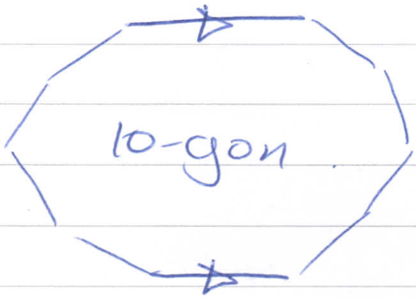
Hamenstädt

Bowen's construction for the Teichmüller flow



Bowen's construction

X = orbifold

ϕ^t continuous flow

$\forall R>0$, $N(R) = \{\gamma \text{ periodic orbit}, |\gamma| \leq R\}$

$$e^{-hR} \sum_{|\gamma| \leq R} \delta_\gamma = \mu_{R,h} \qquad h>0.$$

Take limit – might be zero?

Assume $\begin{cases} \phi^t = \text{smooth Anosov flow} \\ X = \text{cpt.} \\ h = \text{top entropy of } \phi_t \end{cases}$

Bowen $\mu_{R,h} \xrightarrow[\text{weakly}]{} \mu$ (Prob measure).

Unique measure of max entropy

Cor $N(R) \sim e^{hR}/hR$.

Stable & unstable foliations

(Strong) smooth, transverse Hölder.

Margulis: Probability measure μ

– abs. cont.

– conditional on unstable manifolds

- Invariant under holonomy along strong stable
- $\mu^u \circ \phi^t = e^{ht} \mu^u$ | Anosov flows expansive (specification)

Let S = closed oriented surface, genus g (cf Quadratic Abelian differential differentials on S)
finite set $\Sigma \subset S$
family of charts $\phi_i : U_i \subseteq S - \Sigma \hookrightarrow \mathbb{C}$
such that transition functions are translations.
Cone angle at pts in Σ are of the form $2k_i\pi$, $k_i \geq 2$, | Gauss-Bonnet
$(k_1, \ldots, k_s)$, $s = |\Sigma| \geq 1$.

Deformation space of dimension $2g-1+s$.
Define cohomology classes α_1, α_2
$H_1(S, \Sigma, \mathbb{R})$ $\alpha_1 + i\alpha_2$.

Period coordinates (vary α_1, α_2)
If fix α_1 and vary locally α_2
$\longrightarrow$ subspace of half dimension.

Get $SL(2,\mathbb{R})$-action on stratum $\mathcal{S}$
by a post composition of charts

Diagonal flow $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$

Teichmüller flow (expands α_1, contracts α_2).

Take fixed foliations as strong stable & strong unstable.

Lebesgue measure is invariant under the diagonal flow.

– Measure & Veech = finite

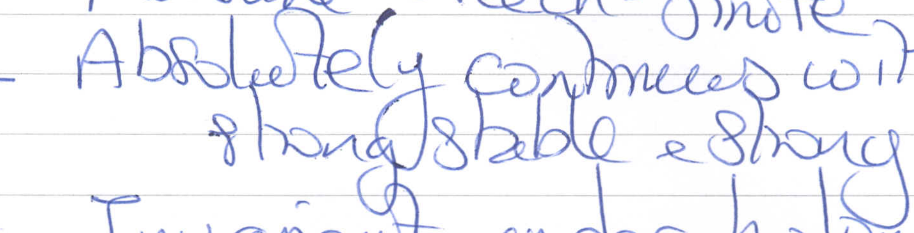
– Absolutely continuous with respect to strong stable & strong unstable fol's.

Invariant under holonomy in the strong stable direction

λ^u = conditional measure on unstable manifold. $\lambda^u \phi_t = e^{ht} \lambda^u$.

$h = 2g - 1 + s$.

shorten

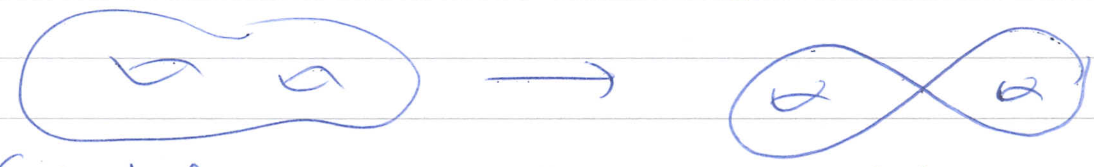


1 singularity cone angle 6π.

Surface
2 singularities of cone angle 4π

(Space of metrics not closed).

Can also move out of surfaces



(Moduli space not compact).

Thm: $e^{-h} \sum_{|\gamma| \leq R} \delta_\gamma \xrightarrow{\text{weakly}}$ Lebesgue $\boxed{h = 2g-1+s}$

Cor $\#\left\{\begin{array}{l}\text{periodic orbits which} \\ \text{meet some fixed cpt set}\end{array} \text{length} \leq R\right\}$
is asymptotic to e^{hR}/hR.

Eskin-Mirzakhani-Rafi: Orbits deep in the cusp grow slower than e^{-hR} ($e^{(h-1+\varepsilon)R}$

Thm 2: If $k_i \neq 2$ for at least one i then $\forall \varepsilon > 0$ $\exists K \subseteq S$ (Stratum) s.t. for sufficiently large $R > 0$,
periodic orbits which do not intersect K $\geq e^{(h-1+\varepsilon)R}$.

If $k_i = 2$ $\forall i \Rightarrow \geq e^{(h-2+\varepsilon)R}$

Strategy of proof (follows classical approac

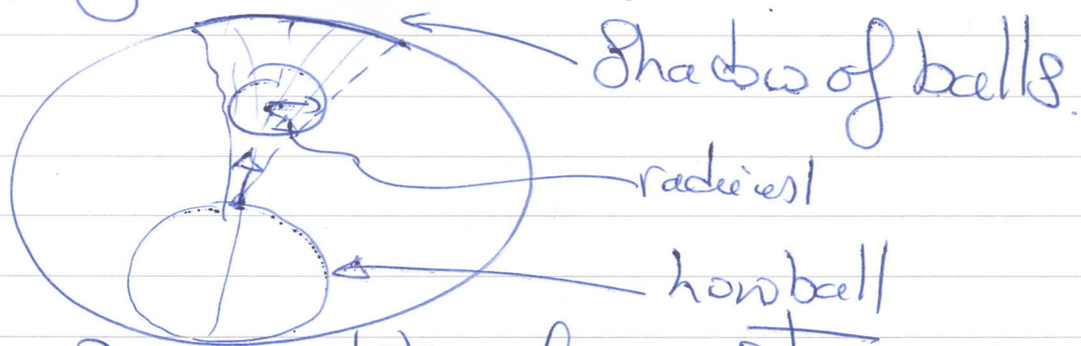
Strategy: $e^{-hR}\sum_{|\gamma|\le R} \delta_\gamma = \mu_R$.

Take a weak limit $\mu = \mu_r + \mu_d$

(μ_r = restriction of μ to recurrent pts,
μ_d = " " divergent pts.)

- Need to show $\mu_d = 0$
- Need to show μ_r = Lebesgue.

For -vely curved manifolds:



Project conditional on strong unstable manifolds to measure class on bd.

Closed balls form a Vitali relation for measure class (allows to show measures abs. cont.)

However – no bdy.

Curve graph – vertices = simple closed curves
connected by edge iff can be realized disjointly

Masur-Minsky: Hyperbolic (sense of Gromov)

Map from "flat surfaces" $\longrightarrow$ curve, graph.

$\rightsquigarrow$ short geodesic.

Unparametrized quasi-geodesics.