfrak{k})$.

10 Limits On Riemannian Homogeneous Manifolds

Let M be a smooth manifold with a transitive action by a Lie group G. A Riemannian metric on M is *G-invariant* if $\bar{L}_a$ for all $a \in G$, are isometries, in which case M is a Riemannian homogeneous space and G is a subgroup of Iso(M). We identify G with the group of actions and M with G/H where $H = G_o$, the subgroup fixing a point o. By declaring $d\pi$ at the identity an isometry, an Ad(H)-invariant inner product on $\mathfrak{m}$ induces a G invariant inner product on T_oM and vice versa. This extends to a G invariant Riemannian metric by defining: $\langle (d\bar{L}_g)_o \pi_* Y_1, (d\bar{L}_g)_o \pi_* Y_2 \rangle_{go} = \langle \pi_* Y_1, \pi_* Y_2 \rangle_o$. Furthermore, G-invariant metrics on M are in one to one correspondence with Ad_H-invariant metrics on $\mathfrak{m}$. We should mention that, by a theorem of S. B. Myers and N. E. Steenrod [53], the set of all isometries of a Riemannian manifold M is a Lie group under composition of maps, and furthermore the isotropy subgroup $\mathrm{Iso}_o(M)$ is compact. See also S. Kobayashi and K. Nomizu [39]. If a subgroup G of Iso(M) acts on M transitively, G/H is a *Riemannian homogeneous space*, in the sense that G acts effectively on M.

A connected Lie group admits an Ad-invariant metric if and only if G is of compact type, i.e. G is isomorphic to the Cartesian product of a compact group and an additive vector group [51, J. Milnor , Lemma 7.5], in which case we choose to use the bi-invariant metric for simplicity. The existence of an Ad_H-invariant metric is less restrictive. If H is compact by averaging we can construct an Ad_H invariant inner product in each irreducible invariant subspace of $\mathfrak{m}$. If $\mathfrak{m} = \mathfrak{m}_0 \oplus \bigoplus_{i=1}^r \mathfrak{m}_i$ is an irreducible invariant decomposition of $\mathfrak{m}$, Ad_H-invariant inner products on $\mathfrak{m}$ are precisely of the form $g_0 + \sum_{i=1}^r a_i g_i$ where g_0 is any inner product on $\mathfrak{m}_0$, a_i are positive numbers and g_i are Ad_H invariant inner products on $\mathfrak{m}_i$. In particular an irreducible homogeneous space with compact H admits a G-invariant Riemannian metric, unique up to homotheties. See [67, J. A. Wolf] and [7, A. L. Besse, Thm. 7.44].

In the remaining of the section we assume that G is given a left invariant Riemannian metric which induces a G-invariant Riemannian metric on M as constructed. We ask the question whether the projection of the limiting stochastic process in Theorem 7.3 is a Brownian motion like process. Firstly, we note that the projections of 'exponentials', $g\exp(tX)$, are not necessarily Riemannian geodesics on M. They are not necessarily Riemannian exponential maps on G. Also, given an orthonormal basis $\{Y_i, 1 \le i \le n\}$ of $\mathfrak{g}$, $\sum_{i=1}^n L_{Y_i^*} L_{Y_i^*}$ is not necessarily the Laplace-beltrami operator.

A connected Lie group G admits a bi-invariant Riemannian metric if and only if its Lie algebra admits an Ad_G invariant inner product, the latter is equivalent to ad(X) is skew symmetric for every $X \in \mathfrak{g}$. It is also equivalent to that G is the Cartesian product of a compact group and an additive vector space. For $X, Y, Z \in \mathfrak{g}$, Koszul's formula for the Levi-Civita connection of the left-invariant metric on G gives:

$$\langle 2\nabla_X Y, Z \rangle = \langle [X,Y], Z \rangle + \langle [Z,Y], X \rangle + \langle [Z,X], Y \rangle.$$

By polarisation, $\nabla_X X = 0$ for all X if and only if

$$\langle [Z,Y], X \rangle + \langle [Z,X], Y \rangle = 0, \quad \forall X, Y, Z, \tag{10.1}$$

in which case $\nabla_X Y = \frac{1}{2}[X,Y]$. In another word, (10.1) holds if and only if ∇^L and ∇ have the same set of geodesics, they are translates of the one parameter subgroups. If the Riemannian metric on G is bi-invariant then translates of the one parameter subgroups are indeed geodesics for the Levi-Civita connection and for the family of connections interpolating left and right invariant connections. A connection is torsion skew symmetric if its torsion $\mathcal{T}$ satisfies: $\langle \mathcal{T}(u,v),w\rangle + \langle \mathcal{T}(u,w),v\rangle = 0$ for all $u,v,w \in T_xM$ and $x \in M$. Denote by $T^L(X,Y)$ the torsion for the flat connection ∇^L. Since $T^L(X,Y) = -[X,Y]$, (10.1) is equivalent to ∇^L being torsion skew symmetric. By Milnor, if a connected Lie group has a left invariant connection whose Ricci curvatures is non-negative then G is unimodular. We expand this in the following lemma, the unimodular case is essentially Lemma 6.3 in [51, J. Milnor].

Lemma 10.1 *Let G be a connected Lie group with a left invariant metric. Then $\sum_{i=1}^n L_{Y_i^*}L_{Y_i^*} = \Delta$ if and only if G is unimodular. If $\mathfrak{m}_l$ is a subspace of $\mathfrak{g}$, then* $\mathrm{trace}_{\mathfrak{m}_l}\,\mathrm{ad}(X) = 0$ *for all $X \in \mathfrak{g}$ if and only if* $\mathrm{trace}_{\mathfrak{m}_l} \nabla d = \mathrm{trace}_{\mathfrak{m}_l} \nabla^L d$.

Proof A Lie group G is unimodular if its left invariant Haar measure is also right invariant. By [30, S. Helgson], a Lie group is unimodular if and only if the absolute value of the determinant of $\mathrm{Ad}(g): G \to G$ is 1 for every $g \in G$. Equivalently $\mathrm{ad}(X)$ has zero trace for every $X \in \mathfrak{g}$, see [51, J. Milnor, Lemma 6.3]. On the other hand $\sum_{i=1}^n L_{Y_i^*}L_{Y_i^*} = \Delta$ if and only if $\sum_{i=1}^n \nabla_{Y_i^*}Y_i^* = 0$. The latter condition is equivalent to $\mathrm{trace}\,\mathrm{ad}(X) = \sum_i \langle [X,Y_i],Y_i\rangle = 0$ for all $X \in \mathfrak{g}$. Similarly if $\{Y_i\}$ is an orthonormal basis of $\mathfrak{m}_l$, then

$$\sum_{i=1}^{\dim(\mathfrak{m}_l)} L_{Y_i^*}L_{Y_i^*} = \sum_{i=1}^{\dim(\mathfrak{m}_l)} \nabla d + \sum_{i=1}^{\dim(\mathfrak{m}_l)} \nabla_{Y_i^*}Y_i^*.$$

The second statement follows from the identity:

$$\langle \sum_{i=1}^{\dim(\mathfrak{m}_l)} \nabla_{Y_i^*}Y_i^*, X^*\rangle = \sum_{i=1}^{\dim(\mathfrak{m}_l)} \langle [X,Y_i],Y_i\rangle.$$

□

We discuss the relation between the Levi-Civita connection and the canonical connection on the Rieamnnian homogeneous manifold. A connection is G invariant if $\{\bar{L}_a, a \in G\}$ are affine maps for the connection, so they preserve parallel vector fields. For $X \in \mathfrak{g}$, define the derivation $A_X = L_X - \nabla_X$. A G-invariant connection is determined by a linear map $(A_X)_o \in \mathbb{L}(T_oM; T_oM)$. Each A_X is in correspondence with an endomorphism $\Lambda_{\mathfrak{m}}(X)$ on $\mathbb{R}^n$, satisfying the condition that for all $X \in \mathfrak{m}$ and $h \in H$, $\Lambda_{\mathfrak{m}}(\mathrm{ad}(h)X) = \mathrm{Ad}(\lambda(h))\Lambda_{\mathfrak{m}}(X)$ where $\lambda(h)$ is the isotropy representation of h, see S. Kobayashi and K. Nomizu [39]. Then

$$\tilde{\nabla}_X Y := L_X Y - u_0 \Lambda_{\mathfrak{m}}(X) u_0^{-1} Y$$

defines a connection. We identified T_oM with $\mathbb{R}^n$ by a frame u_0. If $\Lambda_{\mathfrak{m}} = 0$ this defines the *canonical connection* ∇^c whose parallel translation along a curve is left translation.

Denote by $\frac{D^c}{dt}$ the corresponding covariant differentiation. Parallel translations along $\alpha(t)$ are given by $d\bar{L}_{\gamma(t)}$. This is due to the fact that left translations and π commute. If $X \in \mathfrak{m}$, let $\gamma_t = \gamma_0 \exp(tX)$ and $\alpha(t) = \gamma_0 \exp(tX)o$. Then γ is the horizontal lift of α and $\alpha(t)$ is a ∇^c geodesic,

$$\frac{D^c}{dt}\dot{\alpha} = \frac{D^c}{dt}\pi_* T L_{\gamma(t)}(X) = \frac{D^c}{dt} T\bar{L}_{\gamma(t)}\pi_*(X) = 0.$$

For $x \in \mathfrak{g}$, denote by X_m and $X_{\mathfrak{h}}$ the component of X in $\mathfrak{m}$ and in $\mathfrak{h}$ respectively.

Definition 10.1 A reductive Riemannian homogeneous space is *naturally reductive*, if

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle = -\langle Y, [X,Z]_{\mathfrak{m}}\rangle, \quad \text{for all } X,Y,Z \in \mathfrak{m}. \tag{10.2}$$

If G is of compact type, then G/H is reductive with reductive structure $m = \mathfrak{h}^\perp$ and is naturally reductive, with respect to the bi-invariant metric.

It is clear that (10.1) implies (10.2) and M is naturally reductive if ∇ and ∇^L have the same set of geodesics. In particular, if G admits an ad_G invariant inner product and $\mathfrak{m} = \mathfrak{h}^\perp$, then G/H is naturally reductive with respect to the induced Riemannian metric. A special reductive homogeneous space is a symmetric space with symmetry σ and the canonical decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. For it, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. If the symmetric decomposition is orthogonal, it is naturally reductive.

We collect in the lemma below useful information for the computations of the Markov generators whose proof is included for the convenience of the reader.

Lemma 10.2 *1. ∇^c is torsion skew symmetric precisely if M is naturally reductive.*

2. *The projections of the translates of one parameter family of subgroups of G are geodesics for the Levi-Civita connection if and only if M is naturally reductive.*

3. *Let $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ be defined by*

$$2\langle U(X,Y), Z\rangle = \langle X, [Z,Y]_{\mathfrak{m}}\rangle + \langle [Z,X]_{\mathfrak{m}}, Y\rangle. \tag{10.3}$$

Then $\mathrm{trace}_{\mathfrak{m}_t} \nabla^c df = \mathrm{trace}_{\mathfrak{m}_t} \nabla df$ *if and only if* $\mathrm{trace}_{\mathfrak{m}_t} U = 0$. *In other words, for any* $Z \in \mathfrak{m}$, $\mathrm{trace}_{\mathfrak{m}_t} \mathrm{ad}(Z) = 0$. *In particular* $\mathrm{trace}_{\mathfrak{m}_t} \nabla^c df = \mathrm{trace}_{\mathfrak{m}_t} \nabla df$*, if M is naturally reductive.*

4. $\nabla^c d\pi = 0$.

Proof (1) Let Y^* denote the action field generated by $Y \in \mathfrak{m}$, identified with $\pi_*(Y) \in T_oM$, and $Y^*(uo) = \frac{d}{dt}|_{t=0} \bar{L}_{\exp(tY)} uo = \frac{d}{dt}|_{t=0} \pi(\exp(tY)u)$. If $p = uo$, $\nabla^c_X Y^*(o) = [X^*, Y^*](o) = -[X,Y]_{\mathfrak{m}}$. The torsion tensor for ∇^c is left invariant, its value at o is

$$\mathcal{T}^c_o(X,Y) = (\nabla^c_X Y^*)_o - (\nabla^c_Y X^*)_o - [X^*, Y^*]_o = -[X,Y]_{\mathfrak{m}},$$

We see that $\mathcal{T}^c$ is skew symmetric if and only if (10.2) holds.

(2) A G-invariant connection on M has the same set of geodesics as ∇^c if and only if $\Lambda_{\mathfrak{m}}(X)(X) = 0$, c.f. Kobayashi and Nomizu [39]. It is well known that the function

$$\Lambda_{\mathfrak{m}}(X)(Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y).$$

defines the Levi-Civita connection. In fact it is clear that $\frac{1}{2}[X,Y] - U(X,Y)$ has vanishing torsion and $\langle \Lambda(X,Y), Z\rangle + \langle Y, \Lambda(X,Z)\rangle = 0$, which together with the fact ∇^c is metric implies that it is a Riemannian connection. The Levi-Civita connection ∇ and ∇^c have the same set of geodesics if and only if $U(X,X) = 0$. By polarisation, this is equivalent to M being naturally reductive.

(3) For any $X, Y \in \mathfrak{m}$,

$$(\nabla_X Y^*)_o = \nabla^c_X Y^* + \frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y).$$

If $f : M \to \mathbb{R}$ is a smooth function,

$$\begin{aligned}\langle \nabla_X \nabla f, Y\rangle &= L_Y\left(df(Y^*)\right) - \langle \nabla f, \nabla_X Y^*\rangle \\ &= \langle \nabla^c_X \nabla f, Y\rangle + \langle \nabla f, \nabla^c_X Y^* - \nabla_X Y^*\rangle.\end{aligned}$$

Summing over the basis of $\mathfrak{m}_l$ we see that

$$\mathrm{trace}_{\mathfrak{m}_l} \nabla df - \mathrm{trace}_{\mathfrak{m}_l} \nabla^c df = \langle \nabla f, \sum_i U(Y_i, Y_i)\rangle,$$

which vanishes for all smooth f if and only if $\sum_i U(Y_i, Y_i)$ vanishes. If G is of compact type, it has an Ad_G-invariant metric for which (10.1) holds. This completes the proof.

(4) We differentiate the map $d\pi : G \mapsto \mathbb{L}(TG; TM)$ with respect to the connection ∇^c. Let $\gamma(t)$ be a curve in G with $\gamma(0) = u$ and $\dot\gamma(0) = w$. Then for $X \in \mathfrak{g}$,

$$\left(\nabla^c_w d\pi\right)(X) = /\!/^c_t(\pi\circ\gamma)\frac{d}{dt}(/\!/^c_t(\pi\circ\gamma))^{-1} d\pi\left(/\!/^L_t(\gamma)X\right),$$

where $/\!/^c_t(\pi\circ\gamma)$ and $/\!/^L_t(\gamma)$ denote respectively parallel translations along $\pi\circ\gamma$ and γ with respect to to ∇^c and ∇^L. Since $/\!/^c_t(\pi\circ\gamma) = d\bar L_{\pi\circ\gamma(t)}$ and $/\!/^L_t(\gamma) = dL_{\gamma(t)}$ the covariant derivative vanishes. Indeed π and left translation commutes, $(d\pi)_u dL_u = (d\bar L_u)_o (d\pi)_1$,

$$\nabla^c_w d\pi = /\!/^c_t(\pi\circ\gamma)\frac{d}{dt}|_{t=0}(d\pi)_1\left(dL_{\gamma(-t)}/\!/^L_t(\gamma)\right) = 0.$$

□

In the propositions below we keep the notation in Proposition 6.10, Theorem 7.3 and Proposition 8.2. We identify $\mathfrak{m}_l$ with its projection to T_oM. Let $\bar x_t := \lim_{\epsilon\to 0} x^\epsilon_{\frac{t}{\epsilon}}$.

Proposition 10.3 *Suppose that* $\bar{\mathcal{L}} = \frac{|Y_0|^2}{\lambda(Y_0)\dim(\mathfrak{m}_l)}\Delta_{\mathfrak{m}_l}$. *If* $\mathrm{trace}_{\mathfrak{m}_l} U = 0$, *equivalently* $\mathrm{trace}_{\mathfrak{m}_l}\,\mathrm{ad}(Z) = 0$ *for all* $Z \in \mathfrak{m}$, *then* $(\bar x_t)$ *is a Markov process with generator*

$$\frac{|Y_0|^2}{\lambda(Y_0)\dim(\mathfrak{m}_l)}\mathrm{trace}_{\mathfrak{m}_l}\nabla d.$$

If M is furthermore isotropy irreducible, $(\bar x_t)$ is a scaled Brownian motion.

Proof Let $\{\tilde Y_i, 1 \le i \le m_l\}$ be an orthonormal basis of $\mathfrak{m}_l$ and let $\frac{1}{2}a^2 = \frac{|Y_0|^2}{\dim(\mathfrak{m}_l)\lambda_l}$. Let $Y_i = a\tilde Y_i$. Let (u_t) be a Markov process with generator $\frac{1}{2}\sum_{k=1}^{\dim(m_l)} L_{Y_k^*}L_{Y_k^*}$, and represented by a solution of the left invariant SDE: $du_t = \sum_{k=1}^{m_l} Y_k^*(u_t)\circ dB^k_t$ where $\{B^k_t\}$ are independent one dimensional Brownian motions. Let $y_t = \pi(u_t)$. Denote by ϕ^k_t the integral flow of Y_k, $\phi^i_t(g) = g\exp(tY_i)$. If $f \in C^2_K(M;\mathbb{R})$,

$$\begin{aligned}f\circ\pi(u_t) =& f\circ\pi(u_0) + \sum_{i=1}^{m_l}\int_0^t \frac{d}{dt} f\circ\pi\left(u_r\exp(tY_i)\right)|_{t=0} dB^i_r \\ &+ \frac{1}{2}\sum_{i=1}^{m_l}\int_0^t \frac{d^2}{dt^2} f\circ\pi\left(u_r\exp(tY_i)\right))|_{t=0} dr.\end{aligned}$$

We compute the last term beginning with the first order derivative,

$$\frac{d}{dt} f\circ\left(u_r\exp(tX_i)\right)|_{t=0} = df(\pi_*(L_{u_r\exp(tY_i)}Y_i))|_{t=0} = df((\bar L_{u_r})_* X_i).$$

Let $\frac{D}{dt}$ denote covariant differentiation along the curve (x_t) with respect to the Levi-Civita connection.

$$\sum_{i=1}^{m_l} \frac{d^2}{dt^2} f \circ \pi \left(u_r \exp(tY_i)\right) |_{t=0}$$
$$= \sum_{i=1}^{m_l} \frac{d}{dt} df \left((\bar{L}_{u_r \exp(tY_i)})_* (\pi_*(Y_i)) \right) |_{t=0}$$
$$= \sum_{i=1}^{m_l} \nabla df \left((\bar{L}_{u_r})_*(\pi_*(Y_i)), (\bar{L}_{u_r})_*(\pi_*(Y_i) \right) + \sum_{i=1}^{m_l} df \left(\frac{D}{dt} \left(\bar{L}_{u_r \exp(tY_i)})_*(\pi_*(Y_i) \right) |_{t=0} \right).$$

Since left translations are isometries, for each $u \in G$, $\{(\bar{L}_u)_*(\pi_*(Y_i))\}$ is an orthonormal basis of $\pi_*(u\mathfrak{m}_l) \subset T_{\pi(u)}M$. Thus

$$\sum_{i=1}^{m_l} \frac{d^2}{dt^2} f \circ \pi \left(u_r \exp(tY_i)\right) |_{t=0} = a^2 \text{trace}_{\mathfrak{m}_l} \nabla^c df + \sum_{i=1}^{\mathfrak{m}_l} df \left(\frac{D^c}{dt} (\bar{L}_{u_r \exp(tY_i)})_* Y_i^* |_{t=0} \right).$$

If $\text{trace}_{\mathfrak{m}_l} U = 0$, $\text{trace}_{m_l} \nabla^c df = \text{trace}_{m_l} \nabla df$ by Lemma 10.2. We have seen that the term involving $\frac{D^c}{dt}$ vanishes. This gives:

$$\sum_{i=1}^{m_l} \frac{d^2}{dt^2} f \circ \pi \left(u_r \exp(tY_i)\right) |_{t=0} = a^2 \text{trace}_{\mathfrak{m}_l} \nabla df.$$

Put everything together we see that

$$\mathbf{E} f \circ \pi(u_t) = f \circ \pi(u_0) + \frac{1}{2} a^2 \int_0^t \mathbf{E} (\text{trace})_{\mathfrak{m}_l} \nabla df(\pi(u_r)) dr.$$

For $x = \pi(y)$, define $Q_t f(x) = P_t(f \circ \pi)(y)$. Then $P_t(f \circ \pi) = (Q_t f) \circ \pi$. We apply Dynkin's criterion for functions of a Markov process to see that $(\bar{x}_t)$ is Markovian. The infinitesimal generator associated to Q_t is $\frac{1}{2} a^2 (\text{trace})_{\mathfrak{m}_l} \nabla df$. If M is an irreducible Riemannian symmetric space, $\mathfrak{m}$ is irreducible and $d\pi(\mathfrak{m}) = T_o M$ and the Markov process $(\bar{x}_t)$ is a scaled Brownian motion, concluding the Proposition. □

If M is naturally reductive, the proof is even simpler. In this case, $\sigma_i(t) = \pi\left(u \exp(tY_i)\right)$ is a geodesic with initial velocity Y_i and $\dot{\sigma}(t) = \frac{d}{dt} \pi_* \left(u \exp(tY_i)\right)$. Consequently, the following also vanishes:

$$\frac{D}{dt} (\bar{L}_{u_r \exp(tY_i)})_* (Y_i^*)|_{t=0} = 0.$$

11 Examples

Corollary 9.2 applies to the example in §2.1, where $G = SU(2)$, $H = U(1)$ and the Ad_H-invariant space $\mathfrak{m} = \langle X_2, X_3 \rangle$ is irreducible. For any $Y \in \mathfrak{m}$, $\text{ad}^2(X_1)(Y) = -4Y$. Note that 4 is the second non-zero eigenvalue of Δ_{S^1}, also the killing form of $SU(2)$ is $K(X,Y) = 4\,\text{trace}(XY)$ and $\mathcal{B}_{\text{ad}_{\mathfrak{h}}, \mathfrak{m}}(X_1, X_1) = K(X_1, X_1) = -8$.

Example 11.1 Let (b_t) be a one dimensional Brownian motion, $g_0 \in SU(2)$, $Y_0 \in \langle X_2, X_3 \rangle$ non-zero. Let $(g_t^\epsilon, h_t^\epsilon)$ be the solution to the following SDE on $SU(2) \times U(1)$,

$$dg_t^\epsilon = g_t^\epsilon Y_0 dt + \frac{1}{\sqrt{\epsilon}} g_t^\epsilon X_1 db_t, \tag{11.1}$$

with $g_0^\epsilon = g_0$. Let $\pi(z,w) = (\frac{1}{2}(|w|^2 - |z|^2), z\bar{w})$ and $x_t^\epsilon = \pi(g_t^\epsilon)$. Let $(\tilde{x}_t^\epsilon)$ be the horizontal lift of (x_t^ϵ). Then $(\tilde{x}^\epsilon_{\frac{t}{\epsilon}})$ converges weakly to the hypoelliptic diffusion with generator $\bar{\mathcal{L}} = \frac{|Y_0|^2}{4}\Delta^{hor}$. Furthermore, $x^\epsilon_{\frac{t}{\epsilon}}$ converges in law to the Brownian motion on $S^2(\frac{1}{2})$ scaled by $\frac{1}{2}|Y_0|^2$.

The first part of the theorem follows from Theorem 6.10 and Corollary 9.2. The scaling $\frac{1}{2}$ indicates the extra time needed for producing the extra direction $[X_1, Y_0]$. For the second part we note G is compact, so $\langle [X,Y],[Z]\rangle = -\langle Y,[X,Z]\rangle$ for any $X, Y, Z \in \mathfrak{m}$, and Proposition 10.3 applies. We observe that a similar equation can be set up on $G = SU(n)$ with H the torus group.

Example 11.2 Take $G = SO(4)$ and let $E_{i,j}$ be the elementary 4×4 matrices and $A_{i,j} = \frac{1}{\sqrt{2}}(E_{ij} - E_{ji})$. For $k = 1, 2, 3$ let $Y_k = A_{k4}$ and consider the equations,

$$dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}}A_{1,2}(g_t^\epsilon)\circ db_t^1 + \frac{1}{\sqrt{\epsilon}}A_{1,3}(g_t^\epsilon)\circ db_t^2 + Y_k(g_t^\epsilon)dt.$$

Let $H = SO(3)$. Then $\mathfrak{h} = \{A_{1,2}, A_{1,3}, A_{2,3}\}$, and its orthogonal complement $\mathfrak{m}$ is irreducible w.r.t. Ad_H, in fact $[A_{1,2}, A_{1,3}] = -\frac{1}{\sqrt{2}}A_{2,3}$. Furthermore $\{A_{1,2}, A_{1,3}\}$ is a set of generators and $\mathcal{L}_0 = \frac{1}{2}(A_{1,2})^2 + \frac{1}{2}(A_{1,3})^2$ satisfies strong Hörmander's conditions. It is easy to check that,

$$\frac{1}{2}\mathrm{ad}^2(A_{1,2})(Y_k) + \frac{1}{2}ad^2(A_{1,3})(Y_k) = -\frac{1}{4}(2\delta_{1,k} + \delta_{2,k} + \delta_{3,k})Y_k.$$

For any $Y \in \mathfrak{m}$, $\langle Y, \mathrm{Ad}(h)Y_k\rangle$ is an eigenfunction for $\mathcal{L}_0$ corresponding to the eigenvalue $-\lambda(Y_k)$, where

$$\lambda(Y_1) = \frac{1}{2},\ \lambda(Y_2) = \frac{1}{4},\ \lambda(Y_3) = \frac{1}{4}.$$

Define $\pi : g \in SO(4) \to S^4$ so $\pi(g)$ is the last column of g. Set $x_t^\epsilon = \pi(g_t^\epsilon)$ and denote by $\tilde{x}_t^\epsilon$ its horizontal lift. Then

$$a_{i,j}(Y_k) := \frac{1}{\lambda(Y_k)}\int_H \langle Y_i, \mathrm{Ad}(h)Y_k\rangle \mathcal{L}_0^{-1}(\langle Y_j, \mathrm{Ad}(h)(Y_k)\rangle)dh.$$

For $i \neq j$, $a_{i,j}(Y_k) = 0$ and $a_{i,i}(Y_k) := \frac{1}{3\lambda(Y_k)}$. In Theorem 6.10 take $Y_0 = Y_k$ to see

$$\bar{\mathcal{L}} = \frac{1}{\lambda(Y_k)}\int_{SO(3)} \nabla^L df(\mathrm{Ad}(h)(Y_k))^*, (\mathrm{Ad}(h)(Y_k))^*)dh = \frac{1}{3\lambda(Y_k)}\sum_{i=1}^3 \nabla^L df(Y_i, Y_i).$$

On the other hand, the effective diffusion for the equations

$$dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}}A_{1,2}(g_t^\epsilon)\circ db_t^1 + \frac{1}{\sqrt{\epsilon}}A_{1,3}(g_t^\epsilon)\circ db_t^2 + \frac{1}{\sqrt{\epsilon}}A_{2,3}(g_t^\epsilon)\circ db_t^3 + Y_k(g_t^\epsilon)dt$$

is the same for all Y_k. It is easy to see that $\bar{\mathcal{L}} = \frac{2}{3}\sum_{i=1}^3 \nabla^L df(Y_i, Y_i)$.

Further symmetries can, of course, be explored.

Example 11.3 Let $Y_0 = \sum_{k=1}^3 c_k Y_k$ be a mixed vector. Then,

$$\overline{\alpha_i\beta_j} = \sum_{k,l=1}^3 c_k c_l \overline{\alpha_i(Y_k)\beta_j(Y_l)} = \sum_{k,l=1}^3 c_k c_l \frac{1}{\lambda_l}\overline{\alpha_i(Y_k)\alpha_j(Y_l)}.$$

If $c_1 = 0$ and $c_2 = c_3$, $\overline{\alpha_i\beta_j} = 4(c_2)^2 \sum_{k,l=2}^3 \overline{\alpha_i(Y_k)\alpha_j(Y_l)}$. By symmetry, $\overline{\alpha_i(Y_k)\alpha_j(Y_l)}$ vanishes for $i \neq j$.

Example 11.4 Let $n \geq 2$, $G = SO(n+1)$, $H = \left\{ \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}, R \in SO(n) \right\}$, and $S^n = SO(n+1)/SO(n)$. Then H fixes the point $o = (0, \dots, 0, 1)^T$. The homogeneous space S^n has the reductive decomposition:

$$\mathfrak{h} = \left\{ \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}, S \in \mathfrak{so}(n) \right\}, \quad \mathfrak{m} = \left\{ Y_C = \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix}, C \in \mathbb{R}^n \right\}.$$

Let $\sigma(A) = S_0 A S_0^{-1}$ and $S_0 = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the symmetric space decomposition for σ. We identify Y_C with the vector C, and compute:

$$\mathrm{Ad}\left(\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} Y_C \right) = \begin{pmatrix} 0 & RC \\ -(RC)^T & 0 \end{pmatrix}.$$

This action is transitive on the unit tangent sphere and is irreducible. There is a matrix $R \in SO(n)$ that sends C to $-C$. Then $\bar{Y}_0 = 0$ for every $Y_0 \in \mathfrak{m}$ and the conditions of Proposition 8.2 are satisfied.

Let $A_{i,j} = \frac{1}{\sqrt{2}}(E_{ij} - E_{ji})$, where $i < j$, and E_{ij} is the elementary matrix with the nonzero entry at the (i,j)-th position. Let $Y_0 \in \mathfrak{m}$ be a non-trivial vector. Consider the equation,

$$dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{1 \leq i < j \leq n} A_{i,j}^*(g_t^\epsilon) \circ db_t^{i,j} + Y_0^*(g_t^\epsilon) dt.$$

We define $\lambda = \frac{(n-1)}{4}$. By a symmetry argument it is easy to see that,

$$\sum_{1 \leq i < j \leq n} \mathrm{ad}^2(A_{ij}) = -\frac{1}{2}(n-1)I.$$

For $1 \leq i, j, k \leq n$, $[A_{i,j}, A_{k,n+1}] = -\frac{1}{\sqrt{2}}\delta_{k,i} A_{j,n+1} + \frac{1}{\sqrt{2}}\delta_{j,k} A_{i,n+1}$, which follows from $E_{ij}E_{kl} = \delta_{jk}E_{il}$. Hence for $i \neq j$,

$$2\mathrm{ad}^2(A_{i,j})(A_{k,n+1}) = -(\delta_{i,k} + \delta_{j,k}) A_{k,n+1}. \tag{11.2}$$

Let $x_{\frac{t}{\epsilon}}^\epsilon = g_{\frac{t}{\epsilon}}^\epsilon o$ and $(\tilde{x}_{\frac{t}{\epsilon}}^\epsilon)$ the horizontal lift of $(x_{\frac{t}{\epsilon}}^\epsilon)$ through g_0. By Theorem 6.10, converges to a Markov process whose limiting Markov generator is, by symmetry,

$$\bar{\mathcal{L}} = \frac{|Y_0|^2}{\lambda \dim(\mathfrak{m})} \Delta^{hor} = \frac{4|Y_0|^2}{n(n-1)} \Delta^{hor}.$$

The upper bound for the rate of convergence from Theorem 6.5 holds. By Theorem 6.10 and 10.3 the stochastic processes $(x_{\frac{t}{\epsilon}}^\epsilon)$ converge to the Brownian motion on S^n

with scale $\frac{8|Y_0|^2}{n(n-1)}$. By (11.2), for any $1 \le i,j,k \le n$, $A_{k,n+1}$ are eigenvectors for $\mathrm{ad}^2(A_{i,j})$. Furthermore, for $1 \le i < j \le n$,

$$[A_{i,n+1}, A_{j,n+1}] = -\frac{1}{\sqrt{2}}A_{i,j},$$

and Δ^{hor} satisfies the one step Hörmander condition.

Example 11.5 We keep the notation in the example above. Let Y_0 be a unit vector from $\mathfrak{m}$ and let (g_t^ϵ) be the solutions of the following equation:

$$dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{1\le i<j\le n} A_{i,j}^*(g_t^\epsilon) \circ db_t^{i,j} + \frac{n(n-1)}{8} Y_0^*(g_t^\epsilon)dt. \tag{11.3}$$

Then as $\epsilon \to 0$, $\pi(g_{\frac{t}{\epsilon}}^\epsilon)$ converges to a Brownian motion on S^n.

We finally provide an example in which the group G is not compact.

Example 11.6 Let $n \ge 3$. Let $F(x,y) = -x_0y_0 + \sum_{k=1}^n x_ky_k$ be a bilinear form on $\mathbb{R}^{n+1}$. Let $O(1,n)$ be the set of $(n+1)\times(n+1)$ matrices preserving the indefinite form

$$O(1,n) = \left\{ A \in GL(n+1) : A^TSA = S, S = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \right\}.$$

Let G denote the identity component of $O(1,n)$, consisting of $A \in O(1,n)$ with $\det(A) = 1$ and $a_{00} \ge 1$. This is a $n(n+1)/2$ dimensional manifold, $X \in \mathfrak{g}$ if and only $X^TS + SX = 0$. So

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & \xi^T \\ \xi & S \end{pmatrix}, \xi \in \mathbb{R}^n, S \in \mathfrak{so}(n) \right\}.$$

The map $\sigma(A) = SAS^{-1}$ defines an evolution on G and

$$M = \{x = (x^0, x^1, \dots, x^n) : F(x,x) = -1, x^0 \ge 1\}$$

is a symmetric space. Its isometry group at $e_0 = (1,0,\dots,0)^T$ is $H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \right\}$ where $B \in SO(n)$. The -1 eigenspace of the Lie algebra involution is:

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & \xi^T \\ \xi & 0 \end{pmatrix} \right\}.$$

Let us take $Y_0 \in \mathfrak{m}$ and consider the equation,

$$dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{1\le i<j\le n} A_{i,j}^*(g_t^\epsilon) \circ db_t^{i,j} + Y_0^*(g_t^\epsilon)dt.$$

It is clear that Theorem 6.10 and 7.3 apply. The isotropy representation of H is irreducible. The action of Ad_H on $\mathfrak{m}$ is essentially the action of $SO(n)$ on $\mathbb{R}^n$:

$$\mathrm{Ad}(\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}) \begin{pmatrix} 0 & \xi^T \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & (B\xi)^T \\ B\xi & 0 \end{pmatrix}.$$

The conclusion of Theorems hold. The symmetric space M is naturally reductive and so Proposition 10.3 applies. Note that

$$\left[\begin{pmatrix}0 & \xi^T\\ \xi & 0\end{pmatrix}, \begin{pmatrix}0 & \eta^T\\ \eta & 0\end{pmatrix}\right] = \begin{pmatrix}0 & 0\\ 0 & \xi\eta^T - \eta\xi^T\end{pmatrix}.$$

Since $e_i e_j^T = E_{ij}$, $[\mathfrak{m}, \mathfrak{m}]$ generates a basis of $\mathfrak{h}$. The effective process of the stochastic differential equation (1.1) is a scaled horizontal Brownian motion, which satisfies the one step Hörmander condition. The limit of $\pi(g^\epsilon_{\frac{t}{\epsilon}})$ is a scaled hyperbolic Brownian motion. The rate of convergence stated in Theorem 6.5 is valid here, the scale is $\frac{8}{n(n-1)}$, as for the spheres.

Appendix A Further Discussions and Open Questions

The study of scaling limits should generalise to principal bundles and to foliated manifolds. Let $\pi : P \to M$ be a principal bundle with group action H and give $\mathfrak{h}$ a suitable inner product. For $A \in \mathfrak{h}$ denote by A^* the corresponding fundamental vertical vector field: $A^*(u) = \frac{d}{dt}|_{t=0} u \exp(tA)$. Let us fix an orthonormal basis $\{A_1, \dots, A_p\}$ of $\mathfrak{h}$. Let $\{\sigma_j, j = p+1, \dots, n\}$ be smooth horizontal sections of TP, f_0 a vertical vector field, $\{w_t^1, \dots, w_t^{N_1}, b_t^1, \dots, b_t^{N_2}\}$ independent one dimensional Brownian motions. Let $\{a_j^k, c_i^l\}$ be a family of smooth functions on P. For example, a computation analogous to that in Lemma 4.1, should show lead to a system of SDEs of Markovian type, whose solutions are slow and fast motions, and the following proposition.

Proposition A.1 *Let $u_t \equiv \phi_t(u_0)$ be a solution to*

$$du_t = (\sigma_0 + f_0)(u_t)dt + \sum_{k=1}^{N_1} \sum_{j=p+1}^{n} (c_k^j \sigma_j)(u_t) \circ dw_t^k + \sum_{l=1}^{N_2} \sum_{j=1}^{p} \left(a_l^j A_j^*\right)(u_t) \circ db_t^l. \tag{A.1}$$

Let $x_t = \pi(u_t)$ and $(\tilde{x}_t)$ its horizontal lift. Then $u_t = \tilde{x}_t a_t$ where

$$d\tilde{x}_t = (R_{a_t^{-1}})_* \sigma_0(\tilde{x}_t)dt + \sum_{k=1}^{N_2} \sum_{j=p+1}^{n} c_k^j(\tilde{x}_t a_t)(R_{a_t^{-1}})_*(\sigma_j)(\tilde{x}_t) \circ dw_t^k$$

$$da_t = TR_{a_t}\left(\varpi_{\tilde{x}_t}(f_0)\right)dt + \sum_{l=1}^{N_1} \sum_{j=1}^{p} a_l^j(\tilde{x}_t a_t) A_j^*(a_t) \circ db_t^l.$$

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APPENDIX: REAL PETER-WEYL

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Abstract. We explain how unitary representations of compact groups split into three types. We summarise their properties. As an application we state and prove Peter-Weyl Theorem for real representations of compact groups.

Let G be a compact topological group. We refer the reader to textbooks [1, 2] for standard properties of G and its representations. The group G has a unique biinvariant Haar measure μ such that $\mu(G) = 1$. We utilise the corresponding integral $\int_G \Phi(x)\mu(dx)$.

We study unitary representations of G. It is a representation of G on a real (or complex) Hilbert space V (or $\mathcal{V}$ – we use calligraphic letters for complex Hilbert spaces) such that the action map $G \times V \to V$ is continuous and each $g \in G$ acts by an orthogonal (unitary) operator. A key example is the real regular representation $L^2(G)$ that is the space of all $\mathbb{R}$-valued L^2-functions with the action $[x \cdot \Phi](y) = \Phi(yx)$. By $\mathcal{L}^2(G)$ we denote the complex regular representation, the space of all $\mathbb{C}$-valued L^2-functions with the same action.

Each unitary representation is a direct sum of irreducible unitary representations. Each irreducible unitary representation is finite dimensional. Let $\mathrm{Irr}_{\mathbb{R}}G$ be the set of isomorphism classes of irreducible unitary representations (correspondingly $\mathrm{Irr}_{\mathbb{C}}G$ for complex ones). The character of $V \in \mathrm{Irr}_{\mathbb{R}}G$ is the function $\chi_V \in L^2(G)$ given by $\chi_V(x) = \mathrm{Tr}_V(x)$. The Frobenius-Schur indicator of $\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}}G$ (or $V \in \mathrm{Irr}_{\mathbb{R}}G$) is

$$\mathrm{FS}(\mathcal{V}) = \int_G \chi_{\mathcal{V}}(x^2)\mu(dx) \quad (\text{or } \mathrm{FS}(V) = \int_G \chi_V(x^2)\mu(dx)\ ).$$

A complex unitary representation $(\mathcal{V}, \langle|\rangle)$ can be considered as a real vector space $\mathcal{V}_{\mathbb{R}}$. It is a real unitary representation with a form $\mathrm{Re}\langle|\rangle$. In the opposite direction, $V \otimes_{\mathbb{R}} \mathbb{C}$ is a complexification of $(V, \langle|\rangle)$ with a natural unitary form $\langle x \otimes \alpha | y \otimes \beta\rangle = \overline{\alpha}\beta\langle x|y\rangle$. Notice that $(V \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}} = V \otimes 1 \oplus V \otimes \mathbf{i} \cong 2V$ as real representations of G. This defines a correspondence

$$\mathrm{Irr}_{\mathbb{R}}G \leftrightarrow \mathrm{Irr}_{\mathbb{C}}G :$$

the representations V and $\mathcal{V}$ are in correspondence if $\mathcal{V}$ is a direct summand of $V \otimes_{\mathbb{R}} \mathbb{C}$. By Frobenius reciprocity this is equivalent to V being a direct summand of $\mathcal{V}_{\mathbb{R}}$. The properties of this correspondence depend on the endomorphism algebra $\mathbb{F} = \mathrm{End}_G(V)$. Recall that $\mathbb{F}$ consists of all $\mathbb{R}$-linear operators on V that commute with all elements of G. The algebra $\mathbb{F}$ is a division algebra (Schur lemma), hence, it must be either real numbers $\mathbb{R}$, or complex numbers $\mathbb{C}$, or quaternions $\mathbb{H}$ (Frobenius Theorem). Depending on what $\mathbb{F}$ is, we call each of the representations in correspondence $V \leftrightarrow \mathcal{V}$ *a representation of real, complex or quaternionic type* correspondingly.

Useful properties of representations of each type are summarised in Table 1. The last two rows show that the type can be determined by computing the Frobenius-Schur indicator: it is particularly useful for a finite group G.

The third and forth rows from the bottom are useful for a simple compact Lie group G. An irreducible representation $\mathcal{V}$ of G is parametrised by its highest weight ϖ: we write $\mathcal{V} = \mathcal{L}(\varpi)$, if $\mathcal{V}$ is an irreducible representation with a highest weight vector of weight ϖ. It is known that $\mathcal{L}(\varpi)^* = \mathcal{L}(-w \cdot \varpi)$ where w is the longest element of the Weyl group. The forth row immediately tells you whether $V \leftrightarrow \mathcal{V}$ is of complex type or not: it is easy to verify whether $\varpi = -w \cdot \varpi$ or not in each particular case. For instance, the vector representation of $\mathrm{SU}_n(\mathbb{C})$ is $\mathcal{L}(\varpi_1)$ where ϖ_1 is the first fundamental weight. Then $\mathcal{L}(\varpi_1)^* = \mathcal{L}(-w \cdot \varpi_1) = \mathcal{L}(\varpi_{n-1})$ so that $\mathcal{L}(\varpi_1)$ is of complex type if $n > 2$.

Look at $\mathrm{SU}_2(\mathbb{C})$. Its irreducible representations are $\mathcal{L}(m\varpi_1)$ for natural m and $\mathcal{L}(m\varpi_1)^* = \mathcal{L}(-w \cdot m\varpi_1) = \mathcal{L}(m\varpi)$. No representation of $\mathrm{SU}_2(\mathbb{C})$ is of complex type. The vector representation $\mathcal{L}(\varpi_1)$ admits a symplectic form preserved by $\mathrm{SU}_2(\mathbb{C})$. The representation $\mathcal{L}(m\varpi_1)$ is isomorphic to $S^m(\mathcal{L}(\varpi_1))$, its m-th symmetric power that gets an induced $\mathrm{SU}_2(\mathbb{C})$-invariant form, symmetric if m is even, skew-symmetric if m is odd. Thus, $\mathcal{L}(m\varpi_1)$ is of real type if m is even and is of quaternionic type if m is odd.

The first row helps to determine the type of the adjoint representation $\mathfrak{g}$ of a compact simple Lie group G. Its complexification $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a simple complex Lie algebra. So it is of real type. Alternatively one can use its Killing form or the fact that $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{L}(\alpha_0)$ where α_0 is the highest root.

The field $\mathbb{F}$ is a "natural" field for V. We will write elements of $\mathbb{F}$ acting on vectors from the right (it is essential since $\mathbb{H}$ is non-commutative).

Lemma 1. *(Real Artin-Wedderburn Formula)* $L^2(G) \cong \bigoplus_{V \in \mathrm{Irr}_{\mathbb{R}} G} \dim_{\mathbb{F}}(V) V$.

Proof. Notice that $L^2(G) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathcal{L}^2(G) \cong \bigoplus_{\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}} G} \dim_{\mathbb{C}}(\mathcal{V}) \mathcal{V}$. Thus, we need to prove that $\dim_{\mathbb{F}}(V) V \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\bigoplus_{V \leftrightarrow \mathcal{V}} \dim_{\mathbb{C}}(\mathcal{V}) \mathcal{V}$. This is easily checked case by case for each column in Table 1. □

We fix standard imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{H}$ and $\mathbf{i} \in \mathbb{C}$. The complex conjugation on $\mathbb{F}$ is given by the formula $\overline{\alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}} = \alpha - \beta\mathbf{i} - \gamma\mathbf{j} - \delta\mathbf{k}$. Note that it is identity if $\mathbb{F} = \mathbb{R}$. Given a $\mathbb{F}$-valued function Φ, we want to fix its one, two or four (depending on $\mathbb{F}$) components, the $\mathbb{R}$-valued functions $\Phi^\bullet$ such that

$$\Phi = \Phi^1 1 + \Phi^{\mathbf{i}}\mathbf{i} + \Phi^{\mathbf{j}}\mathbf{j} + \Phi^{\mathbf{k}}\mathbf{k}.$$

Now we are ready to exhibit an explicit orthonormal basis of $L^2(G)$, e.g. state Peter-Weyl Theorem. Let $(V, \langle|\rangle) \in \mathrm{Irr}_{\mathbb{R}} G$, $d = \dim_{\mathbb{F}}(V)$. Consider V as a right vector space over $\mathbb{F}$. Choose a vector space isomorphism with the space of columns

$$V \to \mathbb{F}^d, a \mapsto \underline{a}.$$

This gives a G-invariant positive definite unitary $\mathbb{F}$-form on V:

$$[a|b] = \int_G (\underline{x \cdot a})^* \underline{x \cdot b}\, \mu(dx)$$

where $(\underline{x \cdot a})^*$ is the conjugate-transpose row of the column $\underline{x \cdot a}$. Observe that $[a\mathbf{q}|b\mathbf{r}] = \overline{\mathbf{q}}[a|b]\mathbf{r}$ for all $\mathbf{q}, \mathbf{r} \in \mathbb{F}$.

Lemma 2. *(Real Schur Lemma) Let $\{|\}$ be a G-invariant unitary $\mathbb{F}$-form on V. Then there exists $\mathbf{r} \in Z(\mathbb{F})$ (the centre of $\mathbb{F}$) such that*

$$\{a|b\} = \mathbf{r}[a|b] \quad \textit{for all} \quad a, b \in V.$$

Let $\{|\}^\sharp$ be a G-invariant bilinear $\mathbb{R}$-form on V. Then there exists $\mathbf{q} \in \mathbb{F}$ such that

$$\{a|b\}^\sharp = \langle a|b\mathbf{q}\rangle \quad \textit{for all} \quad a, b \in V.$$

Finally let $\{|\}^\flat$ be a G-invariant symmetric bilinear $\mathbb{R}$-form on V. Then there exists $\mathbf{r} \in \mathbb{R}$ such that

$$\{a|b\}^\flat = \mathbf{r}\langle a|b\rangle \quad \textit{for all} \quad a, b \in V.$$

Proof. The linear operator $\theta : V \to V$ defined by the equation

$$\{a|b\} = [a|\theta(b)] \text{ for all } a, b \in V$$

is a G-module endomorphism since both form are G-invariant. Thus, $\theta(a) = a\mathbf{r}$ for some $\mathbf{r} \in \mathbb{F}$ and $\{a|b\} = [a|b]\mathbf{r}$. The unitarity of $\{|\}$ ensures that $\mathbf{r}$ is in the centre.

A proof for $\{|\}^\flat$ is similar.

Using the second statement we can find $\mathbf{q} \in \mathbb{F}$ such that $\{a|b\}^\flat = \langle a|b\mathbf{q}\rangle$ for all a, b. Since $\{a|b\}^\flat$ is symmetric, $\mathbf{q}$ is self-adjoint:

$$\langle a|b\mathbf{q}\rangle = \{a|b\}^\flat = \{b|a\}^\flat = \langle b|a\mathbf{q}\rangle = \langle a\mathbf{q}|b\rangle.$$

It remains to notice that the adjoint of $\mathbf{q}$ is $\overline{\mathbf{q}}$. This proves that $\mathbf{q} = \overline{\mathbf{q}}$ and $\mathbf{q} \in \mathbb{R}$. □

Notice that the component $[|]^1$ is a G-invariant positive definite symmetric bilinear $\mathbb{R}$-form. Lemma 2 implies that $[|]^1 = \langle|\rangle\mathbf{r}$ for some positive $\mathbf{r} \in \mathbb{R}_{>0}$. Replacing $[|]$ with $[|]\mathbf{r}^{-1}$, we get useful properties connecting the two forms.

Lemma 3. *If the relevant elements of $\mathbb{F}$ exist, then the following identities hold for all $a, b \in V$:*

$$\langle a|b\rangle = [a|b]^1 = \langle a\mathbf{i}|b\mathbf{i}\rangle = \langle a\mathbf{j}|b\mathbf{j}\rangle = \langle a\mathbf{k}|b\mathbf{k}\rangle,$$

$$\langle a\mathbf{i}|b\rangle = [a|b]^\mathbf{i} = -\langle a|b\mathbf{i}\rangle = -\langle a\mathbf{j}|b\mathbf{k}\rangle = \langle a\mathbf{k}|b\mathbf{j}\rangle,$$

$$-\langle a|b\mathbf{j}\rangle = [a|b]^\mathbf{j} = \langle a\mathbf{i}|b\mathbf{k}\rangle = -\langle a\mathbf{k}|b\mathbf{i}\rangle = \langle a\mathbf{j}|b\rangle,$$

$$-\langle a|b\mathbf{k}\rangle = [a|b]^\mathbf{k} = -\langle a\mathbf{i}|b\mathbf{j}\rangle = \langle a\mathbf{j}|b\mathbf{i}\rangle = \langle a\mathbf{k}|b\rangle.$$

Proof. The first identity in the first row is a result of our assumptions. Now $[a\mathbf{i}|b] = [a\mathbf{i}|b]^1 1 + [a\mathbf{i}|b]^\mathbf{i}\mathbf{i} + [a\mathbf{i}|b]^\mathbf{j}\mathbf{j} + [a\mathbf{i}|b]^\mathbf{k}\mathbf{k}$, while at the same time $[a\mathbf{i}|b] = -\mathbf{i}[a|b] = [a|b]^\mathbf{i} 1 - [a|b]^1\mathbf{i} + [a|b]^\mathbf{k}\mathbf{j} - [a|b]^\mathbf{j}\mathbf{k}$. This proves the first identity in the second row. The remaining proofs are similar using properties such as $[a\mathbf{j}|b\mathbf{k}] = -\mathbf{j}[a|b]\mathbf{k}$. □

Let $X_1, X_2 \ldots X_m$ be an orthonormal (with respect to $[|]$) $\mathbb{F}$-basis of V. The scaled coordinate functions in this basis are $\mathbb{F}$-valued functions on G

$${}^V\Phi_{n,l} = {}^V\Phi^1_{n,l}1 + {}^V\Phi^\mathbf{i}_{n,l}\mathbf{i} + {}^V\Phi^\mathbf{j}_{n,l}\mathbf{j} + {}^V\Phi^\mathbf{k}_{n,l}\mathbf{k}, \quad {}^V\Phi_{n,l}(g) = \sqrt{\dim_\mathbb{R}(V)}[X_n|g \cdot X_l].$$

Theorem 1. *(Peter-Weyl Theorem) The components of the scaled coordinate functions $\{{}^V\Phi^\bullet_{n,l} \mid V \in \mathrm{Irr}_\mathbb{R} G\}$ form an orthonormal basis of $L^2(G)$.*

Proof. We deduce this theorem from the usual (complex) Peter-Weyl Theorem. This states by equipping each $\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}} G$ with an orthonormal basis $Y_1, Y_2 \cdots$ we obtain an orthonormal basis of $\mathcal{L}^2(G)$ consisting of scaled coordinate functions

$$\{ {}^{\mathcal{V}}\Psi_{n,l} \mid \mathcal{V} \in \mathrm{Irr}_{\mathbb{C}} G\} \text{ where } {}^{\mathcal{V}}\Psi_{n,l}(g) = \sqrt{\dim_{\mathbb{C}}(\mathcal{V})}\langle Y_n | g \cdot Y_k \rangle.$$

If we fix $\mathcal{V}$, the functions ${}^{\mathcal{V}}\Psi_{n,l}$ form an orthonormal basis of $\dim_{\mathbb{C}}(\mathcal{V})\mathcal{V}$, a direct summand of $\mathcal{L}^2(G)$. It follows that ${}^{V}\Phi^{\bullet}_{n,l}$ and ${}^{W}\Phi^{\bullet}_{n,l}$ for non-isomorphic V and W are orthogonal to each other. It remains to observe that ${}^{V}\Phi^{\bullet}_{n,l}$ form an orthonormal basis of $\dim_{\mathbb{F}}(V)V$. Let $d = \dim_{\mathbb{R}}(V)$.

If $\mathbb{F} = \mathbb{R}$, we can choose $Y_n = X_n \otimes 1$ as an orthonormal basis of $\mathcal{V} = V \otimes_{\mathbb{R}} \mathbb{C}$. Since $d = \dim_{\mathbb{C}}(\mathcal{V})$, ${}^{V}\Phi^{1}_{n,l} = {}^{\mathcal{V}}\Psi_{n,l}$ form an orthonormal basis.

If $\mathbb{F} = \mathbb{C}$, we have two complex representations $\mathcal{V}_+$ and $\mathcal{V}_- \cong \mathcal{V}_+^*$ that we can explicitly pinpoint, writing the two idempotents of $\mathrm{End}_G(V \otimes_{\mathbb{R}} \mathbb{C}) = \mathbb{F} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$:

$$V \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{V}_- \oplus \mathcal{V}_+ \quad \text{where} \quad \mathcal{V}_\pm = (V \otimes_{\mathbb{R}} \mathbb{C})\mathbf{e}_\pm, \quad \mathbf{e}_\pm = \frac{1}{2}(1 \otimes 1 \pm \mathbf{i} \otimes \mathbf{i}).$$

Now we choose $Y_n^\pm = \sqrt{2} X_n \mathbf{e}_\pm$ as an orthonormal basis of $\mathcal{V}_\pm$. It is clear that $\langle Y_n^+ | Y_l^- \rangle = 0$, while inside $\mathcal{V}_\pm$

$$\begin{aligned}\langle Y_n^\pm | Y_k^\pm \rangle = \frac{1}{2}\langle X_n \otimes 1 \pm X_n \mathbf{i} \otimes \mathbf{i} | X_l \otimes 1 \pm X_l \mathbf{i} \otimes \mathbf{i} \rangle = & \frac{1}{2}\langle X_n | X_l \rangle \mp \frac{\mathbf{i}}{2}\langle X_n \mathbf{i} | X_l \rangle \pm \\ \pm \frac{\mathbf{i}}{2}\langle X_n | X_l \mathbf{i} \rangle + \frac{1}{2}\langle X_n \mathbf{i} | X_l \mathbf{i} \rangle = \frac{1}{2}[X_n|X_l]^1 \mp \frac{\mathbf{i}}{2}[X_n|X_l]^{\mathbf{i}} \mp & \frac{\mathbf{i}}{2}[X_n|X_l]^{\mathbf{i}} + \frac{1}{2}[X_n|X_l]^1 = \\ = [X_n|X_l]^1 \mp \mathbf{i}[X_n|X_l]^{\mathbf{i}} = [X_n|X_l] \text{ (or } \overline{[X_n|X_l]}) & = \delta_{n,l}.\end{aligned}$$

In this case $d = 2\dim_{\mathbb{C}} \mathcal{V}_\pm$. We express the functions now:

$$\begin{aligned}{}^{\mathcal{V}_\pm}\Psi_{n,l}(g) = \sqrt{\dim_{\mathbb{C}}(\mathcal{V}_\pm)}\langle Y_n^\pm | g \cdot Y_k^\pm \rangle = & \frac{\sqrt{d}}{2\sqrt{2}}\langle X_n \otimes 1 \pm X_n \mathbf{i} \otimes \mathbf{i} | g \cdot X_l \otimes 1 \pm g \cdot X_l \mathbf{i} \otimes \mathbf{i} \rangle = \\ = \frac{\sqrt{d}}{2\sqrt{2}}\langle X_n | g \cdot X_l \rangle \mp \frac{\mathbf{i}\sqrt{d}}{2\sqrt{2}}\langle X_n \mathbf{i} | g \cdot X_l \rangle \pm & \frac{\mathbf{i}\sqrt{d}}{2\sqrt{2}}\langle X_n | g \cdot X_l \mathbf{i} \rangle + \frac{\sqrt{d}}{2\sqrt{2}}\langle X_n \mathbf{i} | g \cdot X_l \mathbf{i} \rangle = \\ = \frac{\sqrt{d}}{2\sqrt{2}}[X_n | g \cdot X_l]^1 \mp \frac{\mathbf{i}\sqrt{d}}{2\sqrt{2}}[X_n | g \cdot X_l]^{\mathbf{i}} \mp & \frac{\mathbf{i}\sqrt{d}}{2\sqrt{2}}[X_n | g \cdot X_l]^{\mathbf{i}} + \frac{\sqrt{d}}{2\sqrt{2}}[X_n | g \cdot X_l]^1 \\ = \frac{1}{\sqrt{2}}\, {}^{V}\Phi^{1}_{n,l}(g) \mp \frac{\mathbf{i}}{\sqrt{2}}\, {}^{V}\Phi^{\mathbf{i}}_{n,l}(g). &\end{aligned}$$

Orthonormality of ${}^{V}\Phi^{\bullet}_{n,l}$ follows from explicit formulas:

$${}^{V}\Phi^{1}_{n,l} = \frac{1}{\sqrt{2}}\, {}^{\mathcal{V}_+}\Psi_{n,l} + \frac{1}{\sqrt{2}}\, {}^{\mathcal{V}_-}\Psi_{n,l}, \quad {}^{V}\Phi^{\mathbf{i}}_{n,l} = \frac{\mathbf{i}}{\sqrt{2}}\, {}^{\mathcal{V}_+}\Psi_{n,l} - \frac{\mathbf{i}}{\sqrt{2}}\, {}^{\mathcal{V}_-}\Psi_{n,l}.$$

If $\mathbb{F} = \mathbb{H}$, choose a rank 1 idempotent in $\mathrm{End}_G(V \otimes_{\mathbb{R}} \mathbb{C}) = \mathbb{F} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$:

$$V \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{V} \oplus \mathcal{V}^\perp \quad \text{where} \quad \mathcal{V}^\perp \cong \mathcal{V}, \quad \mathcal{V} = (V \otimes_{\mathbb{R}} \mathbb{C})\mathbf{e}, \quad \mathbf{e} = \frac{1}{2}(1 \otimes 1 + \mathbf{i} \otimes \mathbf{i}).$$

Now we choose $Y_n^+ = \sqrt{2} X_n \mathbf{e}$, $Y_n^- = \sqrt{2} X_n \mathbf{j}\, \mathbf{e}$, as an orthonormal basis of $\mathcal{V}$. The calculation in the case of $\mathbb{F} = \mathbb{C}$ makes it clear that $\langle Y_n^+ | Y_n^+ \rangle = 1$ and $\langle Y_n^\pm | Y_k^\bullet \rangle = 0$

if $n \neq k$. Let us do the remaining calculations:

$$\langle Y_n^-|Y_n^-\rangle = \frac{1}{2}\langle X_n\mathbf{j}\otimes 1 - X_n\mathbf{k}\otimes\mathbf{i}|X_n\mathbf{j}\otimes 1 - X_n\mathbf{k}\otimes\mathbf{i}\rangle = \frac{1}{2}\langle X_n\mathbf{j}|X_n\mathbf{j}\rangle - \frac{\mathbf{i}}{2}\langle X_n\mathbf{j}|X_n\mathbf{k}\rangle +$$
$$+\frac{\mathbf{i}}{2}\langle X_n\mathbf{k}|X_n\mathbf{j}\rangle + \frac{1}{2}\langle X_n\mathbf{k}|X_n\mathbf{k}\rangle = [X_n|X_n]^1 + \mathbf{i}[X_n|X_n]^{\mathbf{i}} = 1,$$
$$\langle Y_n^+|Y_n^-\rangle = \frac{1}{2}\langle X_n\otimes 1 + X_n\mathbf{i}\otimes\mathbf{i}|X_n\mathbf{j}\otimes 1 - X_n\mathbf{k}\otimes\mathbf{i}\rangle = \frac{1}{2}\langle X_n|X_n\mathbf{j}\rangle - \frac{\mathbf{i}}{2}\langle X_n|X_n\mathbf{k}\rangle -$$
$$-\frac{\mathbf{i}}{2}\langle X_n\mathbf{i}|X_n\mathbf{j}\rangle - \frac{1}{2}\langle X_n\mathbf{i}|X_n\mathbf{k}\rangle = -[X_n|X_n]^{\mathbf{j}} + \mathbf{i}[X_n|X_n]^{\mathbf{k}} = 0.$$

Let us optimise the notation by $\Psi_{n,l}^{\pm ?} = {}^{\mathcal{V}}\Psi_{n^\pm,l^?}$. The calculation

$$\Psi_{n,l}^{++} = \frac{1}{\sqrt{2}}\,{}^V\Phi_{n,l}^1(g) - \frac{\mathbf{i}}{\sqrt{2}}\,{}^V\Phi_{n,l}^{\mathbf{i}}(g)$$

is as in the complex case. Again $d = 2\dim_{\mathbb{C}}\mathcal{V}$. Let us calculate the remaining functions.

$$\Psi_{n,l}^{--}(g) = \frac{\sqrt{d}}{2\sqrt{2}}\langle X_n\mathbf{j}\otimes 1 - X_n\mathbf{k}\otimes\mathbf{i}|g\cdot X_l\mathbf{j}\otimes 1 - g\cdot X_l\mathbf{k}\otimes\mathbf{i}\rangle =$$
$$= \frac{\sqrt{d}}{2\sqrt{2}}\langle X_n\mathbf{j}|g\cdot X_l\mathbf{j}\rangle - \frac{\mathbf{i}\sqrt{d}}{2\sqrt{2}}\langle X_n\mathbf{j}|g\cdot X_l\mathbf{k}\rangle + \frac{\mathbf{i}\sqrt{d}}{2\sqrt{2}}\langle X_n\mathbf{k}|g\cdot X_l\mathbf{j}\rangle + \frac{\sqrt{d}}{2\sqrt{2}}\langle X_n\mathbf{k}|g\cdot X_l\mathbf{k}\rangle =$$
$$= \frac{1}{\sqrt{2}}\,{}^V\Phi_{n,l}^1(g) + \frac{\mathbf{i}}{\sqrt{2}}\,{}^V\Phi_{n,l}^{\mathbf{i}}(g),$$

$$\Psi_{n,l}^{+-}(g) = \frac{\sqrt{d}}{2\sqrt{2}}\langle X_n\otimes 1 + X_n\mathbf{i}\otimes\mathbf{i}|g\cdot X_l\mathbf{j}\otimes 1 - g\cdot X_l\mathbf{k}\otimes\mathbf{i}\rangle =$$
$$= \frac{\sqrt{d}}{2\sqrt{2}}\langle X_n|g\cdot X_l\mathbf{j}\rangle - \frac{\mathbf{i}\sqrt{d}}{2\sqrt{2}}\langle X_n|g\cdot X_l\mathbf{k}\rangle - \frac{\mathbf{i}\sqrt{d}}{2\sqrt{2}}\langle X_n\mathbf{i}|g\cdot X_l\mathbf{j}\rangle - \frac{\sqrt{d}}{2\sqrt{2}}\langle X_n\mathbf{i}|g\cdot X_l\mathbf{k}\rangle =$$
$$= \frac{-1}{\sqrt{2}}\,{}^V\Phi_{n,l}^{\mathbf{j}}(g) + \frac{\mathbf{i}}{\sqrt{2}}\,{}^V\Phi_{n,l}^{\mathbf{k}}(g),$$

$${}^{-+}\Psi_{n,l}(g) = \frac{\sqrt{d}}{2\sqrt{2}}\langle X_n\mathbf{j}\otimes 1 - X_n\mathbf{k}\otimes\mathbf{i}|g\cdot X_l\otimes 1 + g\cdot X_l\mathbf{i}\otimes\mathbf{i}\rangle =$$
$$= \frac{\sqrt{d}}{2\sqrt{2}}\langle X_n\mathbf{j}|g\cdot X_l\rangle + \frac{\mathbf{i}\sqrt{d}}{2\sqrt{2}}\langle X_n\mathbf{j}|g\cdot X_l\mathbf{i}\rangle + \frac{\mathbf{i}\sqrt{d}}{2\sqrt{2}}\langle X_n\mathbf{k}|g\cdot X_l\rangle - \frac{\sqrt{d}}{2\sqrt{2}}\langle X_n\mathbf{k}|g\cdot X_l\mathbf{i}\rangle =$$
$$= \frac{1}{\sqrt{2}}\,{}^V\Phi_{n,l}^{\mathbf{j}}(g) + \frac{\mathbf{i}}{\sqrt{2}}\,{}^V\Phi_{n,l}^{\mathbf{k}}(g).$$

Orthonormality of ${}^V\Phi_{n,l}^{\bullet}$ follows from explicit formulas:

$${}^V\Phi_{n,l}^1 = \frac{1}{\sqrt{2}}\Psi_{n,l}^{++} + \frac{1}{\sqrt{2}}\Psi_{n,l}^{--}, \quad {}^V\Phi_{n,l}^{\mathbf{i}} = \frac{\mathbf{i}}{\sqrt{2}}\Psi_{n,l}^{++} - \frac{\mathbf{i}}{\sqrt{2}}\Psi_{n,l}^{--},$$
$${}^V\Phi_{n,l}^{\mathbf{j}} = \frac{1}{\sqrt{2}}\Psi_{n,l}^{-+} - \frac{1}{\sqrt{2}}\Psi_{n,l}^{+-}, \quad {}^V\Phi_{n,l}^{\mathbf{k}} = -\frac{\mathbf{i}}{\sqrt{2}}\Psi_{n,l}^{-+} - \frac{\mathbf{i}}{\sqrt{2}}\Psi_{n,l}^{+-}.$$

□

Table 1. Properties of $V \leftrightarrow \mathcal{V}$ depending on $\mathbb{F}$

$\mathbb{F} = \mathrm{End}_G(V)$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$V_{\mathbb{C}} \cong \mathcal{V}$	Yes	No	No
$V_{\mathbb{C}}$	$\mathcal{V}$	$\mathcal{V} \oplus \mathcal{V}^*$	$2\mathcal{V}$
$\mathrm{Irr}_{\mathbb{R}}G$ to $\mathrm{Irr}_{\mathbb{C}}G$ correspondence is	1:1	1:2	1:1
$\mathcal{V}_{\mathbb{R}}$	$2V$	V	V
$\chi_{\mathcal{V}}$ takes values in	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$
relation between characters	$\chi_V = \chi_{\mathcal{V}}$	$\chi_V = 2 \cdot \mathrm{Re}(\chi_{\mathcal{V}})$	$\chi_V = 2 \cdot \chi_{\mathcal{V}}$
$\mathcal{V} \cong \mathcal{V}^*$	Yes	No	Yes
G-invariant form on $\mathcal{V}$	symmetric	only zero	skew-symmetric
$\mathrm{FS}(\mathcal{V})$ is equal to	1	0	-1
$\mathrm{FS}(V)$ is equal to	1	0	-2

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