(+ Benoist)
Quint.

Dynamics on finite volume homogeneous spaces.

G = Lie group
Λ = lattice in G.

G/Λ admits finite G-invariant measure.

$\Gamma < G$ subgp.
$x \in G/\Lambda$.
Describe $\overline{\Gamma x}$?

eg ① G = nilpotent
Λ = cocompact
② $G = SL_d(\mathbb{R})$
$\Lambda = SL_d(\mathbb{Z})$
Γ = any subgp.

③ $\Gamma < SL_d(\mathbb{Z})$
$\Gamma \curvearrowright \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. $\Big\}$ $\xrightarrow{\text{embed}}$ $\dfrac{SL_d(\mathbb{R}) \ltimes \mathbb{R}^d}{SL_d(\mathbb{Z}) \ltimes \mathbb{Z}^d}$.

Special case of Ratner's Thm.

$\Gamma < G$ spanned by Ad-unipotent one parameter subgps

then $\forall x \in G/\Lambda$ $\exists H < G$ closed

$\overline{\Gamma x} = Hx$ and Hx carries a finite H-invariant measure, ie, $x = g\Lambda$. $H \cap g\Lambda g^{-1}$ lattice in H.

One parameter gp: $\begin{pmatrix} 1 & \phi(t) \\ 0 & 1 \end{pmatrix}$

Say that $\overline{\Gamma x}$ is homogeneous.

<u>example</u>: horocycle flow

$\Gamma = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\} \leq SL_2(\mathbb{R})$

$\Lambda = SL_2(\mathbb{Z})$, $G = SL_2(\mathbb{R})$.

(Orbits closed or dense).

When can we make weaker assumptions on Γ?

(Clearly not true for geodesic flow = Anosov flow).

$\mathfrak{g} = Lie(G)$.

$Ad : G \longrightarrow GL(\mathfrak{g})$

<u>Theorem</u>: (Benoist + Q) $Ad\,\Gamma$ has semi-sp(…) Zariski closure. Then $\forall x \in G/\Lambda$ $\overline{\Gamma x}$ is homogeneous.

<u>Ex</u>: $G = SL_2(\mathbb{R})$, $\Lambda = SL_2(\mathbb{Z})$

$\Gamma = \left\langle \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \right\rangle$ - Zariski dense

Every Γ-orbit in $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ is finite or dense.

Every Γ-orbit in $\mathbb{T}^2$ is finite or dense comes from the torus case dealt with by Guivarc'h - Starkov, Muchnik (2003, 2005).

Strategy – follow lines of Ratner's theorem.

Enough to consider $\mathrm{Ad}\,\Gamma \subseteq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

<u>Thm</u> (Ratner–metric theorem)

Γ – spanned by Ad-unipotent 1-parameter subgps then if ν is a Γ-ergodic prob measure on G/Λ $\exists\, x \in G/\Lambda$

$H < G$, $\nu(Hx) = 1$ and $\nu = H$-invariant.

$\Gamma \curvearrowright X$ $\mu =$ prob measure on Γ.

$x \xrightarrow{\gamma} \gamma x$ Random transformation (Markov chain).

$\mu * \nu = \int_\Gamma \gamma_* \nu \, d\mu(\gamma)$ | $\nu = \mu$-stationary if $\mu * \nu = \nu$.

<u>Thm</u> $\mu =$ prob measure on G cptly supported on Zariski closure of the image of μ by Ad is semi-simple. <u>Then</u> every ergodic μ-stationary prob measure on G/Λ is homogeneous.

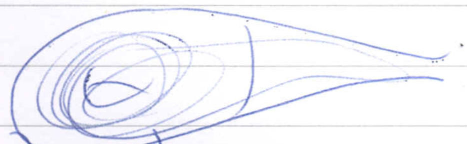
<u>NB</u> Torus case – Bourgain–Furman–Lindenstrauss–Mozes (2007)

Measure theoretic $\Longrightarrow$ Top. result.

<u>Thm</u> (Dani – Margulis)

G, Λ $\quad u_t =$ 1-parameter Ad-unipotent flow then $\forall x \in G/\Lambda$ $\forall \varepsilon > 0$ $\exists K \subseteq G/\Lambda$ s.t. $\forall T > 0$

$$\frac{1}{T} \lambda\{t \in [0,T] \mid u_t x \in K\} \geq 1-\varepsilon$$



Gives limit points have measures.
(Uses structure of cusp $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$)

<u>Thm</u> (Benoist – Q) $\mu =$ cptly supported prob. measure on G: supp$(\mathrm{Ad}\,\mu)$ has a semi simple Zariski closure. Then $\forall x \in G/\Lambda$, $\forall \varepsilon > 0$ $\exists K \subset G/\Lambda$, $\forall n \in \mathbb{N}$

$$\mu^{*n} * \delta_x(K) \geq 1-\varepsilon.$$

[New result]

Eskin – Margulis – special case.

<u>Thm</u> (Nevo + Stein) $\Gamma = \langle g, h \rangle$ free

$\Gamma \circlearrowright (X, \nu)$ ergodic

A $\phi : X \to \mathbb{R}$ $\}$ $\phi \circ g = -\phi = \phi \circ h$.

$S_n \in \Gamma$ sphere word length. v.a.e.

$x \in X$ $\frac{1}{\# S_n} \sum_{\gamma \in S_n} \varphi(\gamma x) \to \int \phi \, d\nu$.

Bufetov - Martingale Convergence
(some subsets of gps).

γx, γy $\quad y = e^{X} x$.

$\gamma y = e^{(\mathrm{Ad}\,\gamma)X} \gamma x$.