LARGE RANKS OF QUADRATIC TWISTS OF THE CONGRUENT NUMBER CURVE

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A joint production with A. Granville
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FOLKLORE

Ranks of elliptic curves over $\mathbb{Q}$ are unbounded

WHY ?

- Function field analogue (Tate · Shafarevich, Ulmer)
- Selmer ranks are unbounded
- Moduli spaces stabilise ($M_{1,28}$ versus $M_{1,29}$)
- Rank 6-11 smallest conductor data

  shows $c \frac{\log N}{\log \log N}$ growth (*)

  Similarly, $r = 28$ curve has $\frac{1}{2} \frac{\log N}{\log \log N} \approx 28.16$

- Mordell-Weil groups $\Leftrightarrow$ unitgroups

  integral pts $\Leftrightarrow$ unit equations $x+y=1$

WHY NOT ?

- Points seem "rare" (average is bounded)
- L-functions don't like to vanish (?!)

  No known L-function of degree 2 and motivic weight ≥ 3 with (observed) analytic rank > 2.

- Probabilistic heuristics and maybe the strong Lang conjecture, suitably malleated

PROBABILISTIC "REASONING"

- We "have" $X^{20/24}$ curves up to abs disc or conductor X

  Half "are" rank 0, the other half rank 1

- Expect $X^{19/24}(\log X)^{\cdot}$ of rank 2.

  when 0, is rank 2

  Here $\Omega_{re} \approx 1/\Delta^{1/12}$, so "$\#Ш$" is a random square up to about $X^{1/12}$

- Expect $X^{18/24}(\log X)^{\cdot}$ of rank 3 ?

  No idea how to model heights of points

$\vdots$

- "Expect" $X^{0/24}(\log X)^{\cdot\cdot}$ of rank 21

  Finitely many of rank ≥ 22. (Same as Granville and/or "strong log")

- Have H^{10} curves up to height H (H^4 c_4's $\times$ H^6 c_6's)
- Have H^9 with a "small" integral point (polynomial in H)
  (rational here should be a log-factor)
- Have $H^8(\log H)^{\cdot}$ with 2 small points

  $\vdots$

- Have $H^0(\log H)^{\cdot\cdot}$ with 10 small points

BUT: Mestre has an infinite family of elliptic curves with 11 independent points of polynomial height

INTEGRAL POINTS ON ELLIPTIC CURVES

$Y^2 = X^3 + AX + B$ Maximize $$\frac{\log X}{\log \max(|A|^{1/2}, |B|^{1/3})}.$$

Consider $A \sim T^2$, $B \sim T^3$ (dyadic intervals, say)

What is the probability RHS is square?

When $X \gg T$, this is about 1 in $X^{3/2}$.

So consider

$$\sum_{X \sim T^\lambda} \sum_{A \sim T^2} \sum_{B \sim T^3} \frac{1}{T^{3\lambda/2}} \sim \frac{T^5}{T^{\lambda/2}}$$

Thus: Expect no solutions when $\lambda > 10$.

(summing over dyadic intervals only gives logs)

This proposes that the ratio should be bounded as $\frac{10 \log T}{\log T} = 10$.

Zagier wrote a paper giving several infinite families with the ratio approaching 9.

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• INTEGRAL POINTS ON ELLIPTIC CURVES

$$Y^2 = X^3 + AX + B \quad \text{Maximize} \quad \frac{\log X}{\log \max\left(|A|^{1/2}, |B|^{1/3}\right)}$$

Lang
Vojta
bounded asymptotically by 10, except for finitely many parametrized families
stark

Elkies (1988)
Take $Q(t)Y(t)^2 = X(t)^3 + A(t)X(t) + B(t)$!

deg	A	B	Q	X	Y	ratio	
	0	1	2	4	5	12	rational solution
	1	1	2	6	8	12	quadratic solution
	1	2	2	8	11	12	none solution
	2	3	2	12	17	12	?

In these cases, equating coefficients gives the same number of equations / unknowns.

Find one specialization with $Q(t) = \square$. (Rescale if necessary)

Pell equation theory gives infinitely many solutions (via specialization), though sparsely.

Gives infinitely many solutions with ratio $\Rightarrow 12$.

Have infinitely many deg(A, B, Q, X, Y) exceeding 10, though field of definition should grow. and 0-dim subvariety

RANKS IN QUADRATIC TWIST FAMILIES

Fix $E: Y^2 = X^3 + AX + B$

Consider $E_d: dY^2 = X^3 + AX + B$ for squarefree d up to D

Expect
D rank 0
D rank 1
$D^{3/4}(\log D)^{\cdot}$ rank 2 (2-torsion affects log-power)
$D^{1/2}(\log D)^{\cdots}$ rank 3 ? (data are unclear)
⋮
$D^0(\log D)^{\cdots}$ rank 5

Finitely many quadratic twists of rank 6 or higher ?!

- Data below suggest "lots" of rank 6 quadratic twists of the Congruent number curve
- However, 5 might be the right transition point when there is no 2-torsion (Granville's heuristic below)
  Not clear why it would not "just" be a log-power
- Rubin/Silverberg give a criterion which indicates a lattice maldistribution when a quadratic twist family has a member of rank exceeding 8.

GRANVILLE'S HEURISTIC

IDEA: "BOUND" INTEGRAL PTS ON

$$Y^2 Z = X^3 + ad^2 X Z^2 + bd^3 Z^3 = f(X, dZ)$$

where a, b are fixed, (d, X, Y, Z) parameters

COMPARE to number of such integral points induced by <u>one</u> rank r twist.

Note: Z is cube

$$X \equiv 0 \pmod{\sqrt[3]{Z}}$$

$$f(X, dZ) \equiv 0 \pmod{Y^2}$$

Take $d \sim D$, $X \sim T$, $Z \sim U/D$

Assume RHS has little cancellation so

$$Y \sim \sqrt{DV^3/U} \quad \text{where } V = \max\left(T, \frac{U}{D}\right)$$

Now estimate the number of points as

$$\ll \sum_{d \sim D} \sum_{Y \sim \sqrt{DV^3/U}} \sum_{\substack{\tilde{Z} \sim \sqrt[3]{U/D} \\ (\tilde{Z}^3 = Z)}} \sum_{\substack{X \sim T \\ \tilde{Z} \mid X \\ f(X, dZ) \equiv 0 \ (Y^2)}} 1$$

The Y^2-congruence has a density of solutions given by $\frac{\sigma_f(Y^2)}{Y^2}$ where $\sigma_f(Y^2)$ is # roots of f modulo Y^2

HEURISTIC GUESSING

Assuming some sort of equidistribution, we get

$$\ll \sum_{d\sim D} \sum_{Y\sim\sqrt{DV^3/U}} \frac{\sigma_f(Y^2)}{Y^2} \sum_{\tilde{z}\sim \sqrt[3]{U/D}} \frac{T}{\tilde{z}}$$

$$\ll TD\sqrt{U/DV^3}\left(\log \frac{DV^3}{U}\right)^{\Omega-1}$$

where $\Omega \in \{1,2,3\}$ is average number of roots of f mod p.

Summing dyadically over T, U (up to B) gives an extra log

Get $C_D(B) \ll \sqrt{D}(\log B)^{\Omega}$

for #pts with $|x|, z \le B$ and $|d|\sim D$.

- Granville notes sieve theory can prove this for $B \ll D^{\delta}$, but he applies it for $B\sim e^{D^{\varepsilon}}$ for $\varepsilon>0$.

Recall: #pts of canonical height $\le h$ is $\frac{h^{r/2}}{\sqrt{R}}$ r rank, R regulator

Assume canonical and naive heights are close

Assume $R \approx \sqrt{D}$ (see BSD)

Get $\frac{h^{r/2}}{D^{1/4}} \ll$ #pts up to height h on one rank r twist of size D $\ll C_D(e^h) \ll \sqrt{D}h^{\Omega}$

FINAL GUESSES

We still need to guess how large h is w.r.t. to D. (how much superequidistribution)
Granville suggests, in analogue to Pell equations, we should be able to take $h \sim D^{\ell}$ with $\ell = \frac{1}{2}$.
So the previous gives (with $h \sim D^{\ell}$)

$$r \le 2\eta + \frac{3}{2\ell} \quad \text{as } D \to \infty$$

Any $\ell > 0$ implies bounded ranks. And $\ell = \frac{1}{2}$ gives 5, 7, 9 respectively for no, part, and full 2-torsion.

Recall every curve with full 2-torsion is isogenous to one with only one 2-torsion point. Maybe 7 thus is the correct bound for full 2-torsion also?

- Direct analogue to cubic twists of $x^3 + z^3 = 1$ Predicts $r \le 9$, while Elkies and Rogers have rank 11 examples
- Less direct analogues for quartic twists $y^2 = x^3 - Dx$ and sextic twists $y^2 = x^3 + k$, and again data seem to disagree. Similarly with family of all elliptic curves.

DATA AND COMPUTATIONS

Gouvêa–Mazur:

Each u/v is an x-coord on a unique $dy^2 = x^3 - x$,

namely $d = uv(u-v)(u+v)$ ← (Take the square-reduced part)

Rogers: Loop over u,v, compute d, find d that appear often, compute 2-Selmer ranks . . . .

For me, once is often enough.

- Apply Monsky matrix to get 2-Selmer rank
- Apply Monsky/Rogers matrices to get it for isogenous curves
- Apply CasselsTatePairing in Magma (4-descent pairing S.R. Donnelly)
- Apply email to T.A. Fisher (slightly finer pairing, degree 32 map using isogs)
- If rank is still possibly 6+ search for points

(u,v) up to 10^8, cap d at 2^{60} though.

6-7 cpu-years, maybe around 10^{14} 2-Selmer tests

Leaving about 40 million to which to apply CasselsTatePairing

About a cpu-year, leaving 21016+71 curves.

Each has rank 6+, or nontrivial Ш[4] on all isogenous curves.

Reduced by Fisher to 2006+16 possible rank 6+ twists

and 1230+13 we found enough pts.

assume parity for r=6

MORE DATA

Rogers found $(u,v) = (134, 779)$ gives $d = 6147134961\,0$

$r=6$ $(u,v) = (2976, 4457)$ gives $d = 6611719866$

$r=7 \to (u,v) = (79873, 235280)$ gives $d = 797507549575$

He probably found more $r=7$, but didn't care enough to record them carefully. We also found

$\underset{\text{smaller } v}{r=7} \to (u,v) = (62936, 207689) \rightsquigarrow d = 2498179915\ldots 285524$

An analysis of the 1230 rank 6 twists is unclear. Growth might be logarithmic, or a small power (maybe $D^{1/4}$?)

RANK 8?

Finding the first rank 7 twist takes about a cpu-hour.

We removed the d-limit, and searched (u,v) up to 10^7.

Another 6-7 cpu-years (still running, in fact).

Found a smaller (u,v) for rank 7 $\quad (u,v) = (34440, 145343)$

$d = 244856702618970820$

What do we expect the (u,v) to be for a rank 8 twist?

What do we expect the d to be for a rank 8 twist?

- Alternative (Elkies) search method: Param square divisors of $uv(u-v)(u+v)$ as $d_1^2 \mid u$, $d_2^2 \mid v$, $d_3^2 \mid (u-v)$, $d_4^2 \mid (u+v)$ $\leftarrow$ first $r=7$

  Takes 5 hours to find $(d_1, d_2, d_3, d_4) = (40, 169, 3, 389)$

- Other curves: no 2-torsion?, curves with 3-isogeny?