Equivalence relations and random graphs: an introduction to \textit{graphical dynamics}

Vadim A. Kaimanovich

University of Ottawa

April 19, 2012

Warwick
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Given an equation involving any number of fluent quantities, to find the fluxions, and vice versa

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Second letter of Newton to Leibniz (1676)

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Given an equation involving any number of fluent quantities, to find the fluxions, and vice versa

It is useful to solve differential equations!
It is useful to consider invariant measures!

Classical examples

- smooth dynamics;
- measurable dynamics;
- symbolic dynamics
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**graphical** dynamics?!
Definition

A graph $\Gamma$ is determined by its set of vertices (nodes) $V$ and its set of edges (links) $E$ connected by an incidence relation (further “decoration” is possible!).

Structured “big” set $\rightarrow$ local structure $\rightarrow$ graph structure

How can one understand a collection of (large) finite objects?

finite objects $\rightarrow$ infinite objects $\rightarrow$ invariant measures

finite words $\rightarrow$ infinite words $A^\mathbb{Z}$ $\rightarrow$ ✓ information theory

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Holonomy invariant measures on foliations

Measured equivalence relations (Feldman–Moore 1977)

\((X, \mu)\) — a Lebesgue probability space

\(R \subset X \times X\) — a Borel equivalence relation with at most countable classes (examples: orbit equivalence relations of group actions, traces on transversals in foliations, etc.)

A partial transformation of \(R\) — a measurable bijection \(\varphi : A \rightarrow B\) with graph \(\varphi \subset R\)

Definition

The measure \(\mu\) is \(R\)-invariant if \(\varphi\mu_A = \mu_B\) for any partial transformation of \(R\).

One can also talk about quasi-invariant measures and the associated Radon–Nikodym cocycle.

Plante 1975

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$$d\#_{\mu}(x, y) = d\mu(x)d\#_{x}(y),$$

where $\#_{x}$ is the counting measure on the fiber $p^{-1}(x)$ of the projection $p : R \to X$ (i.e., on the equivalence class of $x$).

The **involution** $[(x, y) \mapsto (y, x)]$ of $\#_{\mu}$ is the right counting measure $\#^{\mu}$, and $\mu$ is $R$-quasi-invariant $\iff \#_{\mu} \sim \#^{\mu}$

Definition (Feldman–Moore 1977)

$$D(x, y) = \frac{d\#^{\mu}}{d\#_{\mu}}(x, y) = \frac{d\mu(y)}{d\mu(x)}$$

is the (multiplicative) Radon–Nikodym cocycle.

$\mu$ is invariant $\iff D \equiv 1$
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\[ K \subset R \] — a leafwise graph structure on an equivalence relation \( R \); \( (X, \mu, R, K) \) — a graphed equivalence relation.

A discrete analogue of Riemannian foliations. Further “decoration” is possible! (edge length, labelling, colouring etc.). One can consider structures of higher dimensional leafwise abstract simplicial complexes.

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**Definition**

The **simple random walk** on a (locally finite) graph $\Gamma$ is the Markov chain with the transition probabilities

$$p(x, y) = \begin{cases} 
\frac{1}{\deg x}, & x \sim y; \\
0, & \text{otherwise.}
\end{cases}$$

In the same way one defines the simple random walk along classes of a graphed equivalence relation $(X, \mu, R, K)$, cf. leafwise Brownian motion on foliations (Garnett 1983).

**Theorem (K 1988, 1998)**

A measure $\mu$ on a graphed equivalence relation $(X, m, R, K)$ is $R$-invariant $\iff$ the measure $m = \deg \cdot \mu$ is stationary and reversible with respect to the SRW on $X$.

**Idea of proof:** Reversibility $\equiv$ involution invariance of the joint distribution of $(x_0, x_1) \equiv$ involution invariance of $\#_\mu|_K$.  

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**Equivalence relations and random graphs: an introduction to graphical dynamics**

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- Invariance beyond groups
- Counting measure
- Graphed equivalence relations

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**Definition (K)**

A graphed equivalence relation \((X, R, K)\) on a topological state space \(X\) is **continuous** if the map \(x \mapsto \pi_x\) is continuous (with respect to the weak* topology on \(M(X)\)).

**Theorem (K)**

If a graphed equivalence relation \((X, R, K)\) is continuous, then the space of \(R\)-invariant measures is weak* closed.

**Idea of proof:** Use closedness of the space of stationary measures of the simple random walk and correspondence with reversible stationary measures.
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Weaker form: a finite *stationary measure* for the leafwise simple random walk (∝ stationary scenery). Is the same as strong homogenization if the measure is, in addition, reversible.

**Observation**

An *invariant measure* need not exist! Compactness of the state space implies existence of a *stationary* one (cf. Garnett’s *harmonic measures* for foliations).
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Group actions

- Random perturbations of Cayley graphs (extreme case: percolation)
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The equivalence class of a graph $\Gamma$ is the quotient

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$\Gamma$ is vertex transitive $\iff [\Gamma] = \{\cdot\}$

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If a.e. graph in a graphed equivalence relation $(X, \mu, R, K)$ with $R$-invariant measure $\mu$ is rigid, than the image measure $\pi(\mu)$ on $G$ is $R$-invariant.

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For a finite graph the invariant measure is equidistributed on its equivalence class, whereas the unimodular measure is the quotient of the uniform measure on the graph itself.

Theorem (K – uses an appropriate Markov chain on $G$)

The space of unimodular measures on $G$ is weak* closed — the space of invariant ones is not!

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\[ \Delta(x, y) = \frac{|G_x y|}{|G_y x|} \quad \text{— the modular cocycle of } \Gamma. \]

\( \Delta \) determines a multiplicative cocycle of the equivalence relation \( \mathcal{R} \) restricted to the subset \( \mathcal{G}^0 \subset \mathcal{G} \) of graphs \( \Gamma \) with unimodular \( \text{Iso}(\Gamma) \).

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Applications to “real life”
La guerre!!!
C’est une chose trop grave pour la confier à des militaires!
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General Jack D. Ripper
(Dr. Strangelove or: How I Learned to Stop Worrying and Love the Bomb, Stanley Kubrik 1964)

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Thank you!
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Equivalence relations and random graphs: an introduction to graphical dynamics

Vadim A. Kaimanovich

Robert Hooke (1676)

Ut tensio, sic vis!
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celiiinossstttuvw

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La filosofia naturale è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi agli occhi [...] Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi ed altre figure geometriche, senza i quali mezzi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro labirinto.
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Euclidean lattice ($\mathbb{Z}^2$)  Bethe lattice (*free group $\mathcal{F}_2$*)

$A$ is a finite alphabet

$A^G$ — the **space of configurations**

The group $G$ acts on $A^G = \{(a_g)\}_{g \in G}$ by translations

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Dating in a high school
“Roman” encoding (1 ←→ I, 2 ←→ II, 3 ←→ III)
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\[ G \text{ — group, } K \text{ — (symmetric) generating set} \]
\[ \text{Cayley}(G, K) := \text{vertices } V = G, \]
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Edges are labelled!

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Holonomy invariant measures on foliations
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random fields on edges

extreme case: site percolation

extreme case: bond percolation
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For an action $G : X \circlearrowright$

$$X \ni x \mapsto \text{Stab}_x = \{ g \in G : gx = x \} \subset G \quad (*)$$

In the presence of a generating set $K \subset G$ a subgroup $H \subset G$ determines the associated graph $\text{Schreier}(X, G, K)$ on $X = G/H$ rooted at $o = \{ H \} \in X$, and vice versa.

If $m$ is an invariant measure on $X$, then its image under $(*)$ is a $G$-invariant measure on subgroups of $G$ (≡ an invariant measure on the space of Schreier graphs).

**Definition (Vershik 2010)**

An action $G : (X, m) \circlearrowright$ is extremely non-free if $(*)$ is a bijection (mod 0).

Extremely non-free actions of $G$ ≡ invariant measures on the space of Schreier graphs of $G$ ≡ stochastically homogeneous random Schreier graphs.
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Fractal sets arising from \textit{Iterated Function Systems} (e.g., the \textbf{Sierpiński triangle}) give rise to the associated graphs:

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serp.eps
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ext.eps
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\begin{align*}
(T\bar{\omega})_n &= \omega_{n+1} \quad \text{— the shift on } \Omega = \{0, 1\}^\mathbb{Z} = \{\bar{\omega} = (\omega_n)_{n \in \mathbb{Z}}\} \\
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\end{align*}

The skew action \( \alpha(\bar{\omega}, g) = (T\bar{\omega}, \alpha^{\omega_0} g) \) of the free group \( \mathcal{F}_2 = \langle a, b \rangle \) (where \( \alpha = a, b \)) determines a \textbf{stochastically homogeneous Schreier graph} (“slowed down” Cayley tree).

Geometrically: \( \chi = \#a + \#b - \#a^{-1} - \#b^{-1} : \mathcal{F}_2 \to \mathbb{Z} \) — the \textbf{signed letter counting character}. If \( \omega_n = 0 \), then any two edges with a common endpoint between \( \chi^{-1}(n) \) and \( \chi^{-1}(n+1) \) in the Cayley tree of \( \mathcal{F}_2 \) are “glued” together.

\textbf{Another example: } \( m \) — shift-invariant measure on bilateral infinite irreducible words in \( \mathcal{F}_2 \) (invariant measure of the \textbf{geodesic flow}), produces by “doubling” the associated \textbf{stochastically homogeneous Schreier graph} (or consider \( \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \) instead of \( \mathcal{F}_2 \) — Elek 2011).

The associated action of the free group is \textbf{amenable} and \textbf{effective}. 

Vadim A. Kaimanovich
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The associated action of the free group is **amenable** and **effective**.
Realizations of a **branching** (*Galton-Watson*) **process** with offspring distribution \( p = (0, p_1, p_2, \ldots, p_k) \) are rooted trees:

The arising measure \( P \) on rooted trees is **not** invariant (the root is statistically different from other vertices!).

**Solution:** consider **augmented GW trees**: add by force one offspring to the root, i.e., use \( \tilde{p} = (0, 0, p_1, p_2, \ldots) \) for the first generation, and \( p \) otherwise.

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![GW2.eps]

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\tilde{\mathcal{P}}(A) = p_1 \cdot p_2^2 \cdot 4p_1p_2^3 = 4p_1^2p_2^5.
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\[
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3/2 = \frac{\tilde{\mathcal{P}}(A')}{\tilde{\mathcal{P}}(A)} = \frac{\deg o'}{\deg o}
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Invariant and quotient measures on the equivalence class of a finite graph
Equivalence relations and random graphs: an introduction to graphical dynamics

Vadim A. Kaimanovich

Galton–Watson trees

Augmented measure

\[ \frac{2}{3}, \frac{1}{3} \]