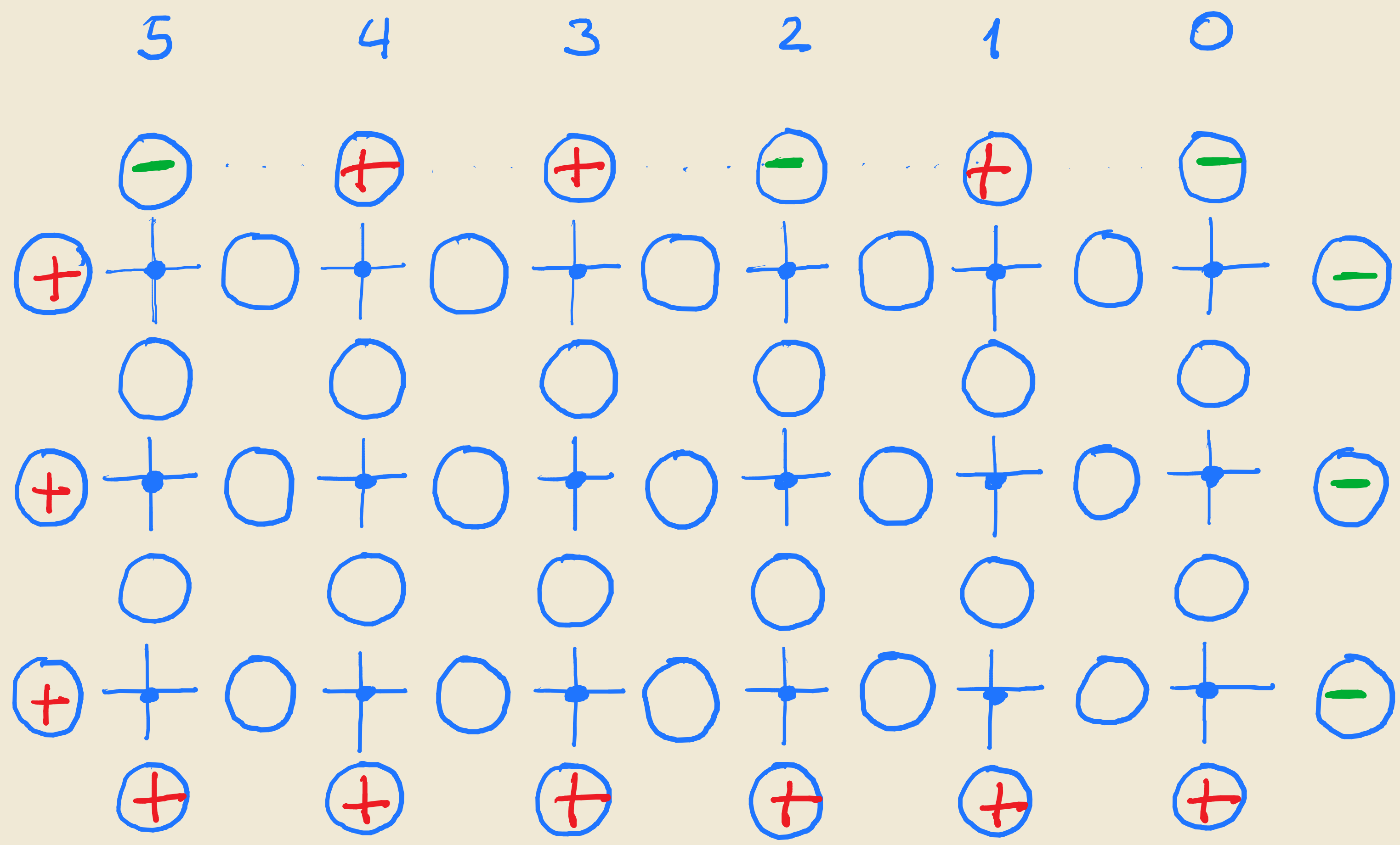
Integrable Probability 8

- Introduction to 6-vertex model, Yang-Baxter & Schur after Brubaker-Bump-Friedberg 0912.0911v3

- Relation to stochastic models after Borodin-Petrov

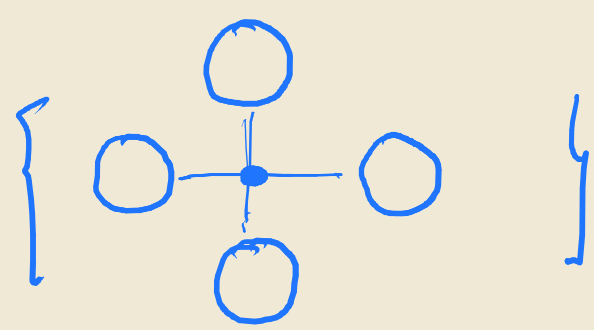
6-vertex or square ice

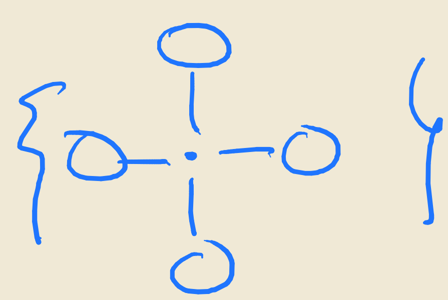


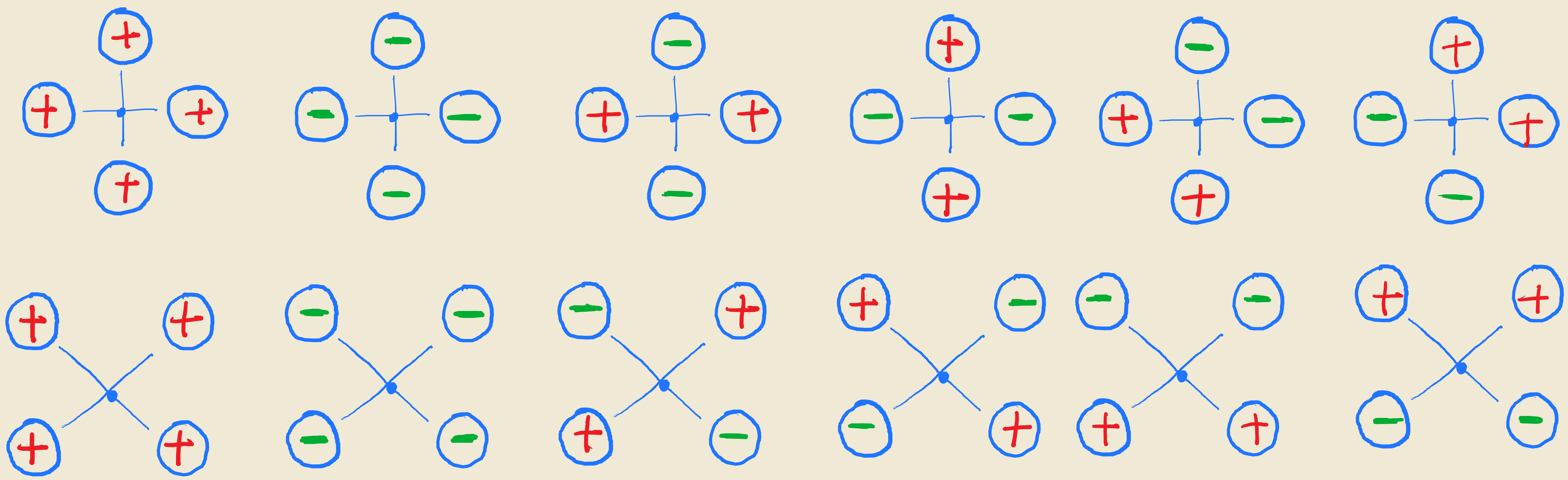
Fill the circles with + or − so that

incoming signs (±) = outgoing signs (±)

prescribe weights or probabilities to each vertex

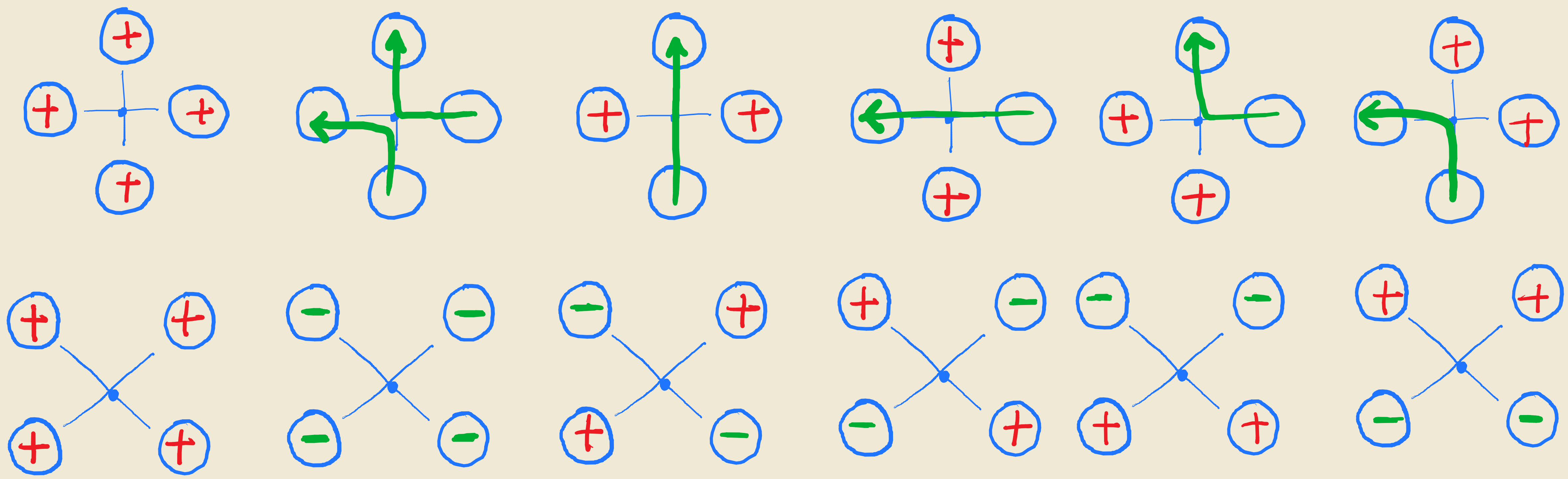
wt 

$$Z_\lambda = \prod_{\substack{\text{vertices}\\ \text{of ice}}} \mathrm{wt}$$ 



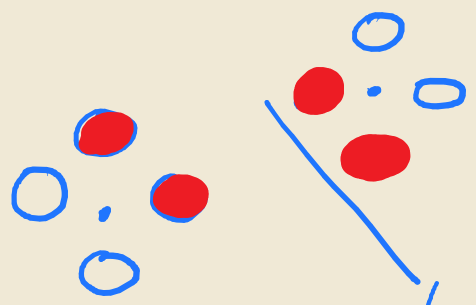
Weights a_1 a_2 b_1 b_2 c_1 c_2

Path representation



Weights a_1 a_2 b_1 b_2 c_1 c_2

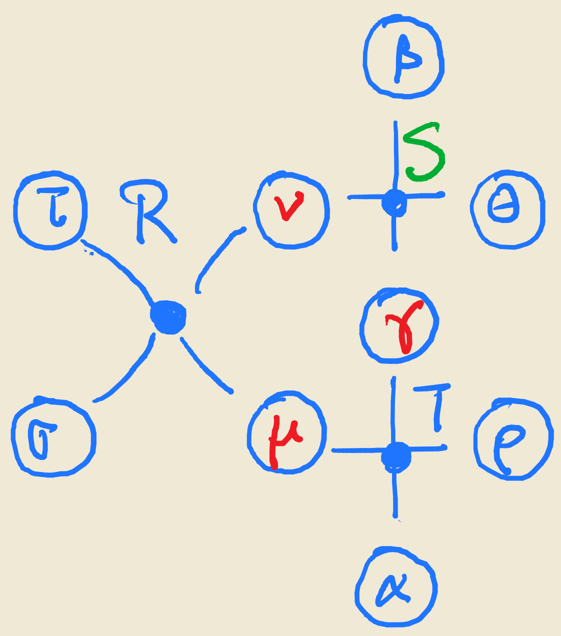
R-matrix & Yang-Baxter



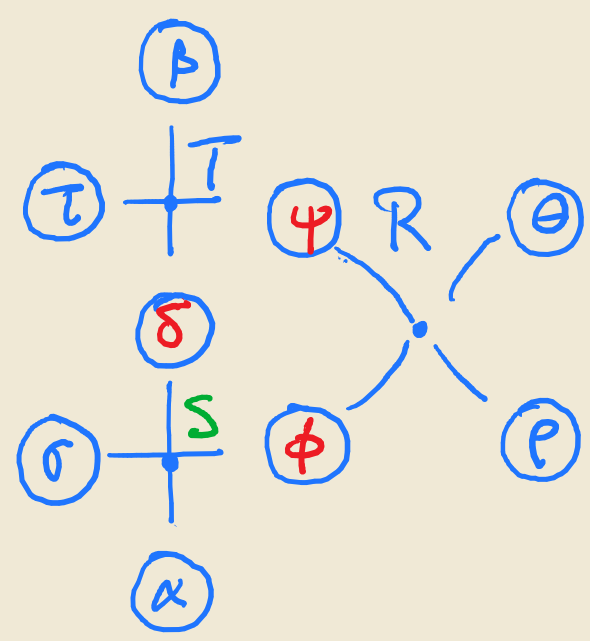
$$R := \begin{array}{c|cccc} & ++ & -+ & +- & -- \\ \hline ++ & a_1 & & & \\ +- & & b_1 & c_1 & \\ -+ & & c_2 & b_2 & \\ -- & & & & a_2 \end{array}$$

Yang-Baxter equation

$\sum_{\nu,\mu,\gamma}$



$= \sum_{\delta,\varphi,\phi}$



Notice: S & T have switched!

In algebraic terms

$$\sum_{\gamma,\mu,\nu} R^{\nu\mu}_{\sigma\tau}\, S^{\theta\gamma}_{\nu\beta}\, T^{\rho\alpha}_{\mu\gamma} = \sum_{\delta,\varphi,\phi} T^{\varphi\delta}_{\tau\beta}\, S^{\phi\alpha}_{\sigma\delta}\, R^{\theta\rho}_{\phi\varphi}$$

In shorthand, denote the YBE as

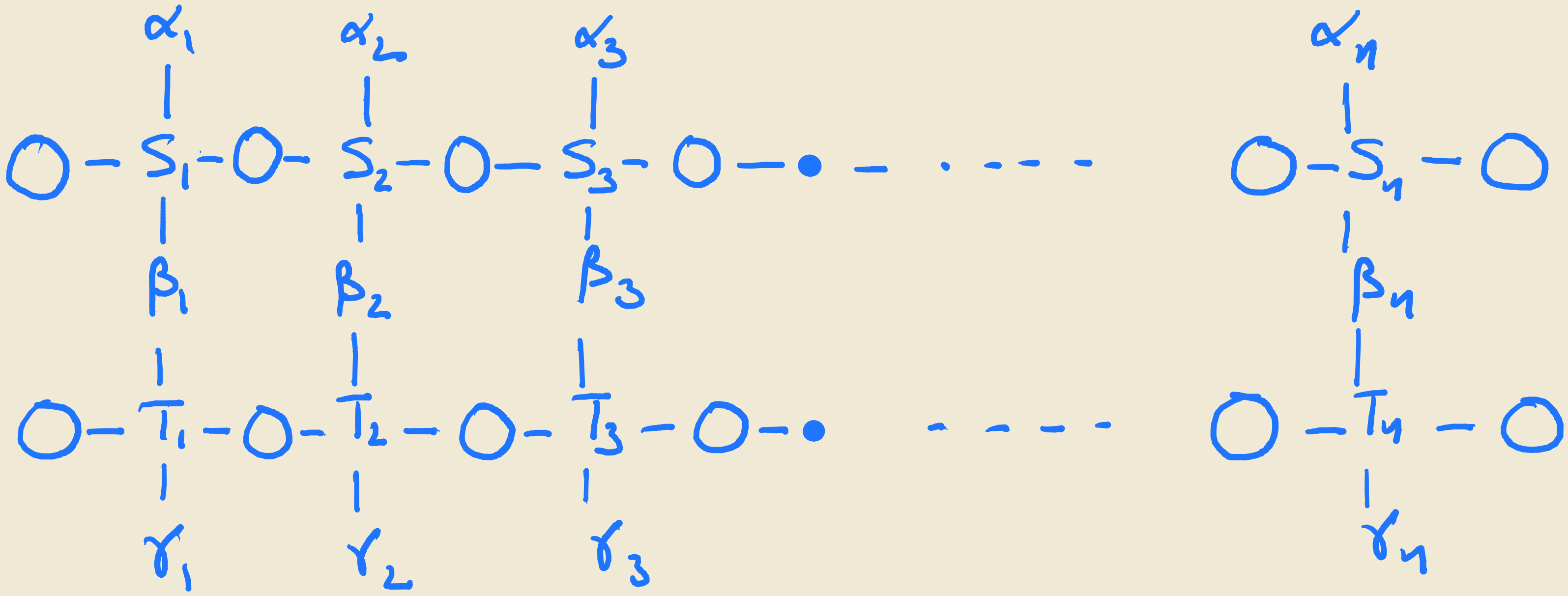
$$[\![R, S, T]\!] = 0$$

In short: given S, T find R s.t.

$$RST = TSR$$

The significance of the YBE

Prob



$V(S) = V(S)_{\alpha,\beta}$ $\qquad \alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$

$V(T) = V(T)_{\beta,\gamma}$ $\qquad \gamma = (\gamma_1, \dots, \gamma_n)$

the $2^n \times 2^n$ rows transfer matrices

<u>then</u> finding an R-matrix s.t. $[\![R, S, T]\!] = 0$

$\Longrightarrow$ $V(S)$ & $V(T)$ commute

Existence of R-matrices

Thm (Brubaker-Bump-Friedberg following Baxter)

Consider GV matrices

$$\begin{pmatrix} a_1 & & & \\ & b_1 & c_1 & \\ & c_2 & b_2 & \\ & & & a_2 \end{pmatrix}$$

associated to vertex operators S, T i.e. entries

$a_1(T), a_2(T), b_1(T), \ldots$

$a_1(S), a_2(S), b_2(S), \ldots$

For a vertex operator T define

$$\Delta_1(T) = \frac{a_1(T)a_2(T) + b_1(T)b_2(T) - c_1(T)c_2(T)}{2a_1(T)b_1(T)}$$

& $$\Delta_2(T) = \frac{\text{same}}{2a_2(T)b_2(T)}$$

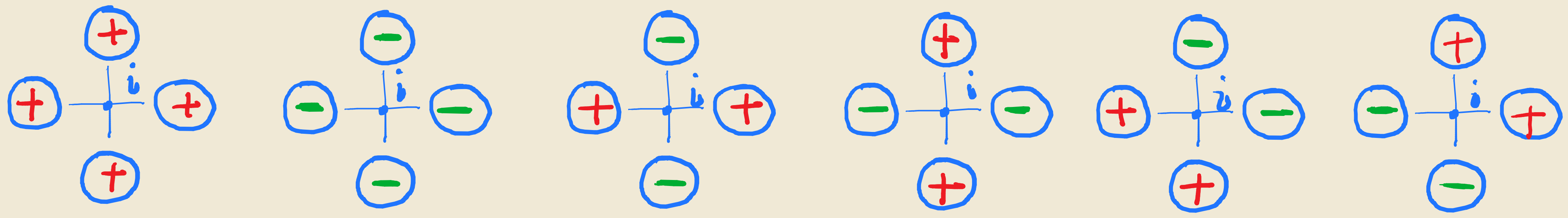
then an R matrix with $[\![R,S,T]\!]=0$ exists iff

$$\Delta_1(S) = \Delta_1(T) \quad \& \quad \Delta_2(S) = \Delta_2(T)$$

Example: Gamma ice, Tokuyama ice & Schur polynomials

$= \Gamma(j)$

$\Gamma(i)$



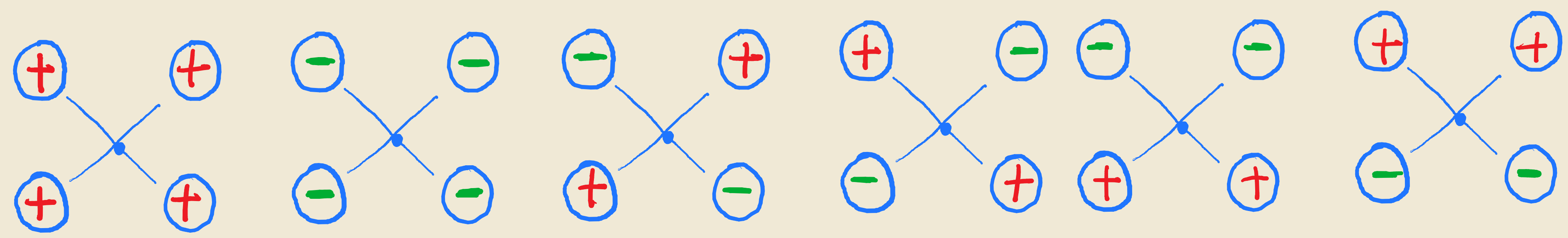
$1 \qquad z_i \qquad t_i \qquad z_i \qquad z_i(t_i+1) \qquad 1$

z_j

$t \quad - t_j$

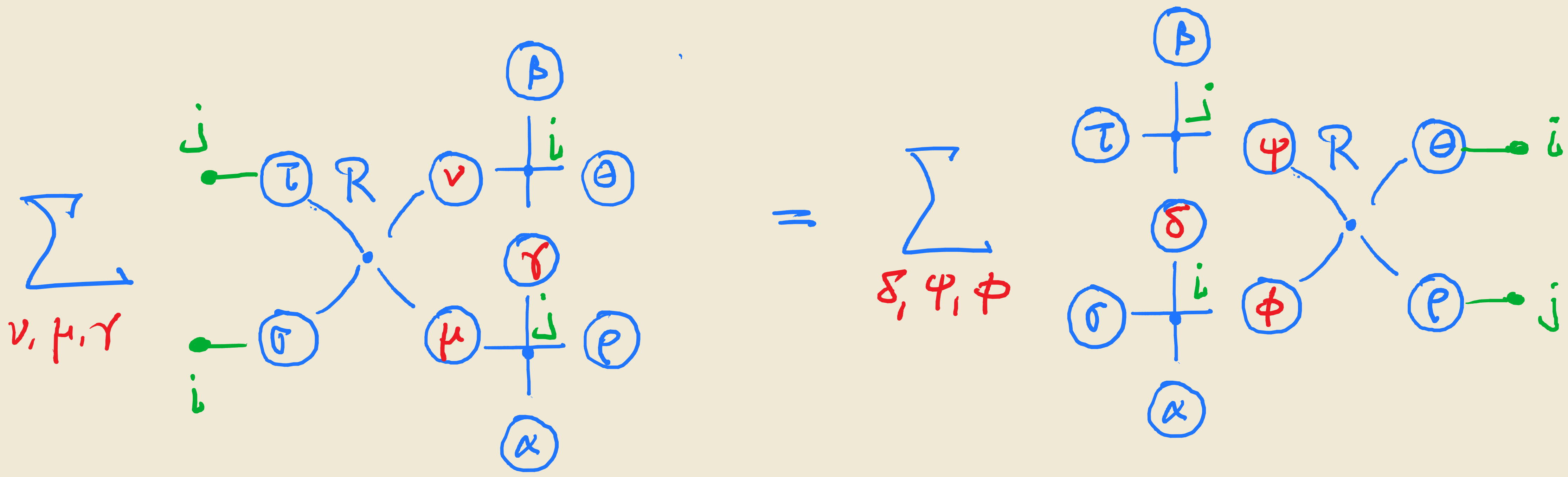
$R_{\Gamma\Gamma}^{(i,j)}$

$1 - b_i$



$t_j z_i + z_j \qquad t_i z_j + z_i \qquad t_i z_j + z_i \qquad z_i - z_j \qquad (t_i+1) z_i \qquad (t_j+1) z_j$

then $\quad [\![R_{\Gamma\Gamma}(i,j), \Gamma(i), \Gamma(j)]\!] = 0 \quad$ or



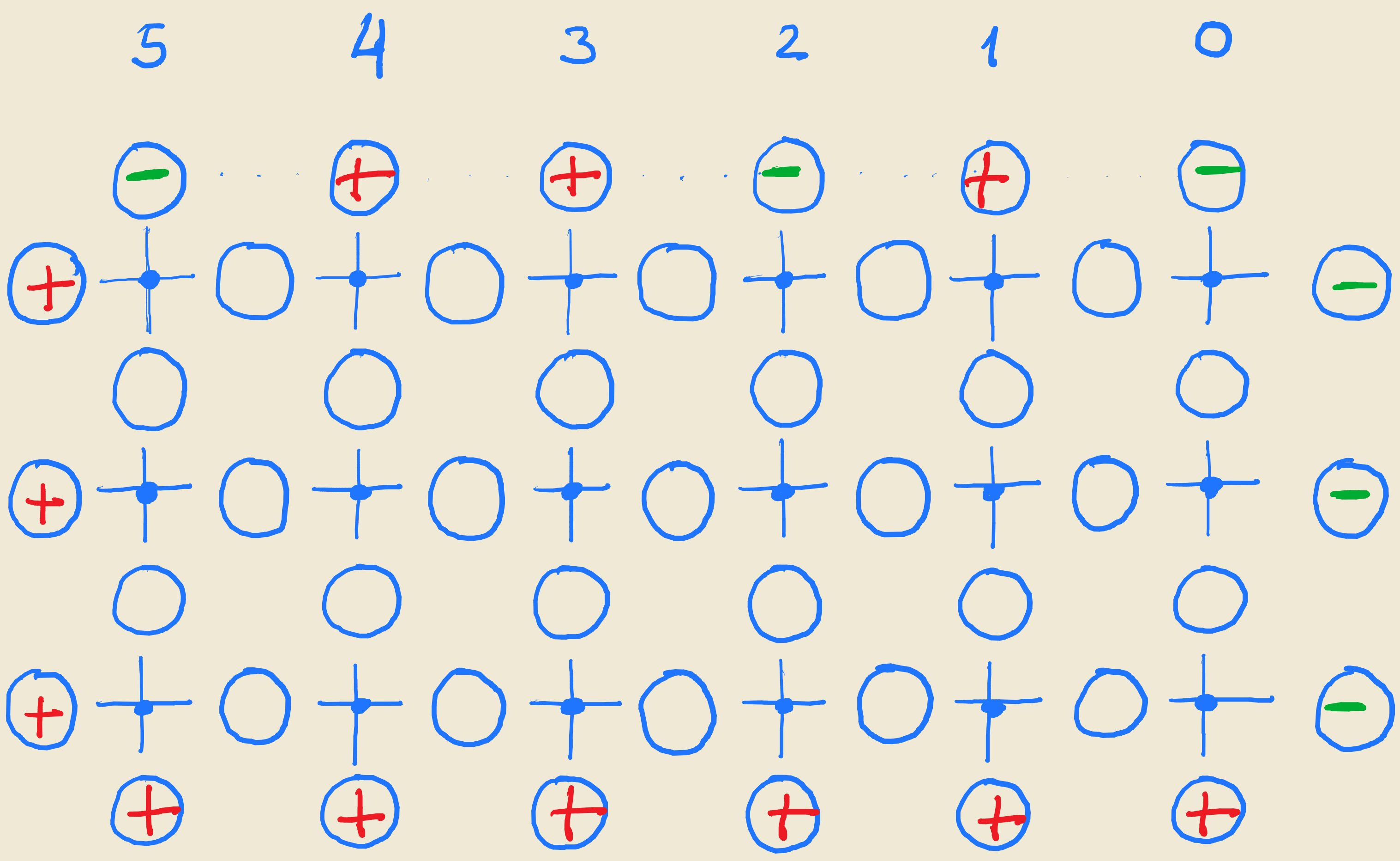
Proof For the proof & a recipe see

Brubaker - Bump - Friedberg.

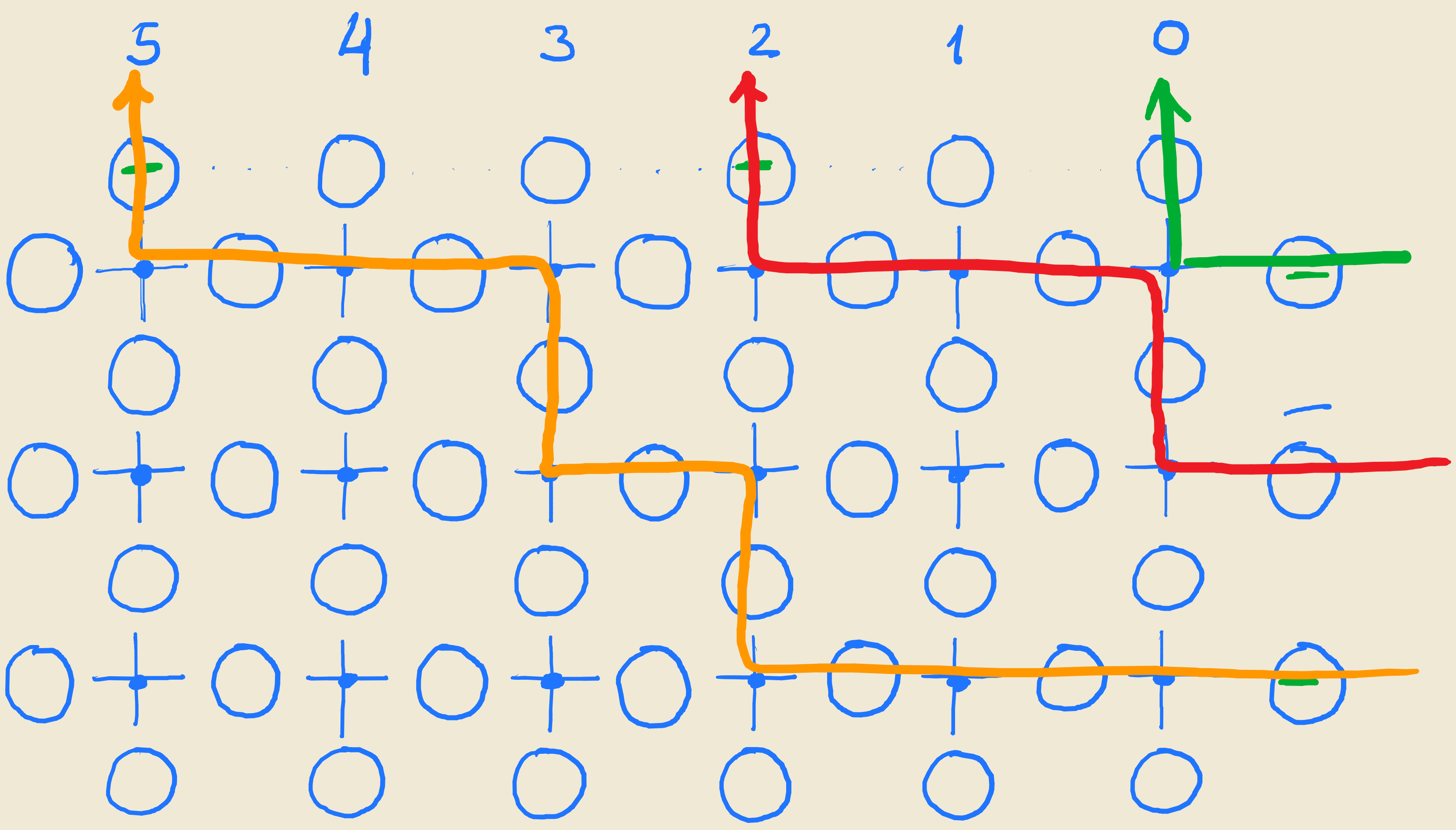
The ice model indexed by partition λ

For a strict partition $\lambda = \lambda_1 > \lambda_2 > \cdots$ e.g. $\lambda = (5,2,0)$

we insert $\ominus$ on the top row (enumeration from left-to-right)



paths



Weight of path ensemble or ice model

$$= \prod_{\substack{\text{vertices} \\ + \\ \text{boundary} \\ \text{conditions}}} wt(\text{vertex}) =: Z_\lambda$$

Thm $$Z_\lambda = \prod_{i<j} (t_i z_j + z_i)\, \underbrace{s_\lambda(z_1, \dots, z_n)}_{\text{Schur}}$$

Proof outline:

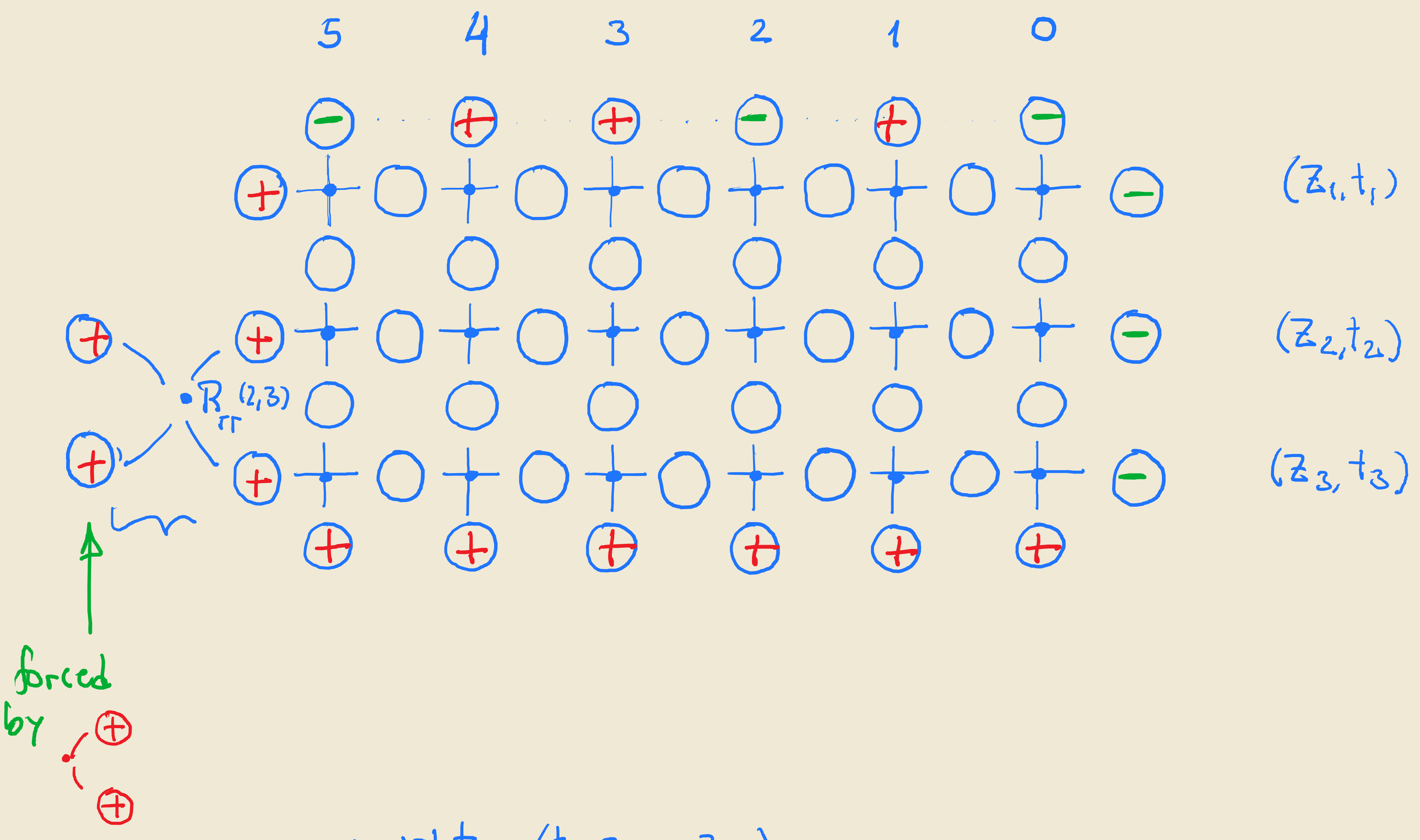
1) Show symmetry of $\prod_{i<j} (t_j z_i + z_j)\, Z_\lambda$ in $z_1, \dots, z_n$ & independence of $t_1 \dots t_n$

2) Set $t_1 = \dots = t_n = -1$ in

$$\prod_{i<j} (t_i z_j + z_i)^{-1}\, Z_\lambda$$

& use Weyl denominator formula

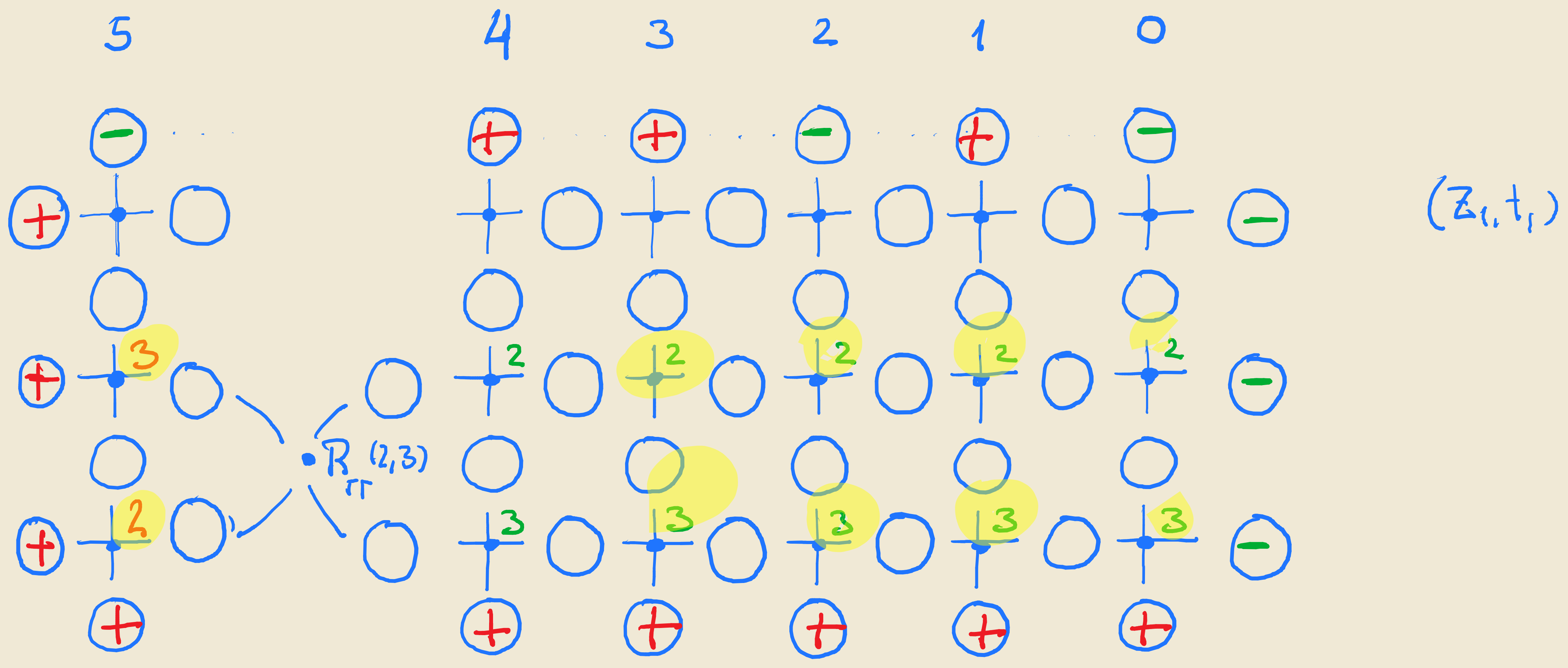
Proof of symmetry



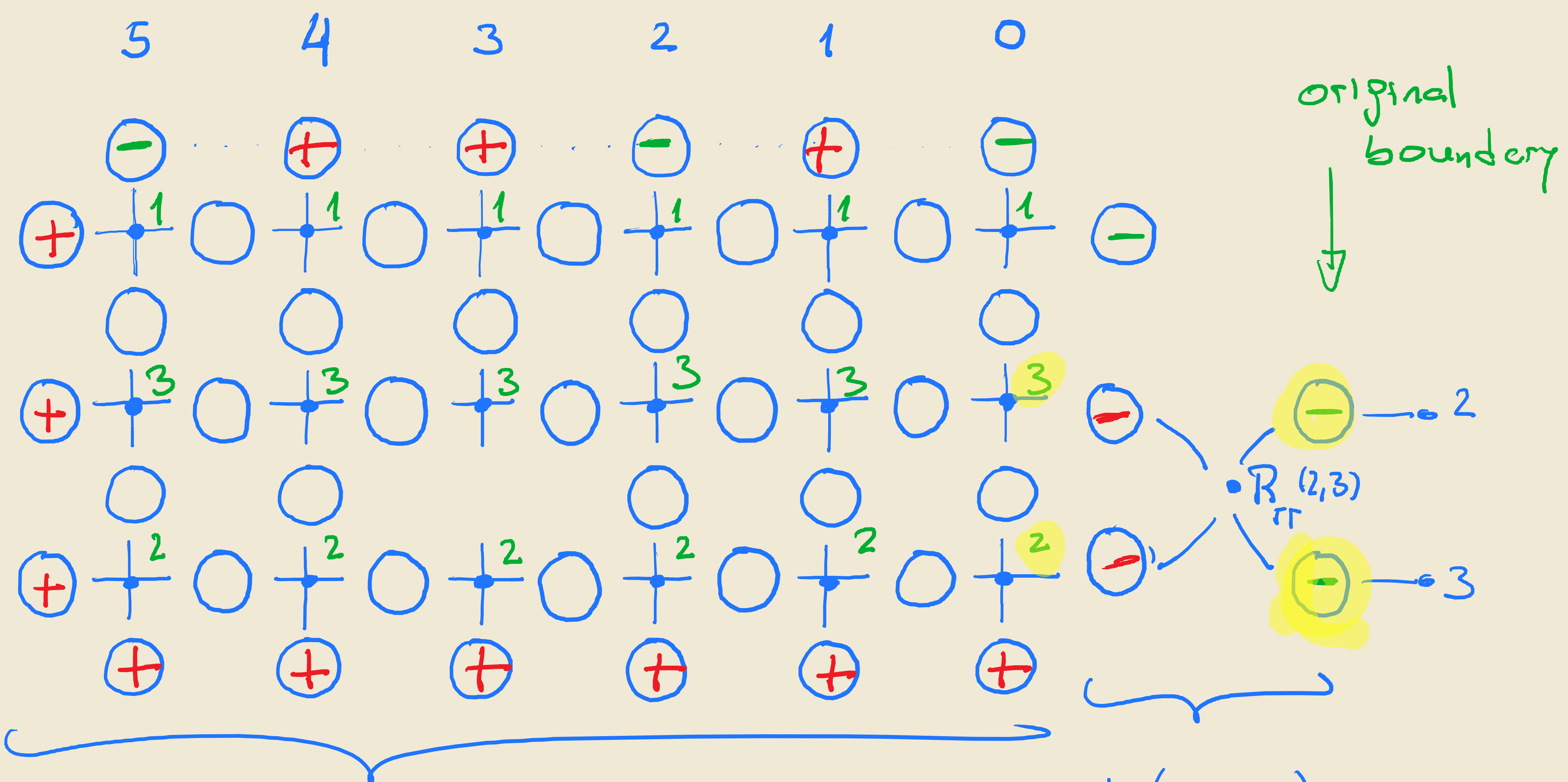
weight $(t_3 z_2 + z_3)$

total weight of above ice $= (t_3 z_2 + z_3)\, Z_\lambda$

Use YBE, the above equals



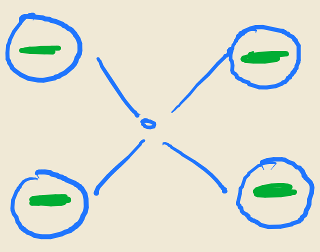
repeat ...



Wt(-‖-) =

wt(-‖-) =

$(t_2 z_3 + z_2)$



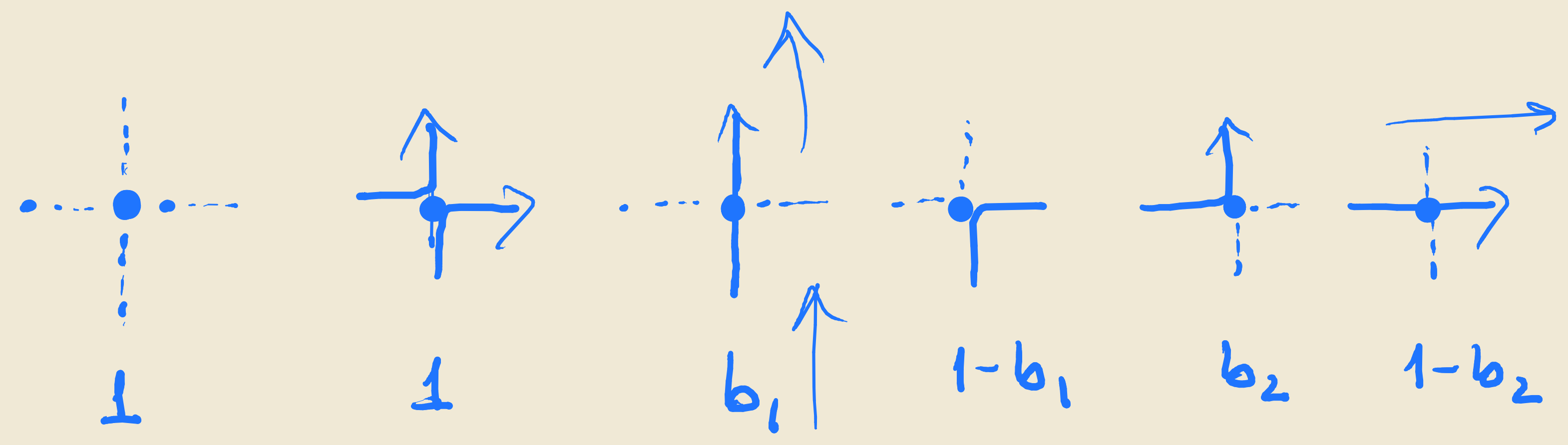
So,

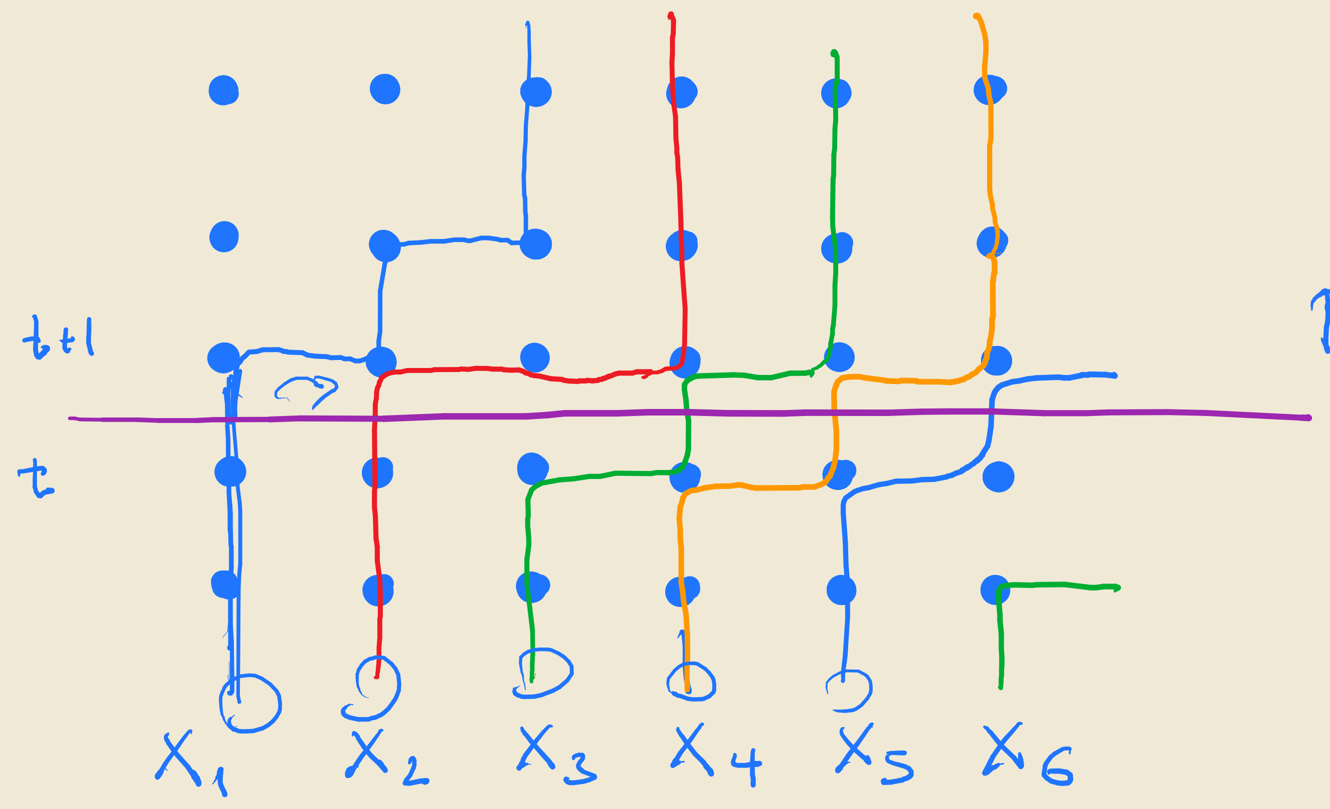
$$(t_3 z_2 + z_3)\quad Z_\lambda(z_1, z_2, z_3, \dots;\ t_1, t_2, t_3, \dots)$$

$$= (t_2 z_3 + z_2)\quad Z_\lambda(z_1, z_3, z_2, \dots;\ t_1, t_3, t_2, \dots)$$

Six Vertex & ASEP

see Borodin-Petrov review
or Borodin-Corwin-Gorin arxiv 1407.6729





particle i jumps with probability b_1 & with jump distribution Geom (b_2) - up to particle $(i+1)$. If jump lands on $(i+1)$, then the latter is pushed by 1.

if $X_{i-1}(t+1) < X_i(t)$

$$\mathbb{P}\left(X_i(t+1) = X_i(t) + k \,\middle|\, \mathbb{X}(t), X_{i-1}(t+1)\right) =$$

$$= \begin{cases} b_1 , & k = 0 \\ (1-b_1)(1-b_2)\, b_2^{k-1} , & 0 < k < X_{i+1}(t) - X_i(t) \\ (1-b_1)\, b_2^{X_{i+1}(t) - X_i(t) - 1} , & k = X_{i+1}(t) - X_i(t) \\ 0 , & \# \end{cases}$$

if $X_{i-1}(t+1) = X_i(t)$

$$\mathbb{P}\left(X_i(t+1) = X_i(t) + k \,\middle|\, \mathbb{X}(t), X_{i-1}(t+1)\right) =$$

$$= \begin{cases} (1-b_2)\, b_2^{k-1} , & 0 < k < X_{i+1}(t) - X_i(t) \\ b_2^{X_{i+1}(t) - X_i(t) - 1} , & k = X_{i+1}(t) - X_i(t) \\ 0 , & \# \end{cases}$$

ASEP limit (see Borodin-Corwin-Gorin 1407.6729 section 2.2)

$$\lim_{\varepsilon \downarrow 0} \mathbb{X}^{b_1^\varepsilon, b_2^\varepsilon}(t/\varepsilon) - t/\varepsilon = \mathrm{ASEP}(t; p, q)$$

for $b_1^\varepsilon = \varepsilon p$ & $b_2^\varepsilon = \varepsilon q$.