Ramsey goodness of bounded degree trees versus general graphs

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Outline

1. Preliminaries
   - The Ramsey goodness problem
   - Known results
   - Main result

2. Base case: $k = 2$
   - $m \gg \Delta$
   - $m \ll \Delta$

3. Induction step: $k \geq 3$
   - T has many leaves
   - T has many bare paths and G is well-connected
   - G is not well-connected

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Ramsey Number

Definition (Ramsey number)

Given two graphs H_1 and H_2, the *Ramsey number* $R(H_1, H_2)$ is defined as the smallest integer N so that for any graph G with N vertices, either G contains either a copy of H_1 or G^c contains a copy of H_2.

- In general, it is difficult to give good bounds on the Ramsey number $R(H_1, H_2)$, let alone finding its exact value.

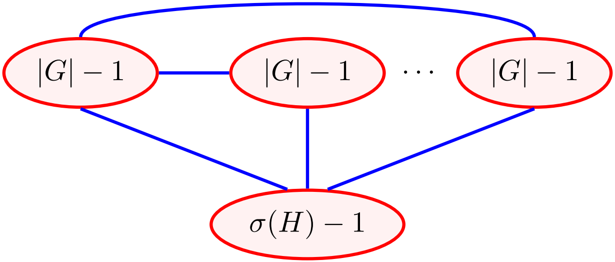
Burr's general lower bound

Definition

For a graph H with chromatic number $\chi(H)$, define $\sigma(H)$ to be the smallest possible size of a colour class in any $\chi(H)$-colouring of H.

Theorem (Burr, 1981)

Suppose G is connected and $|G| \geq \sigma(H)$, then $R(G,H) \geq (|G|-1)(\chi(H)-1) + \sigma(H)$.



Ramsey goodness

Theorem (Burr, 1981)

Given two graphs G and H, if G is connected and $|G| \geq \sigma(H)$, then $R(G, H) \geq (|G| - 1)(\chi(H) - 1) + \sigma(H)$.

Definition (Ramsey goodness)

Given graphs G and H, G is said to be H-good if $R(G, H) = (|G| - 1)(\chi(H) - 1) + \sigma(H)$.

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Known results: paths

Definition (Ramsey goodness)

Given graphs G and H, G is said to be H-good if
$R(G, H) = (|G| - 1)(\chi(H) - 1) + \sigma(H)$.

P_n is H-good when ...

- $H = K_m$. [Erdős, 1947]
- $H = P_m$ and $n \geq m$. [Gerenscér, Gyárfás, 1967]

- $n \geq 4|H|$. [Pokrovskiy, Sudakov, 2017]

Known results: trees

A tree T is H-good when ...

- $H = K_m$. [Chvátal, 1977]
- $\Delta(T) \leq \Delta$ and $|T|$ sufficiently large compared to $|H|$. [Erdős, Faudree, Rousseau, Schelp, 1985]
- Not when $T = K_{1,n}$ and $H = K_{2,2}$ or $K_{1,3}$. [Burr, Erdős, Faudree, Rousseau, Schelp, 1988]
- $\chi(H) = k$, $\Delta(T) \leq \Delta$ and $|T| \geq C_{\Delta,k}|H|\log^4|H|$. [Balla, Pokrovskiy, Sudakov, 2018]

Balla, Pokrovskiy and Sudakov also conjectured that this $\log$ factor can be removed.

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Main Result

We confirm the conjecture of Balla, Pokrovskiy and Sudakov.

Theorem (Montgomery, Pavez-Signé, Y., 2023+)

For any fixed Δ, k*, there exists a constant* $C = C_{\Delta,k}$ *such that for any graph* H *and any tree* T *satisfying* $\chi(H) = k, \Delta(T) \leq \Delta$ *and* $|T| \geq C|H|$*,* T *is* H*-good.*
In other words, $R(T, H) = (|T| - 1)(k - 1) + \sigma(H)$*.*

Reduction to complete multipartite graphs

Theorem (Montgomery, Pavez-Signé, Y., 2023+)

For any fixed Δ, k, there exists a constant $C = C_{\Delta,k}$ such that for any graph H and any tree T satisfying $\chi(H) = k, \Delta(T) \leq \Delta$ and $|T| \geq C|H|$, T is H-good.
In other words, $R(T, H) = (|T| - 1)(k - 1) + \sigma(H)$.

Note that it suffices to prove this for all H of the form $K_{m_1,\cdots,m_k}$.
Because if $\sigma(H) = m_1 \leq \cdots \leq m_k$ are the colour class sizes of a k-colouring of H, then G^c containing $K_{m_1,\cdots,m_k}$ will imply G^c contains H.

Reduction to complete multipartite graphs

Theorem (Montgomery, Pavez-Signé, Y., 2023+)

For any fixed Δ, k, there exists a constant $C = C_{\Delta,k}$ such that for any graph H and any tree T satisfying $\chi(H) = k, \Delta(T) \leq \Delta$ and $|T| \geq C|H|$, T is H-good.
In other words, $R(T, H) = (|T| - 1)(k - 1) + \sigma(H)$.

Therefore, it suffices to prove the following, with μ corresponding to $1/kC$.

Theorem (Montgomery, Pavez-Signé, Y., 2023+)

For any fixed Δ, k, there exists a constant $\mu = \mu_{\Delta,k}$ such that for any $m \leq \mu n$ and any tree T on n vertices satisfying $\Delta(T) \leq \Delta$, T is $K_{m,\mu n,\cdots,\mu n}$-good.
In other words, $R(T, K_{m,\mu n,\cdots,\mu n}) = (n - 1)(k - 1) + m$.

Proof Outline

Setting: T is a tree on n vertices with $\Delta(T) \leq \Delta$. G is a graph on $(k-1)(n-1)+m$ vertices, and G^c contains no copy of $K_{m,\mu n,\cdots,\mu n}$.

Goal: Find a copy of T in G.

Outline: Induction on k.

- Base case $k = 2$:
  - $m \gg \Delta$ is large. Build a vortex structure. $\leftarrow$ Focus of the talk.
  - $m \ll \Delta$ is small.
- Inductive step $k \geq 3$:
  - T has many leaves.
  - T has many bare paths and G is well-connected.
  - G is not well-connected.

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Expansion condition: (m, m')-joined

Definition

A graph G is (m, m')-joined if for any disjoint subsets $U, U' \subset V(G)$ with $|U| = m, |U'| = m'$, there exists an edge between U and U' in G.

Observation

$$\begin{aligned} G^c \text{ contains no } K_{m,m'} &\iff G \text{ is } (m, m')\text{-joined} \\ &\iff |N(U)| \geq |G| - m - m' \\ &\qquad \text{for every } U \subset V(G) \text{ of size } m \end{aligned}$$

Key embedding lemma

- Setting: G has $n + m - 1$ vertices and is $(m, \mu n)$-joined. T is a tree with n vertices and $\Delta(T) \leq \Delta$. We need to find a copy of T in G.
- Main Tool: a vertex-by-vertex embedding technique of bounded degree trees into expander graphs.
  Expansion condition + Spare vertices = Tree embedding

Lemma (Balla, Pokrovskiy, Sudakov, 2018)

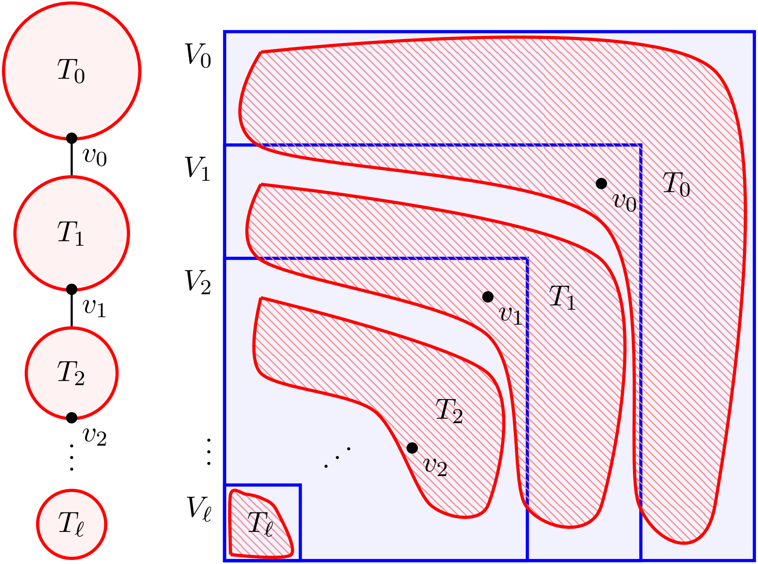
If $|G| \geq |T| + 13\Delta m + m'$, G is (m, m')-joined and $\Delta(T) \leq \Delta$, then we can find a copy of T in G.

- Main difficulty: manage the limited amount of spare vertices. Currently, $m - 1$ spare vertices, but $13\Delta m + \mu n$ needed.

Vortex

Main difficulty: manage the limited amount of spare vertices.
Currently, $m-1$ spare vertices, but $13\Delta m+\mu n$ needed.

Idea: Use a vortex $V(G)=V_0\supset V_1\supset\cdots\supset V_\ell$ to gradually reduce the number of spare vertices needed.


T_0
v_0
T_1
v_1
T_2
v_2
T_ℓ
V_0
V_1
V_2
V_ℓ
T_0
v_0
T_1
v_1
T_2
v_2
T_ℓ

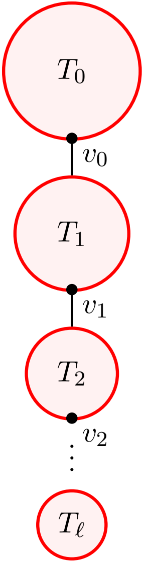
Vortex conditions

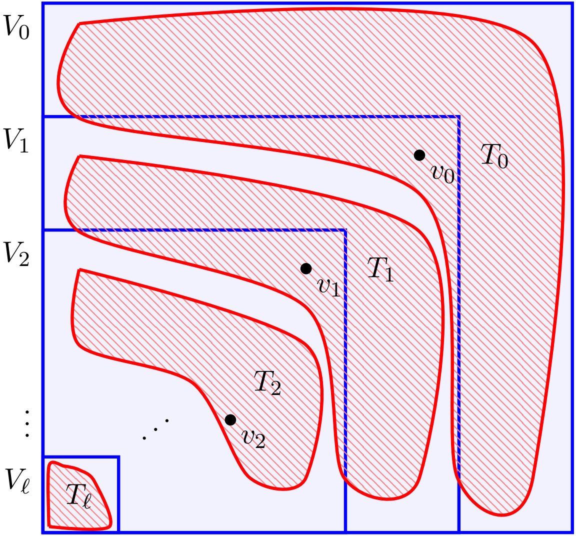
Main difficulty: manage the limited amount of spare vertices.
Currently, $m-1$ spare vertices, but $13\Delta m + \mu n$ needed.

Pick a nested sequence of subsets $V(G) = V_0 \supset V_1 \supset \cdots \supset V_\ell$ of appropriate sizes uniformly at random. Using probablistic methods, we can guarantee the following conditions.

- For some $\lambda > 0$ and every $i \leq \ell - 1$, $G[V_i]$ is $(m, \lambda|V_i|)$-joined.
  $13\Delta m + \lambda|V_i|$ spare vertices needed, decreasing with i.
- For some $D \gg \Delta$, $G[V_\ell]$ is $(\frac{m}{D}, \frac{m}{D})$-joined,
  only $13\Delta\frac{m}{D} + \frac{m}{D} < m-1$ spare vertices needed.

Embed T into the vortex

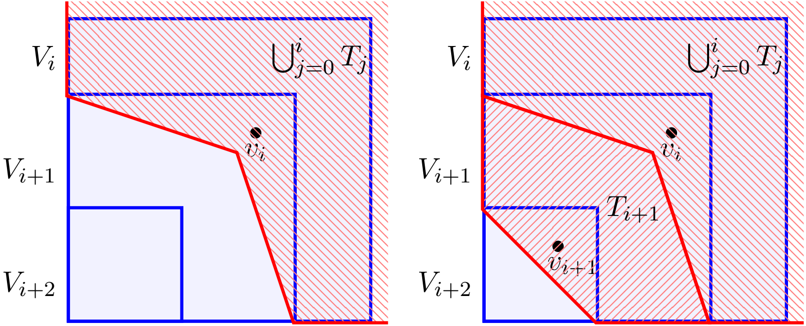




Embed T into the vortex

Key conditions to maintain throughout the embedding process:

- T_i covers all that remains in $V_i \setminus V_{i+1}$ (difficult!),
- The rest of T_i, including v_i, is in $V_{i+1} \setminus V_{i+2}$,



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A switching property

Observation

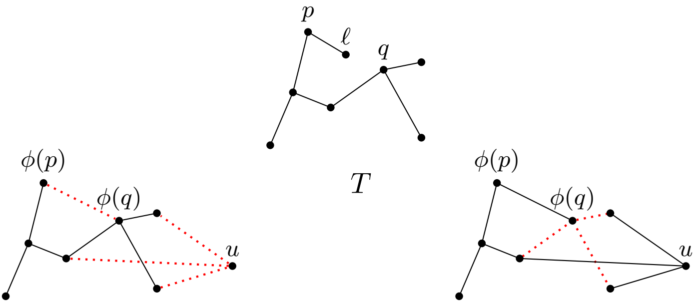
Since $m \ll \Delta$ is quite small, and the graph G is $(m, \mu n)$-joined, G is quite dense with at least $\Theta(n^2/m)$ edges.

If we embed a small portion T_0 of the tree T randomly to $\phi(T_0)$ in G, this enables us to obtain a switching property satisfied by $\phi(T_0)$.

A switching property

Suppose we are trying to embed a vertex ℓ whose parent in T is p.

- either $\phi(p)$ has a neighbour in G that is unused,
- or there exists $q \in T_0$ and an unused vertex $u \in G$, such u can take the place of $\phi(q)$, freeing up $\phi(q)$ to be the image of ℓ.



G before switch

G after switch

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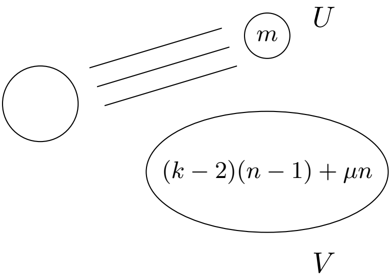
Using induction hypothesis

Setting: T is a tree on n vertices with $\Delta(T) \leq \Delta$. G is a graph on $(k-1)(n-1)+m$ vertices, and G^c contains no copy of $K_{m,\mu n,\cdots,\mu n}$. Need to find a copy of T in G.

Lemma

Either G contains a copy of T, or G is $(m,(k-2)(n-1)+\mu n)$-joined.



- $G[V]$ cannot contain T as G doesn't
- $G[V]^c$ cannot contain $K_{\mu n,\cdots,\mu n}$ otherwise G^c contains $K_{m,\mu n,\cdots,\mu n}$
- this contradicts induction applied to $G[V]$

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Dichotomy between leaves and bare paths

Definition

A path P in a tree T is a bare path if all vertices in P has degree exactly 2.

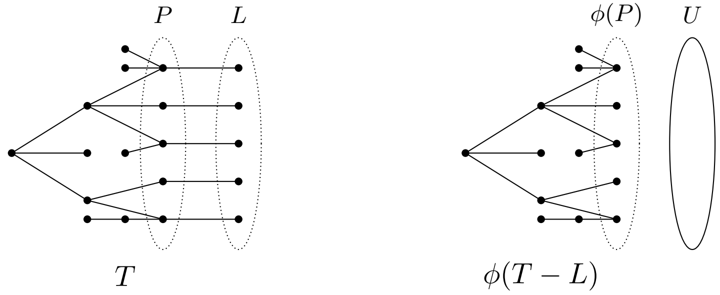
Lemma (Krivelevich, 2010)

Let T be a tree on n vertices, then

- *either T contains at least ℓ leaves,*
- *or T contains at least $\frac{n}{s+1} - 2\ell$ bare paths of length s.*

Embedding T with many leaves

- Remove a set L of leaves, such that each $\ell \in L$ has a distinct parent in T and $|L| = \Theta(n)$.
- Now $|G| \geq |T - L| + 13\Delta m + (k-2)(n-1) + \mu n$, so we can find an embedding ϕ of $T - L$.
- To add the leaves in, use expansion properties to show Hall's matching conditions hold between $\phi(P)$ and the set U of unused vertices.



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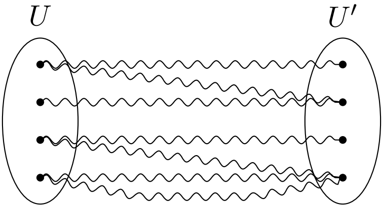
A connecting property

Definition

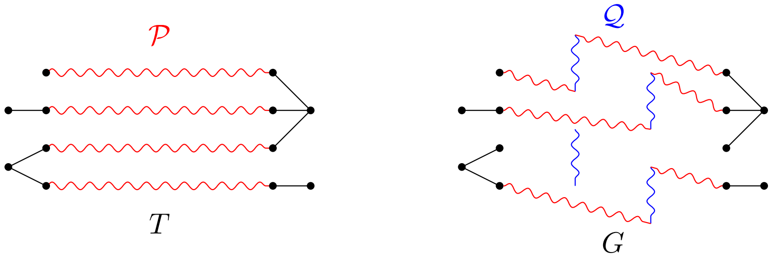
G is well-connected if for any partition $V(G) = V_0 \cup V_1 \cup V_2$ satisfying $|V_0| \leq \lambda n$ and $|V_1|, |V_2| \geq m$,there exists an edge between V_1 and V_2.

We use this to get the following connecting property.

There exists δ, ℓ such that for any disjoint $U, U' \subset V(G)$ of size m, there are δn disjoint paths of the same length ℓ connecting them.



- Let $\mathcal{P}$ be a large collection of bare paths in T.
- Use Ramsey goodness of path to find a LONG path in G, and divide it into a collection $\mathcal{Q}$ of shorter paths.
- Use the connecting property to embed most paths in $\mathcal{P}$ via $\mathcal{Q}$.
- Use the expansion property to embed the rest of T.



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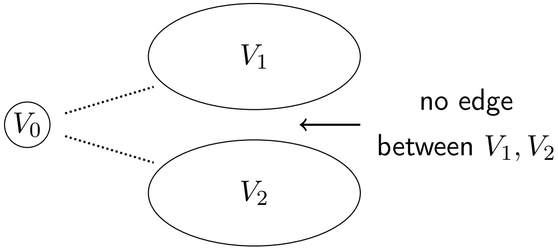
Not well-connected

Definition

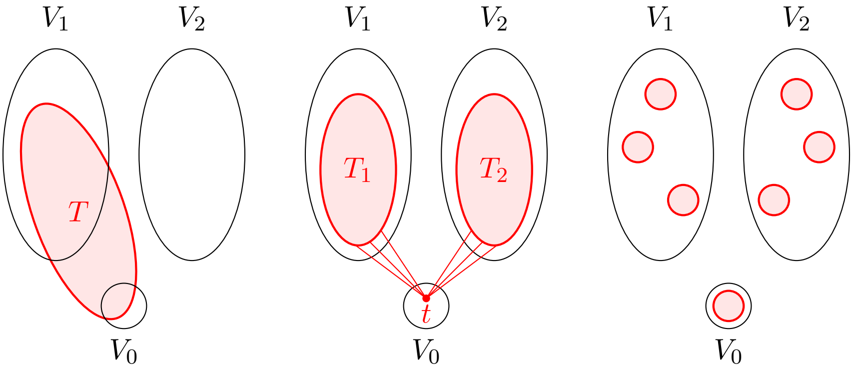
A graph G on n vertices is not well-connected if there exists a partition $V(G) = V_0 \cup V_1 \cup V_2$, such that

- $|V_0| \leq \lambda n$.
- $|V_1|, |V_2| \geq m$.
- There is no edge between V_1 and V_2.



Embed T into a not well-connected G

If G is not well-connected, then one of the following is true.



T in $V_0 \cup V_1$ or $V_0 \cup V_2$

Parts of T in V_1 and V_2 connected via $t \in V_0$

$K_{m,\mu n,\cdots,\mu n}$ in G^c