+ Forni

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Equi dist^n on some (sigh) nilmanifolds
(Joint work with G. Forni)

General problem: Quantitative estimates on the convergence of Birkhoff averages → for uniquely ergodic systems.

- Surface flows (Forni 2002)
- Horocycle flows (F + Forni 2003)
- Flows on nilmanifolds.

N = nilpotent connected, simply connected Lie gp.

$\Gamma < N$ lattice $\quad X \in \mathrm{Lie}(N)$.

Nilflow: $\Phi_X^t(\Gamma y) = \Gamma y \exp tX$

$$\Gamma\backslash N \xrightarrow{\pi} \Gamma^{(1)}\backslash N^{(1)} \cong \mathbb{T}^d \qquad N^{(1)} = [N, N] \quad \Gamma^{(1)} = [\Gamma, \Gamma]$$

$\Phi_X^t \longmapsto \phi_X \leftarrow$ linear flow

Auslander, Green, Hahn

Φ_X^t uniquely ergodic $\Leftrightarrow$ $\phi_{\bar{X}}^t$ uniquely ergodic $\Leftrightarrow$ $\bar{X}$ irrational vector.

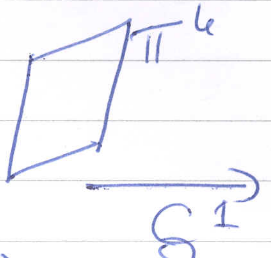
$\mathcal{F}$ = space of metrics on $\Gamma\backslash N$

$\mathcal{I}$ functions on $\mathcal{F}$ with zero average along fibres.

e.g. $L^2_0(\Gamma\backslash N) = \pi^* L^2(N^{(1)}/\Gamma^{(1)})^{\perp}$

A) $Heis^3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & p & r \\ & 1 & q \\ & & 1 \end{pmatrix} \,\middle|\, p,q,r \in \mathbb{R} \right\}$.

$\Gamma = Heis^3(\mathbb{Z})$, $X = \begin{pmatrix} 0 & 1 & * \\ & 0 & \alpha \\ & & 0 \end{pmatrix}$.

B) $M(k) =$ 

$= \mathbb{T}^k \times_B S^1$

$B(y_1, \dots, y_k) = (y_1, y_2 + y_1, \dots, y_k + y_{k-1})$.

$\mathfrak{M}(k) = Fil(k)$

$Lie(Fil(k)) = \langle \xi, \eta_1, \dots, \eta_k \rangle$.

$X = \xi + \alpha_1 \eta_1 + \dots + \alpha_k \eta_k$, $[\xi, \eta_i] = \eta_{i+1}$.

(cf Weyl sums: $\sum_{n=0}^{N} e^{i\alpha n^k}$)

<u>Thm</u>: Suppose α_1 satisfies
$$\|n\alpha_1\| \geq \frac{C}{|n|^{k/2}}$$
Then the flow Φ^t_X on $\mathfrak{M}(k)$ satisfies
$\exists\, \Omega \subset M(k)$ of full measure | $r = r(k)$
such that $\forall f \in C_0^r(M_k)$ $\forall x \in \Omega$
and sufficiently large T we have.

$$\left|\frac{1}{T}\int_0^T f\circ\Phi^t(x)\,d\right| \leq C_\varepsilon\, T^{-\frac{1}{2k(k+1)}+\varepsilon}$$
$$\forall \varepsilon>0$$

Ingredients: C^∞ stability of Φ_X^t
C-ending procedure.

Defn. A flow Φ_X^t is C^∞ stable if $\{Xf \mid f\in C^\infty\}$ is C^∞ closed

HB $\iff$ $Xf=g$ admits a soln iff $g\in \ker(D)$.

$\forall D\in \mathcal{D}_X$ = space of X-inv distns. $\subseteq (C^\infty)'$

C^∞ stable with tame estimates, $\exists r\in\mathbb{R}$ such that

$$\|f\|_t \leq C_{tr}\,\|g\|_{t+r}.$$

Thm Nilflows with diophantine generator are C^∞ stable.

Question: How general is C^∞-stability in the homogeneous setting?

Let γ be the Birkhoff average along a segment of orbit of length T, starting at x, ending at y

$\gamma = D + R$, D = orthogonal projection in H^s on space $\mathcal{D}_X$

$f = f_D + f_R$, $f_D \in \ker D$, $\forall D \in \mathcal{D}_x$

$R(f) = R(f_D + f_R) = R(f_D) = \gamma(f_D)$

$[f_D = X_g]$. $\gamma(X_g) = \frac{1}{T}(g(y) - g(x))$

$$\|R f\|_s \leq \frac{2}{T}(|g(x)| + |g(y)|)$$
$$\leq \frac{2}{T}|g|_\infty \leq \frac{2}{T} C_{sob}(m). \|g\|_\sigma$$
$$\leq \frac{2}{T}. C_{sob}(m) \|f\|_{\sigma+r}.$$

Thus: $\|R\|_{\sigma-r} \leq \frac{C_{sob}}{T}$

Rescaling

$M = {}_{\Gamma}\backslash Heis^3(\mathbb{R})$: $\begin{cases} Aut(Heis^3(\mathbb{R})) \supseteq SL_2(\mathbb{R}) \\ X = \text{eigenvector} \\ A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}) \end{cases}$

A preserves Γ

$\Longrightarrow$ A induces a diffeo ϕ_A ~~of~~ of M
which expands orbits by a factor $\lambda > 1$

ϕ_A leaves invariant $\mathcal{D}_X$

ϕ_A is an homothety of ratio $\lambda^{1/2}$.

($\|\gamma\| \sim \sqrt{\text{length orbit}}$)   Consider $\exp t \log A$.

Main ingredients

- Use Sobolev norms for subelliptic operators which control only transverse direction
- Control the ~~volume~~ transverse volume of a box which injects in a manifold about the segment of length 1