Integrable Probability 7

- Macdonald polynomials
- Macdonald processes

Macdonald polynomials

Definition via inner products

For partitions λ, μ define partial ordering

$$\lambda \geq \mu \iff |\lambda| = |\mu| \ \& \ \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$$

Monomial symmetric functions

$$m_\lambda(x) = \sum_{\sigma:\text{perm}} x^{\sigma(\lambda)} = \sum_{\sigma} \prod x_i^{\lambda_{\sigma(i)}}$$

Power symmetric functions

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

$(x_1, x_2, \ldots)$

where $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots$ & $p_r(x) = \sum_i x_i^r$

Schur functions can also be defined via orthogonalisation:

Define an inner product $\langle \cdot , \cdot \rangle$ via

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} \, z_\lambda$$

$\delta_{\lambda,\mu}$: Kronecker delta & $z_\lambda = \prod_{r \geq 1} r^{m_r} \cdot m_r!$

Then Schur are characterised by

change of base matrix triangular.

1) $$s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu$$

2) $$\langle s_\lambda, s_\mu \rangle = 0 \quad \text{if} \quad \lambda \neq \mu$$

Definition (Macdonald symmetric fcts)

Inner product $\langle \cdot , \cdot \rangle_{q,t}$ s.t. $\langle P_\lambda , P_\mu \rangle_{q,t} = \underbrace{z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}}_{z_\lambda(q,t)} \delta_{\lambda,\mu}$

Macdonald symm. fcts are the unique symmetric functions s.t.

change of base matrix
triangular

A. $P_\lambda = m_\lambda + \sum_{\mu < \lambda} \boxed{u_{\lambda\mu}}\, m_\mu$

B. $\langle P_\lambda , P_\mu \rangle_{q,t} = 0$ if $\lambda \neq \mu$

Prop (Cauchy identity)

Let $\underline{H(x;y)} := \prod_{i,j} \frac{(t x_i y_j ; q)_\infty}{(x_i y_j ; q)_\infty}$

with $(a;q)_\infty = \prod_{r=0}^{\infty} (1 - a q^r)$ q-Pochhammer

If (u_λ), (v_λ) are bases on ring of symmetric fct's

then (I) $\Longleftrightarrow$ (II) with

(I) $\langle u_\lambda , v_\lambda \rangle_{q,t} = \delta_{\lambda,\mu}$ (biorthogonality or Plancherel Thm)

(II) $\sum_\lambda u_\lambda(x) v_\lambda(y) = H(x;y)$ (Cauchy Identity)

Proof $P_\lambda^* := Z_\lambda(q,t)^{-1} \underbrace{P_\lambda}$ then $\underbrace{\langle P_\lambda^*, P_\mu \rangle_{q,t}} = \delta_{\lambda,\mu}$

Expand u_λ, v_λ in (P_λ)

$$\underline{u_\lambda} = \sum_\rho \overbrace{a_{\lambda\rho}}^{A} P_\rho^* \quad \& \quad v_\lambda = \sum_\sigma \overset{B}{b_{\mu\sigma}} P_\sigma$$

then
$$\langle u_\lambda, v_\lambda \rangle_{q,t} = \Big\langle \sum_\rho a_{\lambda\rho} P_\rho^*, \sum_\sigma b_{\mu\sigma} P_\sigma \Big\rangle_{q,t}$$

$$= \sum_{\rho,\sigma} a_{\lambda\rho} b_{\mu\sigma} \langle P_\rho^*, P_\sigma \rangle_{q,t}$$

$$= \sum_\rho a_{\lambda\rho} b_{\mu\rho} = (AB^T)_{\lambda,\mu}$$

On the other hand:

first one can show (check or look at Macdonald)

$$H(x;y) = \sum_\lambda \frac{1}{Z_\lambda(q,t)} P_\lambda(x) P_\lambda(y)$$

$$=: \sum_\lambda P_\lambda^*(x) P_\lambda(y)$$

$\boxed{\text{So}}$ $\sum_\lambda u_\lambda(x) v_\lambda(y) = H(x;y) \iff$

$$\iff \sum_\lambda u_\lambda(x) v_\lambda(y) = \sum_\lambda P_\lambda^*(x) P_\lambda(y) \iff$$

$$\iff \sum_\lambda \sum_{\rho,\sigma} a_{\lambda\rho} b_{\lambda\sigma} P_\rho^*(x) P_\sigma(y) = \sum_\lambda P_\lambda^*(x) P_\lambda(y)$$

$$\iff \sum_\lambda a_{\lambda\rho} b_{\lambda\sigma} = \delta_{\rho,\sigma}$$

So if $A := (a_{\lambda\rho})$ & $B = (b_{\lambda\rho})$

$$(\mathrm{I}) \Longleftrightarrow \sum_{\rho} a_{\lambda\rho} b_{\mu\rho} = \delta_{\lambda,\mu} \Longleftrightarrow$$

$$\Longleftrightarrow AB^T = Id$$

$$(\mathrm{II}) \Longleftrightarrow \sum_{\lambda} a_{\lambda\rho} b_{\lambda\sigma} = \delta_{\rho,\sigma} \Longleftrightarrow$$

$$\Longleftrightarrow A^T B = Id.$$

$\boxtimes$

$$A^T B = Id \Longleftrightarrow AB^T = Id.$$

How to show existence

Construct a (difference) operator D s.t.

- $D m_\lambda = \sum_{\mu \leq \lambda} c_{\lambda\mu} m_\mu$ triangular
- $c_{\lambda\lambda} \neq c_{\mu\mu}$ for $\lambda \neq \mu$
- $\langle Df, g \rangle_{q,t} = \langle f, Dg \rangle_{q,t}$ self-adjoint
- $D P_\lambda = c_{\lambda\lambda} P_\lambda$ Macdonald fcts eigenfunctions

then

$$c_{\lambda\lambda} \langle P_\lambda, P_\mu \rangle_{q,t} = \langle D P_\lambda, P_\mu \rangle_{q,t} \underset{\text{adj.}}{\overset{\text{self-}}{=}} \langle P_\lambda, D P_\mu \rangle_{q,t}$$

$$= c_{\mu\mu} \langle P_\lambda, P_\mu \rangle_{q,t}$$

$$\xRightarrow{c_{\lambda\lambda} \neq c_{\mu\mu}} \quad \langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad \text{if} \quad \lambda \neq \mu.$$

MACDONALD OPERATOR

$$D = \frac{1}{\Delta} \sum_{i=1}^{n} \left(T_{t,x_i} \Delta \right) T_{q,x_i}$$

with $\quad \Delta = \prod_{1 \leq i < j \leq n} (x_i - x_j) \quad$ &

$$\left(T_{q,x_i} f \right)(x_1, \ldots, x_n) = f(x_1, \ldots, q x_i, \ldots, x_n)$$

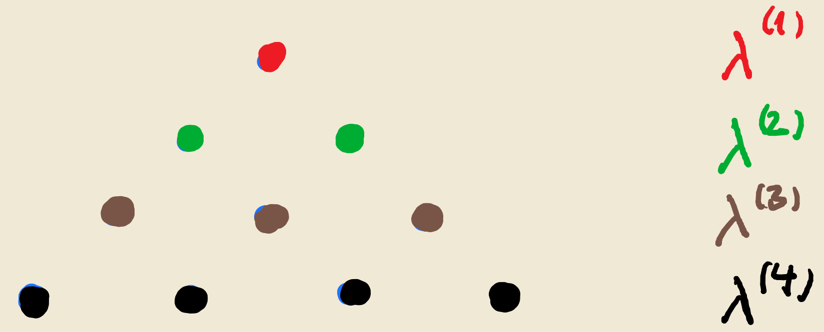
$$P_\lambda \;,\; Q_\lambda \quad \rightsquigarrow \quad (u_\lambda) \quad (v_\lambda)$$

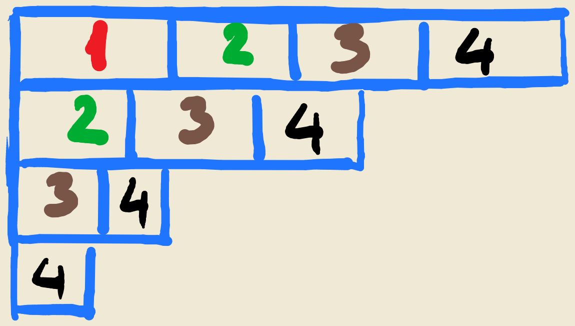
$$Q_\lambda = \frac{P_\lambda}{\underbrace{\|P_\lambda\|_2^2}}$$

$$\sum P_\lambda(x) Q_\lambda(y) = H(x;y)$$

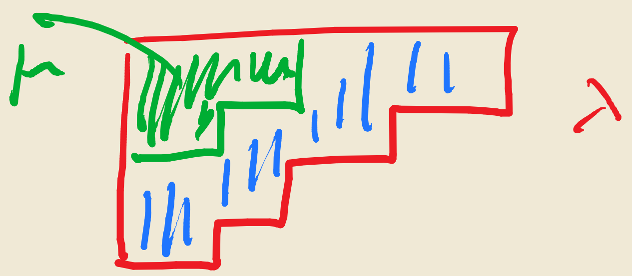
$$\langle P_\lambda, Q_\mu \rangle_{q,t} = \delta_{\lambda,\mu}.$$

Macdonald Processes





λ/μ skew partition



Branching rule

$$P_\lambda(x) = \sum_{\lambda^{(1)}<\lambda^{(2)}<\cdots<\lambda^{(k)}=\lambda} P_{\lambda^1}(x_1)\, P_{\lambda^2/\lambda^1}(x_2)\cdots P_{\lambda^k/\lambda^{k-1}}(x_k)$$

with $P_{\lambda/\mu}(x_1) = \varphi_{\lambda/\mu}\, x_1^{|\lambda|-|\mu|}$ for λ/μ horizontal strip

General form for $\varphi_{\lambda/\mu}$ can be found in the notes

if $t=0$ then

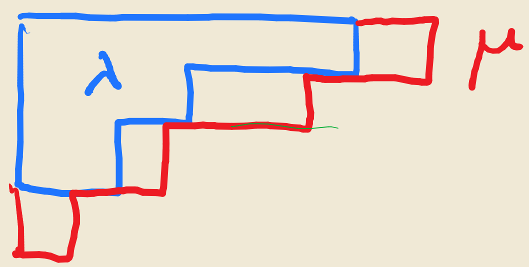
$$\varphi_{\lambda/\mu} = (q;q)_\infty^{-\ell(\mu)} \prod_{i=1}^{\ell(\mu)} \frac{(q^{\lambda_i-\mu_i+1};q)_\infty\,(q^{\mu_i-\lambda_{i+1}+1};q)_\infty}{(q^{\lambda_i-\lambda_{i+1}+1};q)_\infty}$$

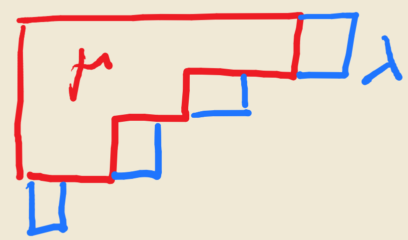
if also $|\lambda/\mu|=1$ then

$$\varphi_{\lambda/\mu} = \frac{1-q^{\mu_j-\mu_{j+1}+1}}{1-q}$$

Skew Cauchy identity

$$\sum_{\mu} P_{\mu/\lambda}(x)\, Q_{\mu/\nu}(y) = H(x;y) \sum_{\mu} Q_{\lambda/\mu}(y)\, P_{\nu/\mu}(x)$$





In particular, if $\underline{\lambda = \nu = \emptyset}$ then

$$\sum_{\lambda} P_{\lambda}(x)\, Q_{\lambda}(y) = H(x;y)$$

& if $\lambda = \emptyset$ then

$$\sum_{\mu} P_{\mu}(x)\, Q_{\mu/\nu}(y) = H(x;y)\, P_{\nu}(x)$$

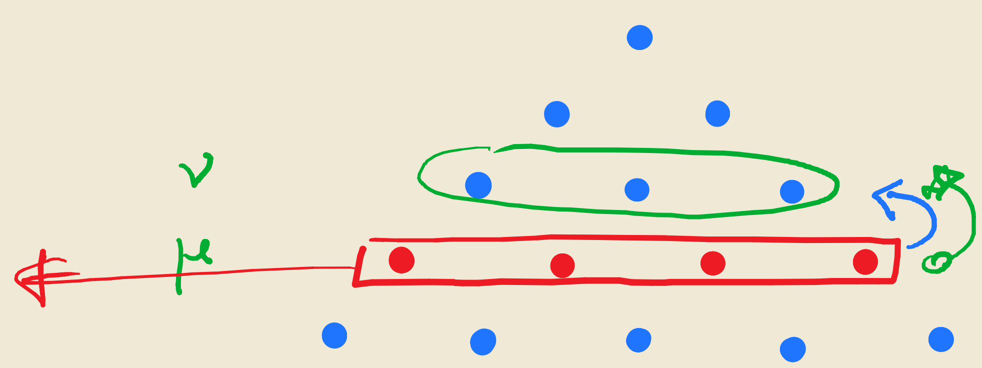
Towards intertwining & Markovian dynamics
(Borodin - Corwin)

Define a Markovian kernel on partitions with k parts [Pieri or Skew Cauchy]

$$P_k(\mu,\nu) := \frac{1}{H(x_1,\dots,x_k,\rho)} \cdot \frac{P_\nu(x_1,\dots,x_k)}{P_\mu(x_1,\dots,x_k)} \cdot Q_{\nu/\mu}(\rho)$$

ν/μ horizontal strip

$\rho \in \mathbb{R}_+$ & will play the role of time



$\Lambda^k_{k-1}(\mu,\nu)$

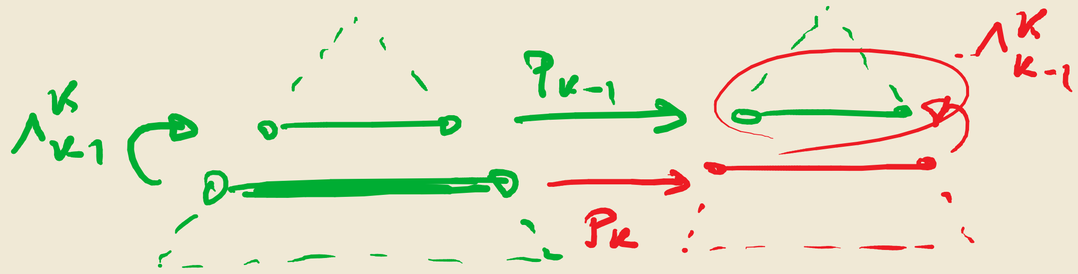
$P_k(\mu,\nu)$ is a prob. kernel thanks to skew-Cauchy.

Define also stochastic links [Branching rule]

$$\Lambda^k_{\underset{\text{k-parts}}{k-1}}(\underset{\text{k-parts}}{\mu},\underset{\text{(k-1)-parts}}{\nu}) := \frac{P_\nu(x_1,\dots,x_{k-1})}{P_\mu(x_1,\dots,x_k)} \underbrace{P_{\mu/\nu}(x_k)}_{\mu/\nu \text{ horizontal strip}}$$

← probability due to branching rule

Proposition (2-step intertwining)



$$\Delta^k_{k-1} := \boxed{\Lambda^k_{k-1} P_{k-1} = P_k \Lambda^k_{k-1}}$$

Proof

$$(\Lambda^k_{k-1} P_{k-1})(\lambda, \nu) = \frac{1}{H(X^{k-1};\rho)} \sum_{\mu} \frac{\cancel{P_\mu(x_1, \dots x_{k-1})}}{P_\lambda(x_1, \dots x_k)} P_{\lambda/\mu}(x_k) \cdot \frac{P_\nu(x_1 \dots x_{k-1})}{\cancel{P_\mu(x_1 \dots x_{k-1})}} Q_{\nu/\mu}(\rho)$$

$$X^{k-1} = (x_1, \dots, x_{k-1})$$

$$= \frac{1}{H(X^{k-1};\rho)} \cdot \frac{P_\nu(x_1 \dots x_{k-1})}{P_\lambda(x_1 \dots x_k)} \sum_\mu P_{\lambda/\mu}(x_k) Q_{\nu/\mu}(\rho)$$

$$(P_k \Lambda^k_{k-1})(\lambda, \nu) = \frac{1}{H(X^k;\rho)} \sum_\mu \frac{\cancel{P_\mu(x_1 \dots x_k)}}{P_\lambda(x_1 \dots x_k)} Q_{\mu/\lambda}(\rho) \cdot \frac{P_\nu(x_1 \dots x_{k-1})}{\cancel{P_\mu(x_1 \dots x_k)}} P_{\mu/\nu}(x_k)$$

$$X^k = (x_1, \dots x_k)$$

$$= \frac{1}{H(X^k;\rho)} \frac{P_\nu(x_1 \dots x_{k-1})}{P_\lambda(x_1 \dots x_k)} \sum_\mu Q_{\mu/\lambda}(\rho) P_{\mu/\nu}(x_k)$$

$$\underset{\text{Cauchy}}{\overset{\text{Skew}}{=}} \frac{H(x_k;\rho)}{H(x_1 \dots x_k;\rho)} \frac{P_\nu(x_1 \dots x_{k-1})}{P_\lambda(x_1 \dots x_k)} \sum_\mu P_{\lambda/\mu}(x_k) Q_{\nu/\mu}(\rho)$$

Full intertwining on GT patterns

Define transition of GT patterns

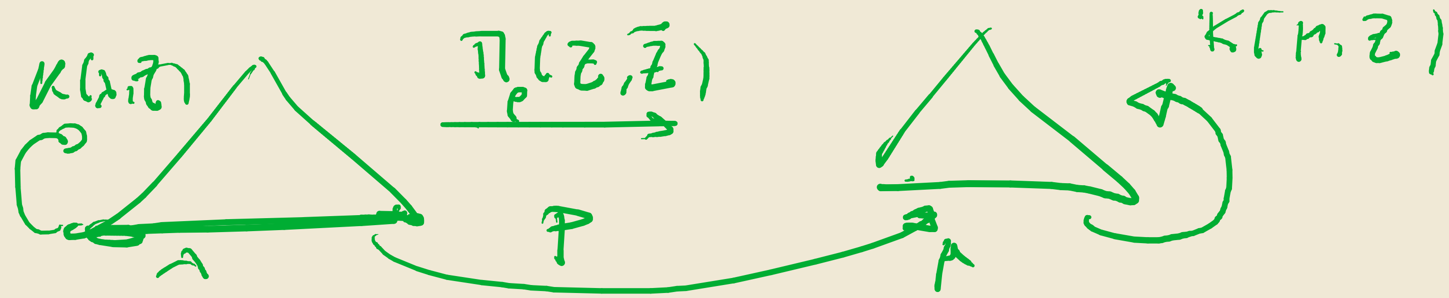
$$\Pi_\rho(Z, \tilde{Z}) = P_1(z^1, \tilde{z}^1) \prod_{k=2}^{n} \frac{P_k(z^k, \tilde{z}^k)\, \Lambda^k_{k-1}(\tilde{z}^k, \tilde{z}^{k-1})}{\Delta^k_{k-1}(z^k, \tilde{z}^{k-1})}$$

&

$$K(\lambda, Z) = \prod_{i=2}^{n} \Lambda^i_{i-1}(z^i, z^{i-1}) \, \mathbb{1}_{z^n = \lambda}$$

then

$$K \Pi_\rho = P K$$



where here if the height of the GT pattern is h, then

$$P = P_h$$

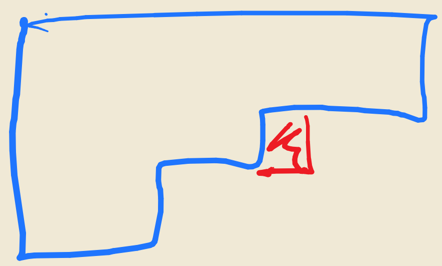
Example : q-Whittaker 2d model & q-TASEP's

After some simple cancellations & using the explicit coefficients from branching rule, we have

$$\Pi_\rho(Z,\tilde{Z}) = \prod_{k=1}^{n} U_\rho\left(Z^k, \tilde{Z}^k \mid Z^{k-1}, \tilde{Z}^{k-1}\right)$$

with $U_\rho\left(Z^k, \tilde{Z}^{k-1} \mid Z^k, \tilde{Z}^{k-1}\right)$

$$= \frac{1}{H(x_k;\rho)} \frac{\psi_{\tilde{Z}^k/\tilde{Z}^{k-1}}}{\psi_{Z^k/\tilde{Z}^{k-1}}} \phi_{\tilde{Z}^k/Z^k} \cdot (\rho x_k)^{|\tilde{Z}^k|-|Z^k|}$$



& if $\lambda_j = \mu_j + 1$ & $\lambda_i = \mu_i \quad \forall\, i \neq j$

$$\phi_{\lambda/\mu} = \frac{1-q^{\mu_{j-1}-\mu_j}}{1-q}, \quad \psi_{\lambda/\mu} = \frac{1-q^{\mu_j-\mu_{j+1}+1}}{1-q}$$

find the generator of the process Π_ϱ

recall that we will interpret the parameter ϱ as time

$$\mathcal{L} = \frac{d}{d\varrho} \Pi_\varrho \Big|_{\varrho=0} = \frac{d}{d\varrho} \prod_{k=1}^{n} U_\varrho \left(z^k, \tilde{z}^{k-1} \,\middle|\, z^k, \tilde{z}^{k-1} \right)$$

$$= \sum_i \prod_{k \neq i} U_\varrho \left(z^k, \tilde{z}^{k-1} \,\middle|\, z^k, \tilde{z}^{k-1} \right)$$

$$\cdot \frac{d}{d\varrho} U_\varrho \left(z^k, \tilde{z}^{k-1} \,\middle|\, z^k, \tilde{z}^{k-1} \right)$$

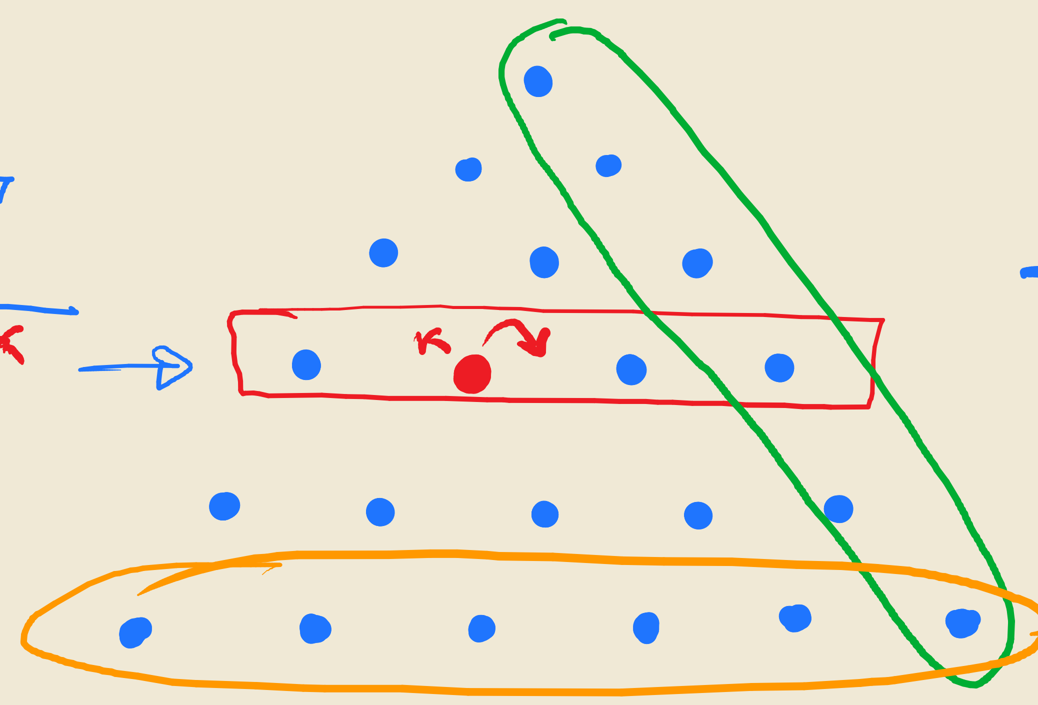
To proceed we simplify & write

$$U_\rho\left(z^{k},\tilde z^{k-1}\,\middle|\,z^{k},\tilde z^{k-1}\right)=\frac{\rho x_k}{(\rho x_k;q)_\infty}\cdot\frac{1-q^{\tilde z^{k-1}_{j-1}-z^k_j}}{1-q^{z^k_j-\tilde z^{k-1}_j+1}}\cdot\frac{1-q^{z^k_j-z^k_{j+1}+1}}{1-q}$$

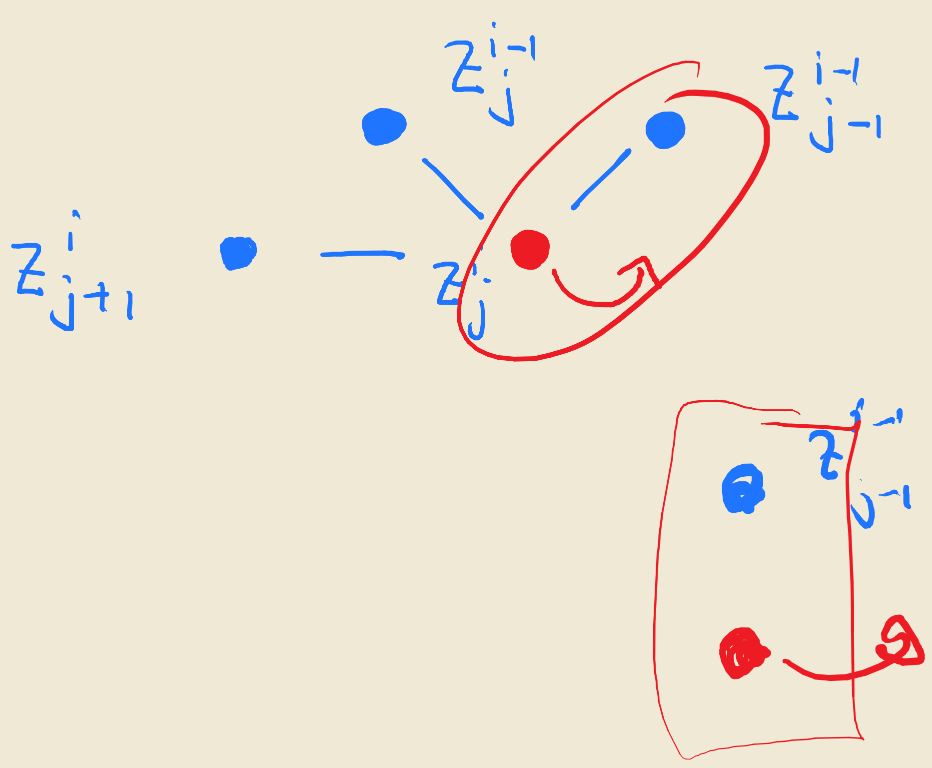
& doing the differentiations or Taylor expansion

q-Whittaker 2d particle system.

$$\frac{d}{d\rho}U_\rho=\frac{x_i}{1-q}\left\{\sum_{k}\rightarrow\qquad-\ \mathrm{Id}\right\}$$



q-whittaker process.



rate $\displaystyle\frac{x_i}{1-q}\cdot\frac{\left(1-q^{z^{i-1}_{j-1}-z^i_j}\right)\cdot\left(1-q^{z^i_j-z^i_{j+1}+1}\right)}{1-q^{z^i_j-z^i_{j-1}+1}}$

q-push-TASEP

$q\to1$ then bottom row Whittaker diffusion

$$\mathcal{L}=\frac12\Delta+\nabla\log\underbrace{\psi_\lambda(x)}_{}\cdot\nabla$$

Whittaker function.

$$\Delta=\frac{\partial^2}{\partial x_1^2}+\dots+\frac{\partial^2}{\partial x_n^2}$$

$$\Delta-\sum e^{x_i-x_{i+1}}$$