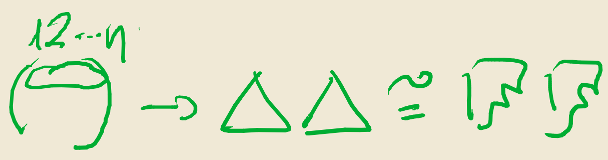
Integrable Probability : Lecture 6

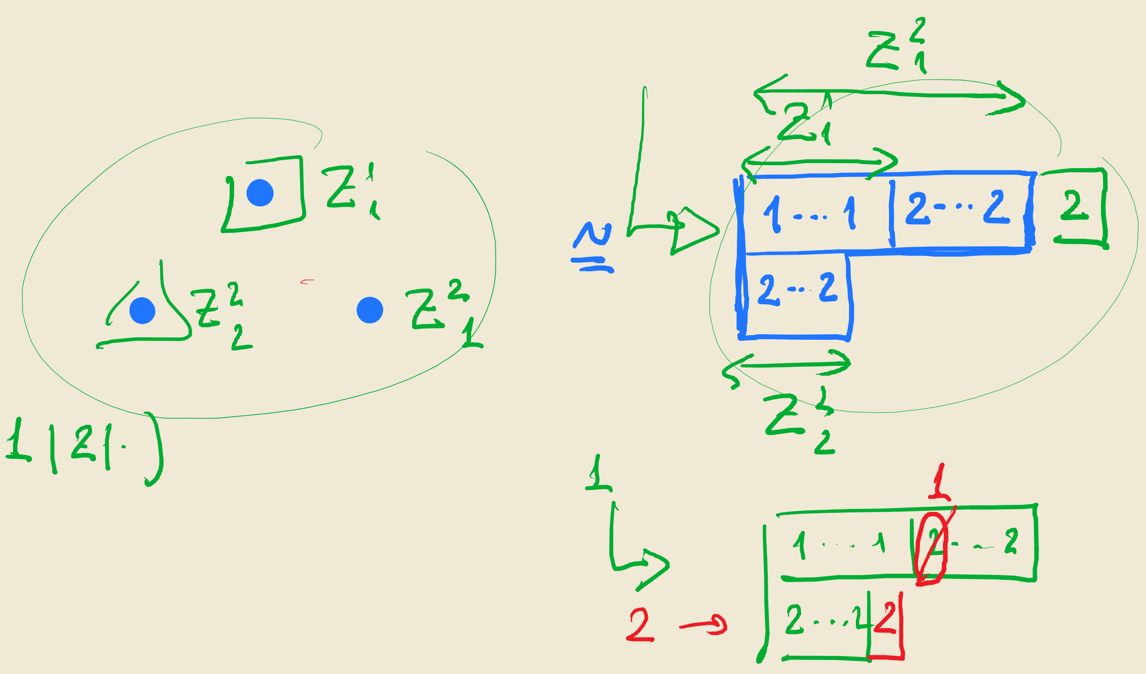
- RSK dynamics
- Branching rule
- Pieri rule
- Intertwining



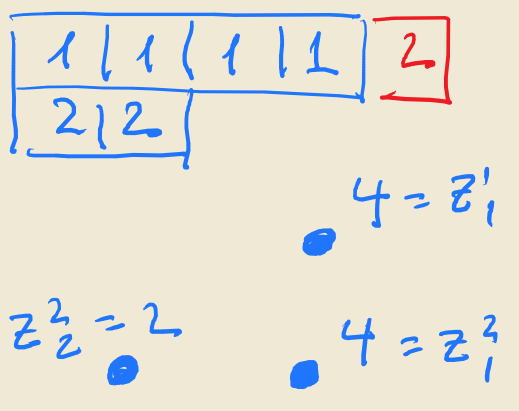
RSK induced dynamics

$$W = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{array}\right) \end{array} \cong (2|1|1|1|\times|1|2|\cdot)$$

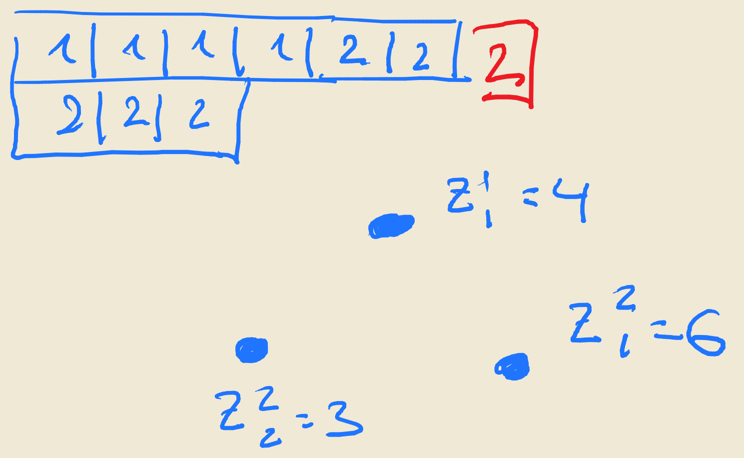
$\xrightarrow{RSK}$



1)



2)

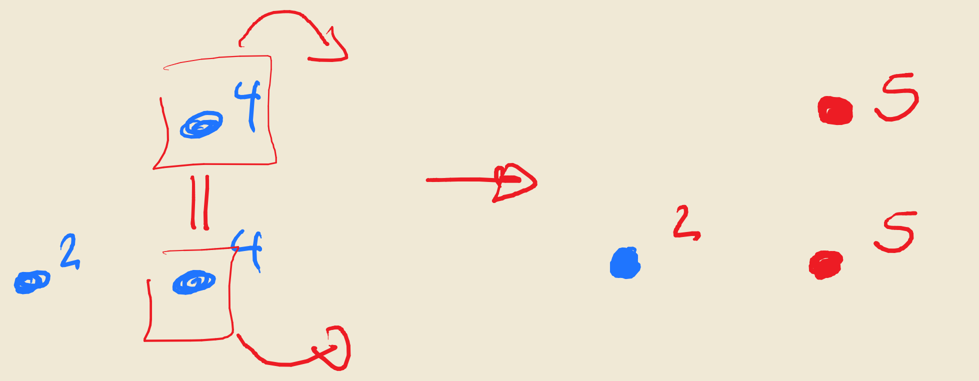


Let's insert the next row (word) of W

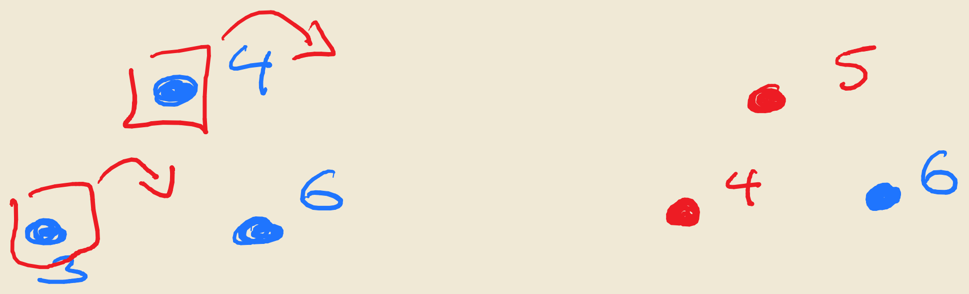
a) $(1,0)$ or $(0,1)$

i.e. insert a 1 or insert a 2.

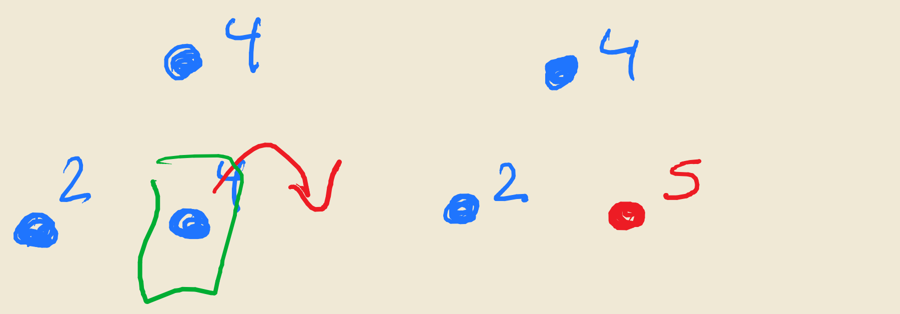
$\boxed{1a)}$



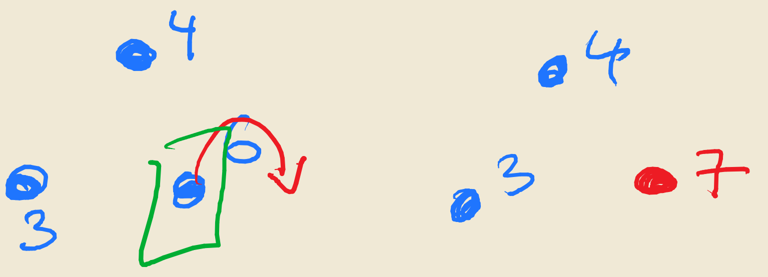
$\boxed{2a)}$



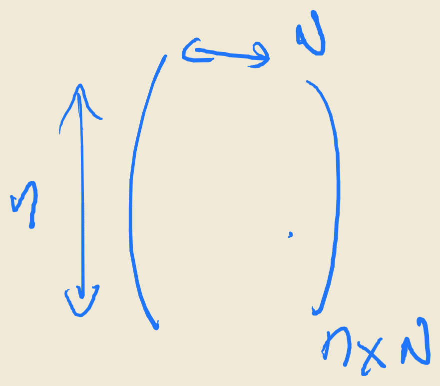
1b) insert a 2



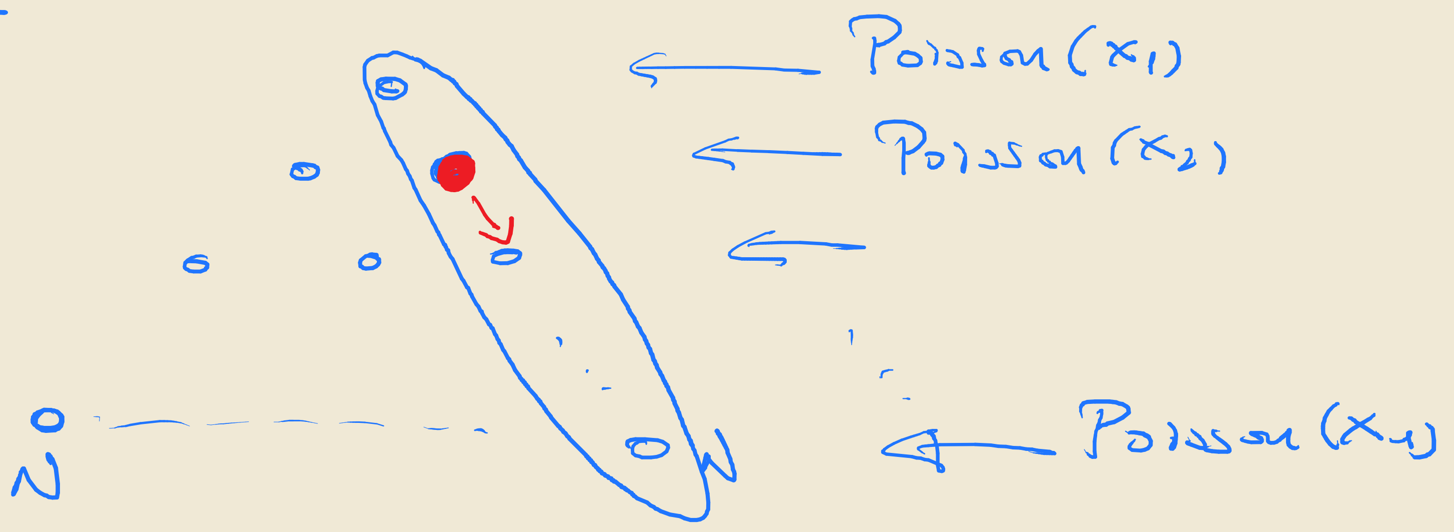
2b)



Summary – general rule

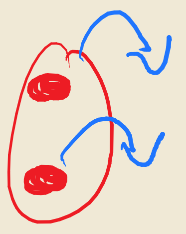


$\longrightarrow$



A| only particle at the edge jump independently

B| if

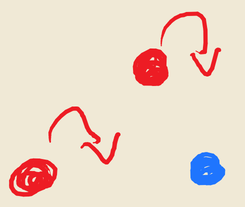


$\boxed{\text{if}}$ particle that decides to jump is at the same location like its right-down neighbour

$\boxed{\text{then}}$ it will push the neighbour together

e.g. case (1a)

C| if



$\boxed{\text{if}}$ right neighbour stands away to the right

$\boxed{\text{then}}$ the top particle will pull together the left neighbour

e.g. case (2a)

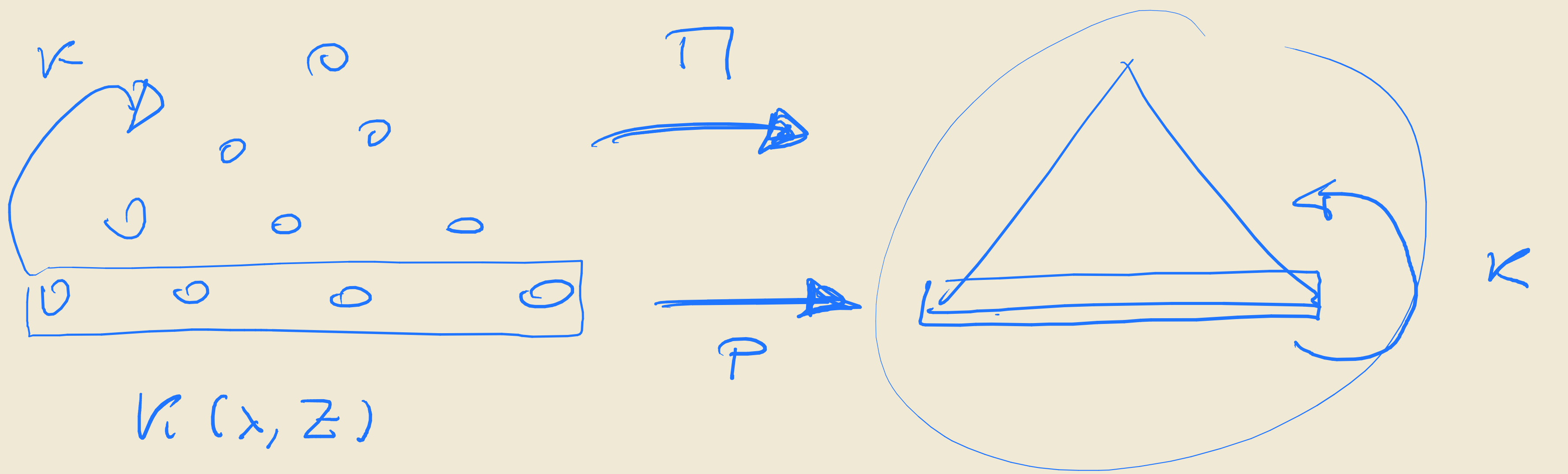
Ingredients to dynamics

- Branching rule, Pieri's rule

- Markov generators, Doob's h-tranform

$$\sum_{z'} \Pi(z, z') \cdot 1 = 1$$

$$P(K\,1) = K\,\underbrace{\Pi\,1} = (K1)$$



Branching Rule for Schur

In terms of Gelfand-Tsetlin patterns

Schur polynomial

$$S_\lambda(x) := S_\lambda(x_1, x_2, \dots, x_n) = \sum_{\substack{Z:GT \\ sh(Z)=\lambda}} \prod_{i=1}^{n} x_i^{|z^i| - |z^{i-1}|}$$

$\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

with $\quad Z = (z^i_j)_{i \leq j \leq n}$

$$sh(Z) = (z^n_1, z^n_2, \dots, z^n_n)$$

$$|z^i| := \sum_{j=1}^{i} z^i_j$$

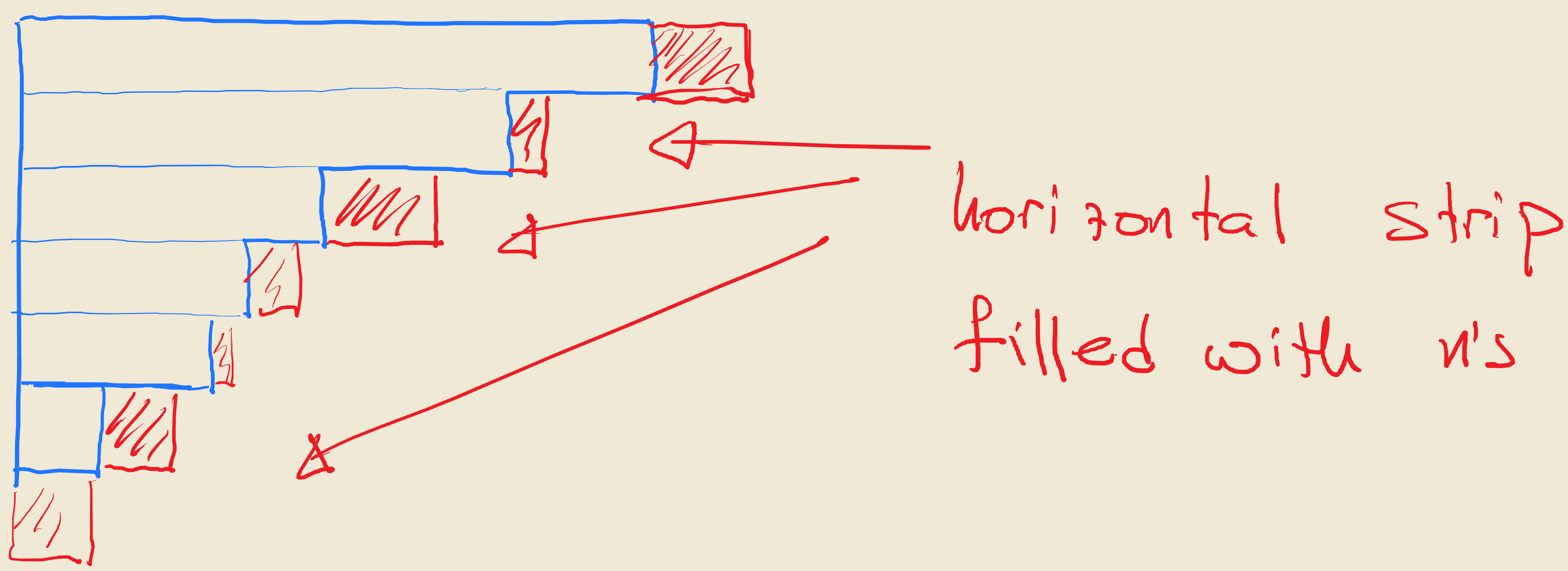
Rewrite

$$s_\lambda(x_1,\dots,x_n) = \sum_{\substack{Z:GT\\ sh(Z)=\lambda}} \prod_{i=1}^{n} Q^{i}_{i-1}(z^i, z^{i-1}; x_i)$$

$$= \sum_{\substack{z^n=\lambda\\ z^{n-1}\prec z^n}} Q^{n}_{n-1}(z^n, z^{n-1}; x_n) \sum_{z^1,\dots,z^{n-2}} \prod_{i=1}^{n-1} Q^{i}_{i-1}(z^i, z^{i-1}; x_i)$$

$$= \sum_{\mu:\mu\prec\lambda} \underbrace{Q^{n}_{n-1}(z^n, z^{n-1}; x_n)}_{s_{\lambda/\mu}(x_n)} s_\mu(x_1,\dots,x_{n-1})$$

In pictures



The Kernel K

$Z \in \mathbb{R}^{\frac{n(n+1)}{2}}$ Gelfand - Tsetlin pattern

$\lambda \in \mathbb{R}^n$ shape of GT pattern

$$K(\lambda, Z) = \prod_{i=1}^{n} Q_{n-1}^{n}\left(z^n, z^{n-1}; x_n\right) \mathbb{1}_{z^n = \lambda}$$

We have

$$S_\lambda(x) = S_\lambda(x_1, \dots, x_n) = (K \mathbb{1})(\lambda)$$

$$= \sum_{Z : GT} \prod_{i=1}^{n} Q_{n-1}^{n}\left(z^n, z^{n-1}; x_n\right) \mathbb{1}_{z^n = \lambda}$$

Towards intertwinning : if one finds an operator P st.

$$PK1 = K1 \iff \sum_{\mu} P(\lambda, \mu) S_\mu(x) = S_\lambda(x)$$

then P will be the natural canditate for the intertwinning

$$PK = K\Pi$$

which will of course have to be checked.

Pieri Rule

Complete symmetric polynomials

$$h_k(x_1, \dots, x_n) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} x_{i_1} \cdots x_{i_k}$$

Denote $x = (x_1, \dots, x_n)$

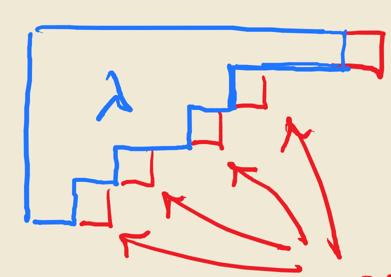
$$h_k(x) s_\lambda(x) = \sum_{\nu \succ_k \lambda} s_\nu$$

with $\nu \succ_k \lambda$ a horizontal strip ν/λ with k boxes

- We will use the $k=1$ case i.e.

$$h_1(x) s_\lambda(x) = \sum_{\nu \succ_1 \lambda} s_\nu(x)$$

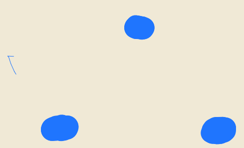
$$\Longleftrightarrow \quad (x_1 + \cdots + x_n) s_\lambda(x) = \sum_{\nu \succ_1 \lambda} s_\nu(x)$$

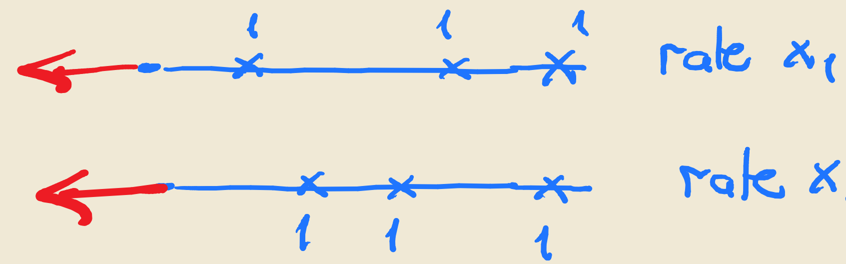


one of the boxes added to give ν

The intertwining

1) Run the RSK dynamics & write down $\Pi(Z,\tilde{Z})$

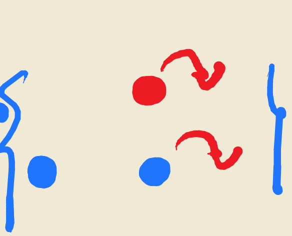
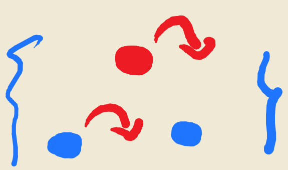
$Z =$  $=$ Z^1_2 Z^1_1 Z^2_1

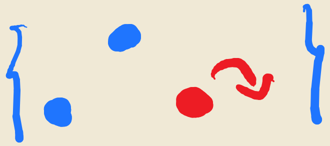


to write $\Pi(Z,\tilde{Z})$ notice that Z^1_1 will jump at rate x_1

Z^2_1 — " — x_2

& move will trickle down

$\Pi(Z,\tilde{Z}) = x_1 \mathbb{1}\{$  $\} + x_1 \mathbb{1}\{$  $\}$

$+ x_2 \mathbb{1}\{$  $\}$

Notice that

$$\sum \Pi(z, \tilde{z}) = x_1 + x_2 \qquad (\neq 1 \text{ in general})$$
$$= h_1(x_1, x_2)$$

2) Put Pieri rule into action

(A) $$\sum_i s_{\lambda+e_i}(x) = \sum_i \sum_{\tilde{z}} K(\lambda+e_i, \tilde{z}) \qquad \text{(Branching)}$$
$$= \sum_\nu \sum_{\tilde{z}} P(\lambda,\nu)\, K(\nu, \tilde{z})$$

with $$P(\lambda,\nu) = \sum_i \mathbb{1}_{\nu=\lambda+e_i}$$

(B) $$\sum_i s_{\lambda+e_i}(x) = h_1(x)\, s_\lambda(x) = h_1(x) \sum_z K(\lambda, z)$$
$$= \sum_z K(\lambda, z) \sum_{\tilde{z}} \Pi(z, \tilde{z}).$$

Markovian dynamics & Doob's transform

We have checked intertwining for

$\Pi(Z,\tilde{Z})$ rate matrix of RSK dynamics

$$P(\lambda,\nu) = \sum_{i=1}^{n} 1_{\nu=\lambda+e_i}$$ Pieri operator

$$K(x,Z) = \prod_{i=1}^{n} x_i^{|z^i|-|z^{i-1}|}$$ Schur kernel or branching rule.

But these are <u>not</u> probability kernels & strictly speaking not in the Pitman-Rogers frame.

Normalisation

define $$\tilde{K}(\lambda, Z) = \frac{K(\lambda, Z)}{\sum_Z K(\lambda, Z)} = \frac{K(\lambda, Z)}{S_\lambda(x)}$$

$$\tilde{P}(\lambda, \nu) = \frac{S_\nu(x)}{S_\lambda(x)} P(\lambda, \nu)$$

We will see that $\tilde{P}(\lambda, \nu)$ is the transition kernel of a Poissonian r.w. in the Weyl chamber

$$W_n = \{ \lambda \in \mathbb{R}^n : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}$$

1) Consider a random walk $\lambda(t) \in W_n$

killed upon hitting the boundary of W_n

$\lambda(t)$: $\lambda \longrightarrow \lambda + e_i$ at rate x_i

transition kernel $\mathbf{Q}(\lambda, \nu) = \sum_{i=1}^{n} x_i \, \mathbb{1}_{\nu = \lambda + e_i}$

2) harmonic functions

let $H(\lambda) := x_1^{-\lambda_1} \cdots x_n^{-\lambda_n} \, s_\lambda(x)$ $\big(= 0$ outside $W_n \big)$

then $(\mathbf{Q} H)(\lambda) = \left(\sum_{i=1}^{n} x_i \right) H(\lambda) \iff$

$\iff \sum_{i=1}^{n} x_i \, H(\lambda + e_i) = \left(\sum x_i \right) H(\lambda)$

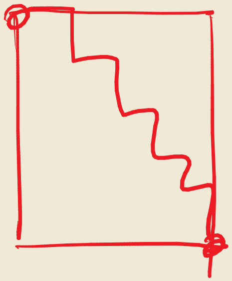
(this is a consequence of Pieri)

3) an easy computation shows

$$\mathbf{Q}(\lambda, \nu) = \frac{x_1^{\nu_1} \cdots x_n^{\nu_n}}{x_1^{\lambda_1} \cdots x_n^{\lambda_n}} \, P(\lambda, \nu)$$

with $P(\lambda, \nu) = \sum_{i=1}^{n} \mathbb{1}_{\nu = \lambda + e_i}$

4) $$\mathbf{P}(\lambda, \nu) = \frac{H(\nu)}{H(\lambda)} \, Q(\lambda, \nu) = \frac{s_\nu(x)}{s_\lambda(x)} \, P(\lambda, \nu)$$

$\max_{\pi} \sum_{x\in\pi}$  $= \tau_N$

$$\frac{\tau_N - fN}{\sigma N^{1/3}} \xrightarrow{d} TW_{GUE}$$

$\sum_{\pi} \prod_{x\in\pi}$

$$\frac{\log Z_N - f' N}{\sigma' N^{1/3}} \xrightarrow{d} \ ? \quad TW_{GUE}$$

YES

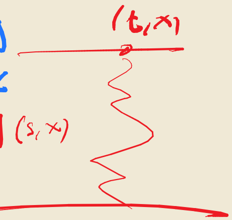
$\beta^{-1} \log Z_{N,\beta}$ $\quad \beta \nearrow \infty$

Random Polymer Model

B.M. $\longrightarrow$ SRW

$\dot W(t,x) \longrightarrow (\omega(t,x))_{t\in\mathbb{N},\, x\in\mathbb{Z}}$

i.i.d.



$$Z_{N,\beta} = E_x^{SRW}\left[e^{\sum_{n=1}^{N} \beta\,\omega(n,S_n)}\right]$$

$\beta_N = N^{-\frac{1}{4}}$

$\downarrow d$

(KPZ)

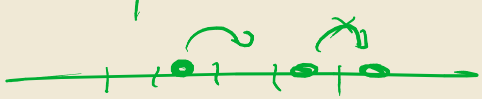
$$\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}|\nabla h|^2 + \dot W$$

$$h(t,x) = \log u(t,x)$$

$$\partial_t u = \frac{1}{2}\Delta u + \beta \dot W u \quad \text{(SHE)}$$

Feynman–Kac $\quad \cancel{u(t,x) = E_x\left[e^{\int_0^t \dot W(t-s,\, B(s))\, ds}\right]}$

$\longrightarrow$ TASEP



$1/2-\varepsilon \quad 1/2+\varepsilon$

ASEP