Infinite-dimensional stochastic differential equations related to random matrices

- Ginibre RPF, Sine RPF, Bessel RPF, Airy RPF
- General theory for ISDEs:
  quasi-Gibbs property & log derivative
- Ginibre Interacting Brownian motions
- Palm measures of Ginibre RPF
- Homogenization of diffusion in 2D Coulomb environment

Let $S = \mathbb{R}^d$, $\mathbb{C}$, $[0, \infty)$.

S: Configuration space over S

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\, ;\ s_i \in S,\ \ \mathsf{s}(|s| < r) < \infty\ \ (\forall r \in \mathbb{N})\}$$

μ: RPF over S. i.e. prob meas. on S.

Prob: (1) To construct a *natural* stochastic dynamics

$$\mathbf{X}_t = (X_t^i)_{i\in\mathbb{N}} \quad \text{(labeled dynamics)}$$

related to μ, *i.e.*

$$\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i} \quad \text{(unlabeled dynamics)}$$

is reversible w.r.t. μ.

(2) To find the ∞-dim. SDE that $\mathbf{X}_t$ satisfies.

• ρ^n is called the n-correlation function of μ w.r.t. Radon m. m if

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(\mathbf{x}_n) \prod_{i=1}^{n} m(dx_i) = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!} d\mu$$

for any disjoint $A_i \in \mathcal{B}(S)$, $k_i \in \mathbb{N}$ s.t. $k_1 + \ldots + k_m = n$.

• μ is called the determinantal RPF generated by (K, m) if its n-corraltion fun. ρ^n is given by

$$\rho^n(\mathbf{x}_n) = \det[K(x_i, x_j)]_{1\le i,j\le n}$$

• Ginibre RPF $S = \mathbb{C}$. μ_{gin} is generated by $(K_{\mathsf{gin}}, \mathrm{g})$

$$K_{\mathsf{gin}}(x,y) = e^{x\bar{y}} \qquad \mathrm{g}(dx) = \pi^{-1} e^{-|x|^2} dx$$

Property of Ginibre RPF

(g1) μ_{gin} is translation and rotation invariant

(g2) μ_{gin} is the weak limit of μ_{gin}^N:

the labeled expression $\check{\mu}_{\text{gin}}^N$ of μ_{gin}^N is

$$\check{\mu}_{\text{gin}}^N = \frac{1}{Z} \prod_{i<j}^{N} |x_i - x_j|^2 \prod_{k=1}^{N} \mathrm{g}(dx_k) \tag{1}$$

μ_{gin}^N is the determinantal RPF gen. by $(K_{\text{gin}}^N, \mathrm{g})$, where

$$K_{\text{gin}}^N(x,y) = \sum_{i=0}^{N-1} \frac{(x\bar{y})^i}{i!}$$

Non rigorous expression of μ_{gin} as a measure $\bar{\mu_{\mathsf{gin}}}$ on $\mathbb{C}^{\mathbb{N}}$:

From (g2)

$$\bar{\mu_{\mathsf{gin}}} = \frac{1}{Z} \prod_{i<j}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} \frac{e^{-|x_k|^2}}{\pi} dx_k \tag{2}$$

From the translation invariance we have another informal expression:

$$\bar{\mu_{\mathsf{gin}}} = \frac{1}{Z} \prod_{i<j}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} dx_k \tag{3}$$

Which representations are correct?

Non rigorous expression of μ_{gin} as a measure $\mu_{\bar{\text{gin}}}$ on $\mathbb{C}^{\mathbb{N}}$:

From (g2)

$$\mu_{\bar{\text{gin}}} = \frac{1}{Z} \lim_{r\to\infty} \prod_{i<j,|x_i|,|x_j|<r}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} \frac{e^{-|x_k|^2}}{\pi} dx_k \qquad (4)$$

From the translation invariance we have another informal expression:

$$\mu_{\bar{\text{gin}}} = \frac{1}{Z} \lim_{r\to\infty} \prod_{i<j,|x_i-x_j|<r}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} dx_k \qquad (5)$$

Which representations are correct?

Both

Log derivative

- Let μ^1 be the 1-Campbell measure on $\mathbb{R}^d \times \mathsf{S}$:

$$\mu^1(A \times B) = \int_A \rho^1(x)\mu_x(B)dx$$

Here $\mu_x(\cdot) = \mu(\cdot - \delta_x | \mathsf{s}(x) \geq 1)$ is the Palm m. at x.

- $\mathsf{d}_\mu \in L^1(\mathbb{R}^d \times \mathsf{S}, \mu^1)$ is called the log derivative of μ if

$$\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^1 = - \int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}_\mu d\mu^1 \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}$$

Here ∇_x is the nabla on $\mathbb{R}^d$, $\mathcal{D}$ is the space of local smooth functions on S with compact support.

- Very informally

$$\mathsf{d}_\mu = \nabla_x \log \mu^1$$

- Ginibre RPF: $\mathsf{d}_{\mu_{\mathrm{gin}}}$ has plural representations

$$\mathsf{d}_{\mu_{\mathrm{gin}}}(x,\mathsf{y}) = -2x + 2\lim_{r\to\infty}\sum_{|y_i|<r}\frac{x-y_i}{|x-y_i|^2} \qquad \text{in } L^2_{\mathrm{loc}}(\mu^1)$$

$$\mathsf{d}_{\mu_{\mathrm{gin}}}(x,\mathsf{y}) = 2\lim_{r\to\infty}\sum_{|x-y_i|<r}\frac{x-y_i}{|x-y_i|^2} \qquad \text{in } L^2_{\mathrm{loc}}(\mu^1)$$

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- These corespond to the following:

$$\bar{\mu_{\mathsf{gin}}} = \frac{1}{Z} \lim_{r\to\infty} \prod_{i<j,|x_i|,|x_j|<r}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} \frac{e^{-|x_k|^2}}{\pi} dx_k \tag{4}$$

$$\bar{\mu_{\mathsf{gin}}} = \frac{1}{Z} \lim_{r\to\infty} \prod_{i<j,|x_i-x_j|<r}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} dx_k \tag{5}$$

(A1) ρ^k are locally bounded for all $k \in \mathbb{N}$
(A2) The log derivative $\mathsf{d}_\mu \in L^1_{\mathrm{loc}}(\mu^1)$ exists
(A3) μ is a quasi-Gibbs measure
(A4) $\{X_t^i\}$ do not collide each other (non-collision)
(A5) each tagged particle X_t^i never explode (non-explosion)

Let $\mathfrak{u}: S^{\mathbb{N}} \to \mathsf{S}$ such that $\mathfrak{u}((s_i)) = \sum_i \delta_{s_i}$.

Thm 1. *Assume* (A1)–(A5). *Then* $\exists \mathsf{S}_0 \subset \mathsf{S}$ *such that*

$$\mu(\mathsf{S}_0) = 1, \tag{6}$$

and that, for $\forall \mathsf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$, $\exists \mathfrak{u}^{-1}(\mathsf{S}_0)$*-valued pr.* $(X_t^i)_{i\in\mathbb{N}}$ *and* $\exists S^{\mathbb{N}}$*-valued Brownian m.* $(B_t^i)_{i\in\mathbb{N}}$ *satisfying*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}_\mu(X_t^i, \sum_{j\neq i} \delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s} \tag{7}$$

How to relate the SDE with the equilibrium state μ.

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}_\mu(X_t^i, \sum_{\mathsf{j}\neq\mathsf{i}} \delta_{X_t^j})dt, \qquad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

Thm 2 (O. (JMSJ 09)). (1) *The family of processes* $\{(X_t^i)_{i\in\mathbb{N}}\}$ *is a diffusion with state space* $\mathfrak{u}^{-1}(\mathsf{S}_0)$.
(2) *The associated unlabeled process* $\{\mathsf{X}_t\}$ *is a diffusion with state space* S_0. *Here* $\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i}$.
(3) $\{\mathsf{X}_t\}$ *is reversible w.r.t.* μ.

Remark 1. (1) (A1)–(A5) can be checked for Ginibre RPF ($\beta = 2$), Sine RPFs, Airy RPFs and Bessel RPFs ($\beta = 1, 2, 4$).
(2) We can calcurate the log derivatives of these measures.
(3) We have general theorems for quasi-Gibbs property and the log derivatives (O. PTRF, AOP). The statements are too messy to be omited here.

Examples: By Theorem 1 and 2 we have the following:

Ginibre RPF: When $\mu = \mu_{\text{gin}}$,

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{\substack{|X_t^i - X_t^j| < r, \\ j \neq i}} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \tag{8}$$

and also

$$dX_t^i = dB_t^i - X_t^i dt + \lim_{r\to\infty} \sum_{\substack{|X_t^j| < r \\ j \neq i}} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \tag{9}$$

Sine$_\beta$ RPF: $S = R$, $\beta = 1, 2, 4$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^j|<r,\ j\neq i} \frac{1}{X_t^i - X_t^j} dt$$

Since $d = 1$, we have

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\ j\neq i} \frac{1}{X_t^i - X_t^j} dt$$

Spohn (1987) considered the case $\beta = 2$:

$$dX_t^i = dB_t^i + \sum_{j\neq i} \frac{1}{X_t^i - X_t^j} dt$$

He constructed the dynamics as a Markov semigroup by Dirichlet form.

Bessel RPF (with Honda):

$S = [0, \infty)$, $\beta = 1, 2, 4$, $a > 1$

$$dX_t^i = dB_t^i + \frac{a}{2X_t^i}dt + \lim_{r\to\infty}\frac{\beta}{2}\sum_{\substack{|X_t^j|<r \\ j\neq i}}\frac{1}{X_t^i - X_t^j}dt$$

Airy RPF is more complicated.

(Joint work with Tanemura)

quasi-Gibbs m.: Ψ: Ruelle class interaction potensial,

$Q_r = \{|x| \le r\}$, $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap Q_r)$, $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap Q_r^c)$

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | \mathsf{s}(Q_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi))$$

- μ is called (Φ, Ψ)-Gibbs m. if it satisfies DLR eq:

$$d\mu_{r,\xi}^m = \frac{1}{z_{r,\xi}} e^{-\mathcal{H}_r - \mathcal{W}_{r,\xi}} \prod_{k=1}^m e^{-\Phi(s_k)} ds_k$$

$$\mathcal{H}_r(\mathsf{s}) = \sum_{\substack{s_i, s_j \in Q_r, \\ i<j}} \Psi(s_i - s_j), \quad \mathcal{W}_{r,\xi} = \sum_{\substack{s_i \in Q_r, \\ \xi_j \in Q_r^c}} \Psi(s_i - \xi_j)$$

$$\mu = \frac{1}{Z} e^{-\sum_{i<j}^{\infty} \Psi(x_i - x_j)} \prod_{k=1}^{\infty} e^{-\Phi(x_k)} dx_k \quad \text{(informally)}$$

- $\Phi = \Psi = 0$: Poisson rpf: $\Lambda = \frac{1}{Z} \prod_{i \in \mathbb{N}} dx_i$.
- Ginibre RPF: $\Phi = 0$ $\Psi(x) = -2 \log |x|$

$\mathcal{W}_{r,\xi}$ diverge, so DLR does not make sense

Gibbs m. Let $\nu_r^m = \prod_{k=1}^m 1_{Q_r}(s_k) e^{-\Phi(s_k)} ds_k$

$$d\mu_{r,\xi}^m = \frac{1}{z_{r,\xi}^m} e^{-\mathcal{H}_r - \mathcal{W}_{r,\xi}} d\nu_r^m \qquad \text{(DLR eq)}$$

quasi-Gibbs m. $\exists\ c_{r,\xi}^m$

$$\frac{1}{c_{r,\xi}^m} e^{-\mathcal{H}_r} d\nu_r^m \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\nu_r^m$$

- If μ is Ginibre RPF, $\mathcal{W}_{r,\xi}$ and $z_{r,\xi}^m$ diverge. But $e^{-\mathcal{W}_{r,\xi}}/z_{r,\xi}^m$ conv.

$$\frac{1}{c_{r,\xi}^m} \le \frac{e^{-\mathcal{W}_{r,\xi}}}{z_{r,\xi}^m} \le c_{r,\xi}^m$$

- Quasi-Gibbs is very mild restriction. If μ is (Φ, Ψ)-quasi-Gibbs m, then μ is also $(\Phi + f, \Psi)$-quasi Gibbs m for any loc bdd m'able f.

Unlabeled level construction Let $\mathbb{D}$ be the canonical square field on S: $\mathsf{s} = \sum_i \delta_{s_i}$, $\mathbf{s} = (s_i)$.

$$\mathbb{D}[f, g](\mathsf{s}) = \frac{1}{2} \sum_i \nabla_{s_i} \tilde{f}(\mathbf{s}) \cdot \nabla_{s_i} \tilde{g}(\mathbf{s})$$

Let $\mathcal{D}$ be the set of local smooth fun with cpt support.

$$\mathcal{E}^{\mu}(f, g) = \int_{\mathsf{S}} \mathbb{D}[f, g] d\mu$$

Thm 3. (1) *If μ is quasi-Gibbs, then $(\mathcal{E}^{\mu}, \mathcal{D})$ is closable on $L^2(\mathsf{S}, \mu)$.* (2) *If $(\mathcal{E}^{\mu}, \mathcal{D})$ is closable on $L^2(\mathsf{S}, \mu)$ and* (A.1) *is satisfied, then there exists diffusion X_t associated with the closure of $(\mathcal{E}^{\mu}, \mathcal{D})$ on $L^2(\mathsf{S}, \mu)$.*

If μ is Poisson rpf with Lebesgue intensity, then $\mathsf{X}_t = \sum_i \delta_{B_t^i}$.

Thm 4. *Ginibre RPF ($\beta = 2$), Sine RPFs, Airy RPFs and Bessel RPFs ($\beta = 1, 2, 4$) are qausi-Gibbs m. for* $\Psi(x) = -\beta \log |x|$.

- The key point of the proof is to use the small fluctuation property (SFP) of linear statistics for these measures.
- SFP was established by Soshnikov (Sine, Airy, Bessel RPFs), Shirai (Ginibre RPF).
- Proof consists of several parts:

(1) To find a good finite particle approximation $\{\mu^N\}$
(2) To prove uniform *small fluctuation* of $\{\mu^N\}$
(3) To prove uni bounds of 1 & 2 cor funs of $\{\mu^N\}$
(4) To carry out the limiting procedure of d_{μ^N} & quasi-Gibbs property. (General theorems to appear in O. PTRF, AOP)

Related problems:

- Yoo proved that Determinantal RPF with

$$\text{Spec}(K) \subset [0, 1)$$

are *Gibbs* measures. So it is likely all Determinantal RPF are quasi Gibbs measures, i.e., under the condition

$$\text{Spec}(K) \subset [0, 1]$$

To strength Yoo's result like this is important because RPFs in infinite volume appeared in RMT usually satisfy that

$$\text{Spec}(K) = \{0, 1\}$$

- To calculate the log derivative of Determinantal RPFs.

- β ensemble of Sine, Bessel, Airy for general $\beta > 0$:

(Valkó, B.-Virág, B., Ramírez, J.-Rider, B.-Virág, B.)

Good finite approximations are clear: Log gasses.

The problem is to control correlation functions and to prove small fluctuations.

- The spectrum of Gaussian Analitic functions

(Some progress done by Shirai)

- In particular, GAF with Bergmann Kernel

Geometric property of Ginibre RPF

Rider, Goldman, Kostlan, Shirai

Palm meas. For $\mathbf{x} = \{x_1, \ldots, x_{\mathsf{m}}\} \subset S^{\mathsf{m}}$ set

$$\mu_{\mathbf{x}} := \mu(\cdot - \sum_{l=1}^{\mathsf{m}} \delta_{x_l} \mid \mathsf{s}(\{x_l\}) \geq 1(\forall l))$$

Thm 5 (with Shirai). *Let* $\mathsf{m}, \mathsf{n} \in \{0\} \cup \mathbb{N}$. *Then*

(1) *If* $\mathsf{m} = \mathsf{n}$, *then* $\mu_{\mathbf{x}} \sim \mu_{\mathbf{y}}$. *($\sim$ means ab. cont.)*

(2) *If* $\mathsf{m} \neq \mathsf{n}$, *then* $\mu_{\mathbf{x}}$ *and* $\mu_{\mathbf{y}}$ *are singular each other.*

Remark: • In case of Gibbs measures, it holds always

$$\mu_{\mathbf{x}} \prec \mu$$

• In this sense Ginibre RPF is similar to periodic RPF.

Thm 6 (with Shirai). *Suppose* $\mathsf{m} = \mathsf{n}$. *Then for* $\mu_{\mathbf{y}}$*-a.s.* $\mathsf{s} = \sum_i \delta_{s_i}$,

$$\frac{d\mu_{\mathbf{x}}}{d\mu_{\mathbf{y}}} = \frac{\Delta^{\mathsf{m}}(\mathbf{x}) \det[K_{\mathsf{gin}}(x_i, x_j)]_{i,j=1}^{\mathsf{m}}}{\Delta^{\mathsf{m}}(\mathbf{y}) \det[K_{\mathsf{gin}}(y_i, y_j)]_{i,j=1}^{\mathsf{m}}} \lim_{r\to\infty} \prod_{|s_i|<b_r} \frac{|\mathbf{x} - s_i|^2}{|\mathbf{y} - s_i|^2}$$

cpt uni in $\mathbf{x} \in \mathbb{C}^{\mathsf{m}}$.

- $\{b_r\}_{r\in\mathbb{N}}$: $\lim b_r = \infty$
- $|\mathbf{x} - s_i| = \prod_{m=1}^{\mathsf{m}} |x_m - s_i|$ *for* $\mathbf{x} = (x_1, \ldots, x_{\mathsf{m}})$
- $\Delta^{\mathsf{m}}(\mathbf{x}) = \prod_{i<j}^{\mathsf{m}} |x_i - x_j|^2$ *if* $\mathsf{m} \geq 2$, $\Delta^{\mathsf{m}}(\mathbf{x}) = 1$ *if* $\mathsf{m} = 1$.

  In particular, if $\mathsf{m} = 1$, then

$$\frac{d\mu_x}{d\mu_y} = \frac{e^{-|x|^2}}{e^{-|y|^2}} \lim_{r\to\infty} \prod_{|s_i|<b_r} \frac{|x - s_i|^2}{|y - s_i|^2}$$

Index of the number of missing particles:

$D_q = \{z \in \mathbb{C}\,;\, |z| < \sqrt{q}\}\ q \in \mathbb{N}$

$$F_r(\mathsf{s}) = \frac{1}{r}\sum_{q=1}^{r}(\mathsf{s}(D_q) - q). \tag{10}$$

Thm 7. *Let* S *be the configuration space over* $\mathbb{C}$. *Let* $\mathsf{m} \in \mathbb{N}$. *Then for* $\mathbf{x} = (x_1, \ldots, x_{\mathsf{m}})$

$$\lim_{r\to\infty} F_r(\mathbf{s}) = -\mathsf{m} \qquad \textit{weakly in } L^2(\mathsf{S}, \mu_{\mathbf{x}}) \tag{11}$$

Remark: m is the number of the removed particles.

$$\infty - \mathsf{m} \neq \infty$$

Application of stochastic geo. to stochastic dyn.: homogenization of diffusion in 2D Coulomb environment

Let $\mathsf{s} = \sum_i \delta_{s_i} \in \mathsf{S}$. Let $X_t^{\mathsf{s}} \in \mathbb{R}^2$ be the solution of

$$dX_t^{\mathsf{s}} = dB_t + \lim_{q \to \infty} \sum_{i \in \mathbb{N},\ |X_t^{\mathsf{s}} - s_i| < q} \frac{X_t^{\mathsf{s}} - s_i}{|X_t^{\mathsf{s}} - s_i|^2} dt$$

Then

$$\lim_{\varepsilon \to \infty} \varepsilon X^{\mathsf{s}}_{t/\varepsilon^2} = \sqrt{\alpha} B_t \qquad \text{in } \mu_{\mathsf{gin},0}\text{-measure} \qquad ([O.98])$$

The matrix α is called the effective conductivity.

Thm 8. $\alpha = 0$

The key of the proof is the function F_r in (10):

$$F_r(\mathsf{s}) = \frac{1}{r} \sum_{q=1}^{r} (\mathsf{s}(D_q) - q)$$

Representation of α Let $\tilde{\mathcal{E}} : L^2(\mu_{\mathsf{gin},0}) \otimes L^2(\mu_{\mathsf{gin},0}) \to \mathbb{R}$:

$$\tilde{\mathcal{E}}(\mathbf{f}, \mathbf{g}) = \int_{\mathsf{S}} \frac{1}{2} \sum_{i=1}^{2} f_i g_i d\mu_{\mathsf{gin},0} \qquad \text{for } \mathbf{f} = (f_1, f_2)$$

$$\tilde{\mathcal{D}} = \overline{\{(D_1 f, D_2 f); f \in \mathcal{D}_0\}}$$

Here D_i is the generator of the translation, $\mathbf{e}_i$ unit vector. There's a unique $\mathbf{u}_i \in \tilde{\mathcal{D}}$ s.t.

$$\tilde{\mathcal{E}}(\mathbf{u}_i, \mathbf{g}) = \tilde{\mathcal{E}}(\mathbf{e}_i, \mathbf{g}) \text{ for all } \mathbf{g} \in \tilde{\mathcal{D}} \qquad (12)$$

Thm 9 (O.98). *The effective-diffusion matrix* α *is given by*

$$\alpha_{ij} = \tilde{\mathcal{E}}(\mathbf{e}_i - \mathbf{u}_i, \mathbf{e}_j - \mathbf{u}_j) \tag{13}$$

Moreover, $\alpha = 0$ *if and only if* $\mathbf{e}_i \in \tilde{\mathcal{D}}$ *for all* i*.*

- We need to prove $\mathbf{e}_1, \mathbf{e}_2 \in \tilde{\mathcal{D}}$
- One can check $(F_r, 0) \in \tilde{\mathcal{D}}$. Hence

$$\mathbf{e}_1 = \lim_{r\to\infty} (-F_r, 0) \quad \text{weakly in } \tilde{\mathcal{D}}$$

This completes the proof of Thm 8.

Conj: If we replace $\mu_{\mathrm{gin},0}$ by μ_{gin}, then $\alpha > 0$. Indeed, in case of periodic μ, this is the case.