$|\varphi|^4$ spin model in 4 dimensions

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1 Model

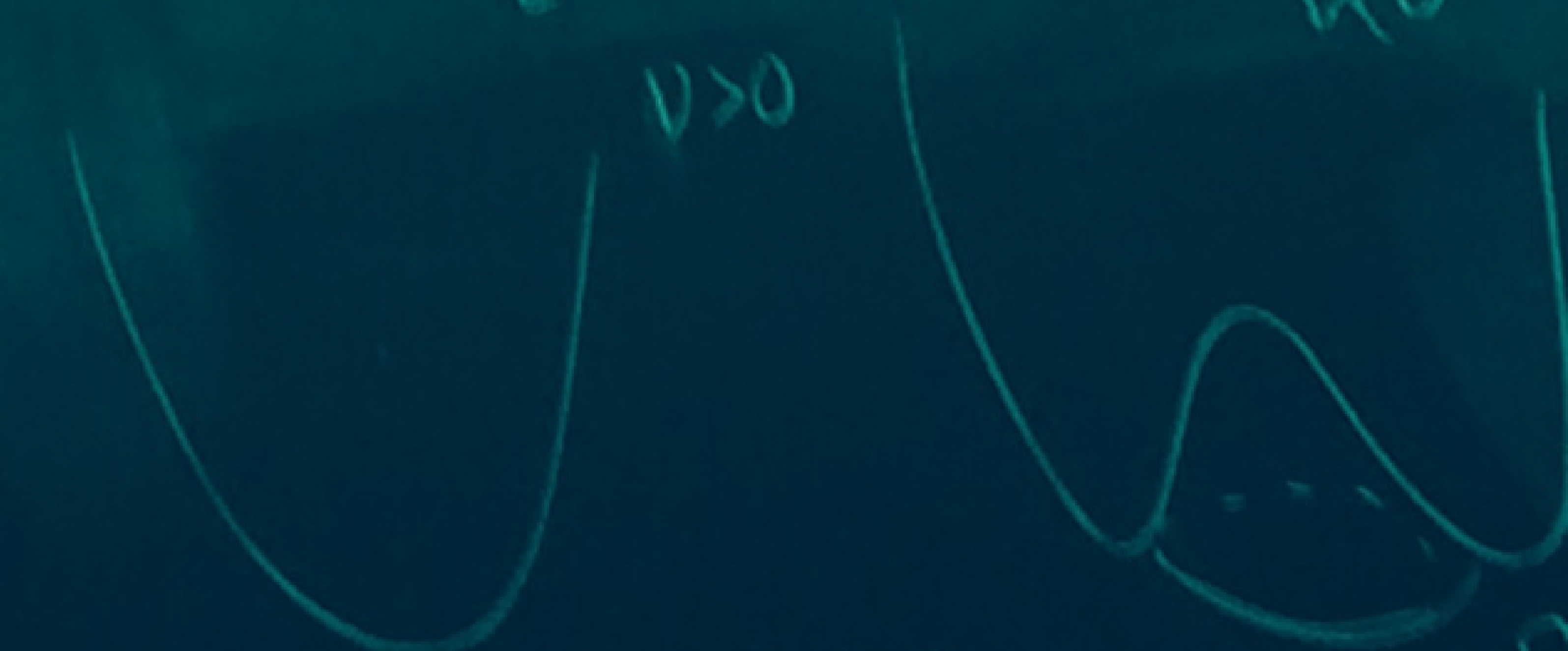
$\Lambda = \Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$

$\varphi_x \in \mathbb{R}^n$ spin, $x \in \Lambda$, $\varphi_x = (\varphi_x^i)_{i=1}^n$

$\langle \cdot \rangle_{g,\nu,\Lambda}$ Leb. on $\mathbb{R}^{\Lambda n}$

$$P_{g,\nu,\Lambda}(d\varphi) = \frac{1}{Z_{g,\nu,\Lambda}} e^{-\sum_{x\in\Lambda}\left(\frac{1}{2}\varphi_x(-\Delta\varphi_x) + \frac{1}{2}\nu|\varphi_x|^2 + \frac{1}{4}g|\varphi_x|^4\right)} d\varphi$$

$\frac{1}{2}\nu|\varphi|^2 + \frac{1}{4}g|\varphi|^4$



$\nu_c = \nu_c(g) < 0$

FSS '76 phase transition in $d \geq 3$, $n \geq 1$

Goal study $\nu = \nu_c$, $\nu \downarrow \nu_c$ in $d = 4$

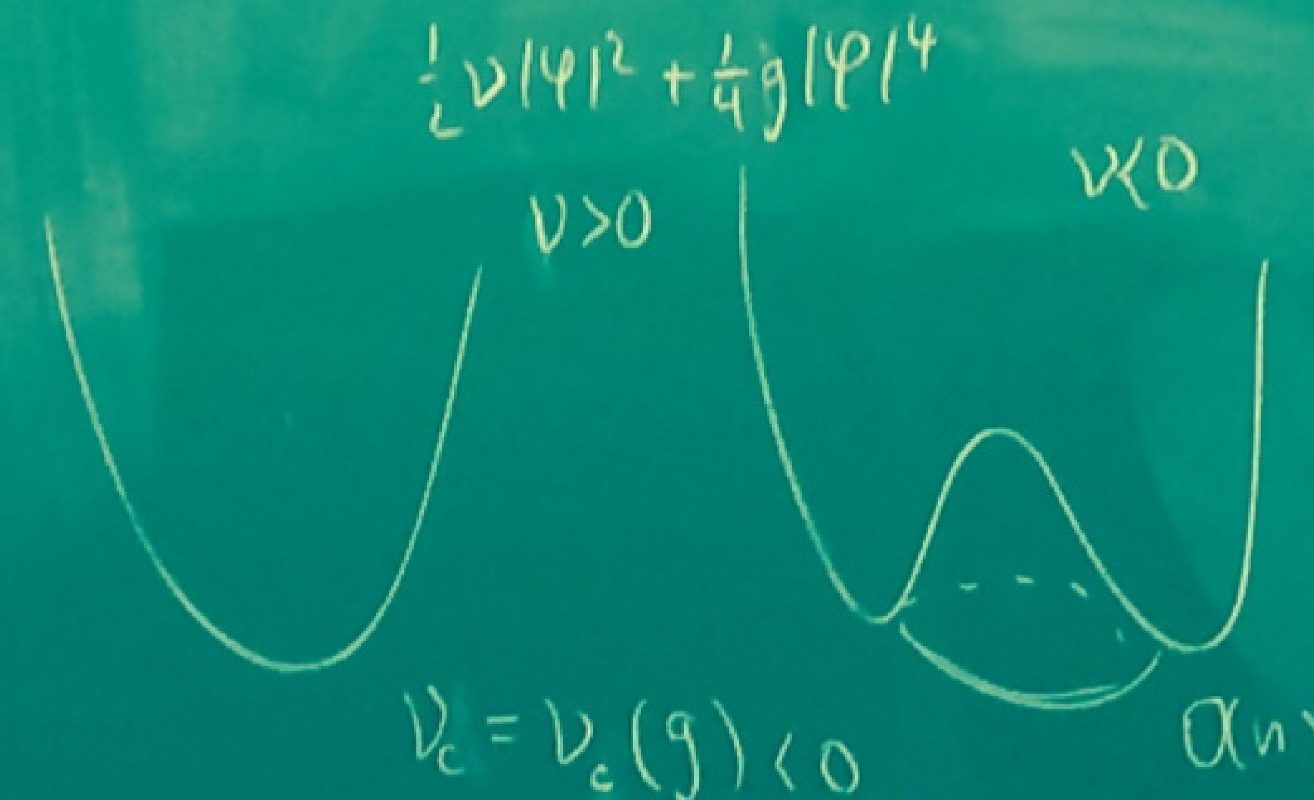
$\to$ RG

L^N

FSS '76 phase transition in $d \geq 3$, $n \geq 1$

Goal study $v = v_c$, $v \downarrow v_c$ in $d=4$

$\to$ RG



Scope of method

- Correlation function: $\Lambda_N \to \mathbb{Z}^4$
- Scaling limit: $L^{-N}\Lambda_N \to \mathbb{R}^d/\mathbb{Z}^d$
- Thermodynamic obs

$$\chi(g,v) = \lim_{N\to\infty} \sum_{x\in\Lambda_N} \langle \varphi_0^1 \varphi_x^1 \rangle \quad \text{— suscept}$$

pressure, specific heat

Thm For $g>0$ small,

$$\chi(g, v_c+\varepsilon) \overset{\varepsilon\downarrow 0}{\sim} A(g)\,\varepsilon^{-1} \left(\log \varepsilon^{-1}\right)^{\frac{n+2}{n+8}}$$

$$\chi(0, 0+\varepsilon) = \varepsilon^{-1}$$

2. Approximation by FF

$$e^{-V_{g,v,1}(\varphi)}$$

$$V_{g,v,1}(\varphi) = \sum_{x\in\Lambda}\Big(\overbrace{\tfrac{1}{2}\cdot 1 \cdot \varphi_x(-\Delta\varphi)_x + \tfrac{1}{2} v|\varphi_x|^2}^{\text{quad}} + \tfrac{1}{4} g|\varphi_x|^4\Big)$$

$$1 = \frac{1}{1+z_0} + \frac{z_0}{1+z_0} \qquad v = \frac{m^2}{1+z_0} + \left(v - \frac{m^2}{1+z_0}\right)$$

$$V_{g,v,1}(\varphi) = V_{0,m^2,1}\left((1+z_0)^{-\frac{1}{2}}\varphi\right) + V_{g_0,v_0,z_0}\left((1+z_0)^{-\frac{1}{2}}\varphi\right)$$

$$g_0 = g(1+z_0)^2$$

$$v_0 = v(1+z_0) - m^2$$

$$C = (-\Delta + m^2)^{-1}$$

$$\mathbb{E}_C$$

$$\langle F(\varphi)\rangle_{g,\nu,\Lambda} = \frac{\mathbb{E}_C\left(Z_0(\varphi)\, F((1+z_0)^{\frac12}\varphi)\right)}{\mathbb{E}_C(Z_0(\varphi))}, \quad Z_0(\varphi) = e^{-V_0(\varphi)}, \quad V_0(\varphi) = V_{g_0,\nu_0,z_0}(\varphi)$$

$$\mathbb{E}_C\left(e^{(f,\varphi)} Z_0(\varphi)\right) = e^{\frac12 (f,Cf)} \underbrace{\mathbb{E}_C(Z_0(\varphi + Cf))}_{Z_N(Cf)}$$

$$Z_N(\varphi) = \mathbb{E}_C(Z_0(\varphi+\varphi')) =: (\mathbb{E}_C \theta Z_0)(\varphi)$$

3 Decomposition

$$\underset{\mathrm{GFF}_{m^2}}{\varphi} \overset{D}{=} \varphi_1 + \cdots\cdots + \varphi_N$$

Gauss., indep.

$$(-\Delta + m^2)^{-1} = \sum_{j=1}^{N} C_j$$

$$C_j(x,y) = 0 \quad \text{if} \quad |x-y| > \tfrac12 L^j$$

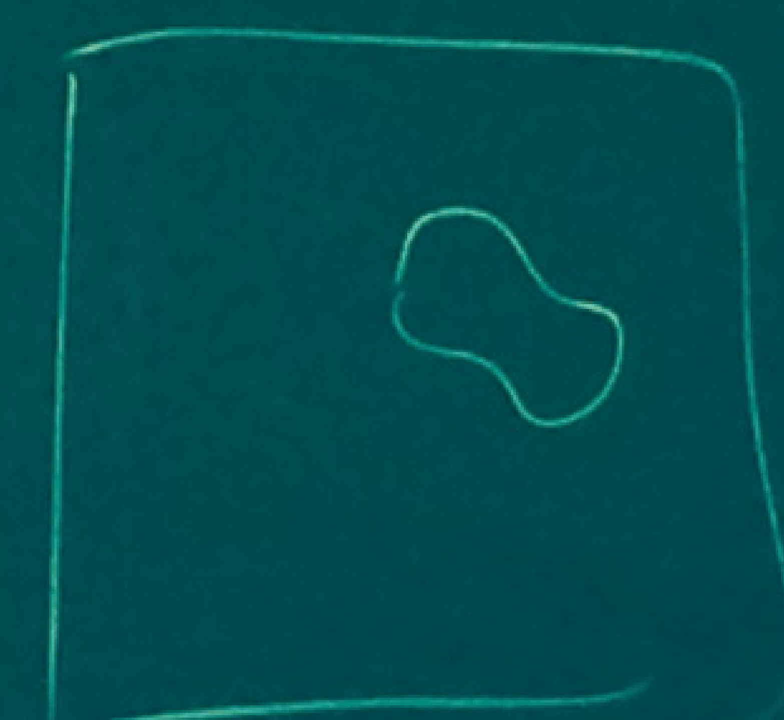
$$\lim_{m^2 \downarrow 0} C_j(x,y) = L^{-(d-2)j} u(L^{-j}|x-y|) + O(L^{-j} L^{-(d-2)j})$$

∇^α $\quad L^{-|\alpha| j}$ $\quad u \in C_c^\infty$ $\quad L^{-|\alpha| j}$

$L^{-|\alpha| j}$

$$Z_0 = e^{-V_0(\varphi)} \xrightarrow{\mathbb{E}_{C_1}\theta} Z_1 \longrightarrow \cdots \longrightarrow Z_N$$

$$Z_{j+1} = \mathbb{E}_{C_{j+1}}\theta Z_j$$



4. RG

normalize $Z_j(0) = 1$

Study: $e^{-u_j} Z_j \xrightarrow{\mathbb{E}_{C_{j+1}}\theta} e^{-u_{j+1}} Z_{j+1}$

coordinates: $Z_j = Z_j(V_j, K_j)$

- $V_j \in \mathbb{R}^3$, detailed knowledge
- K_j ∞-dim, contracting $\longrightarrow$ D.B.

$$(V_j, K_j) \longmapsto (V_{j+1}, K_{j+1}) = \Phi_j(V_j, K_j)$$

$$(V_j, 0) \longrightarrow$$

$V_j = (g_j, \nu_j, z_j) \longrightarrow V_j(\varphi) = \sum_{x\in\Lambda}\left(\frac{1}{2} z_j \varphi_x(-\Delta\varphi)_x + \frac{1}{2}\nu_j |\varphi_x|^2 + \frac{1}{4} g_j |\varphi_x|^4\right)$

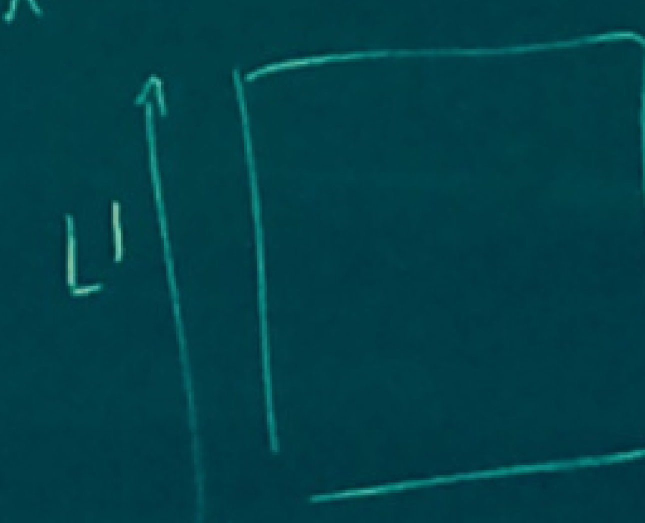
"$K_0 = 0$"

$|\varphi_j(x)| \approx \sqrt{C_j(x,x)} \approx L^{-j}$

$\sum_{x\in B} |\varphi_j(x)|^4 \approx L^{4j} \cdot L^{-j4} \approx 1$

$|\varphi_j(x)|^2 \approx L^{2j}$

$\sum_{x\in B} \varphi_j(-\Delta\varphi_j) \approx 1$



All other terms $\approx L^{-pj}$

$$Z_0(\varphi) = I_0(V_0, \varphi), \quad I_0(V_0,\varphi) = e^{-V_0(\varphi)}$$

<u>Prop</u> $E_{C_{j+1}} \theta I_j(V_j) = I_{j+1}(V_{j+1}) + O(V_j^3)$ where

$$I_j(V_j, \varphi) = e^{-V_j(\varphi)}\left(1 + W_j(V_j, \varphi)\right)$$

relev / marg. <u>irrel.</u>

$$\beta_j = (n+8) \sum_{x \in \Lambda} \left(\left(\sum_{k=1}^{j} C_k(x) \right)^2 - \left(\sum_{k=1}^{j-1} C_k(x) \right)^2 \right)$$

$\longrightarrow$ $g_{j+1} = g_j - \beta_j g_j^2 + (\cdots)$

$\mu_{j+1} = L^2 \mu_j \left(1 - \frac{n+2}{n+8} \beta_j g_j\right) + (\cdots)$

$$\underset{j \to \infty}{\overset{m^2 \downarrow 0}{\sim}} (n+8) \frac{\log L}{16 \pi^2}$$

$\longrightarrow$ $Z_{j+1} = Z_j \cdots$

$g_\infty \overset{m^2 \downarrow 0}{\sim} \dfrac{1}{\sum_j \beta_j} \sim c(\log m^2)^{-1}$

<u>Prop.</u> For $m^2 \geq 0$, $g_0 > 0$ small,

$\exists\ \nu_0 = \nu_0^c(m^2, g_0)$, $z_0 = z_0^c(\cdots)$

s.t. $(V_j, K_j) = (g_j, \mu_j, z_j, K_j) \longrightarrow (g_\infty, 0, 0, 0)$, $g_\infty > 0$

5 Susceptibility

$$(g, \nu) \longrightarrow (m^2, g_0, \nu_0, z_0)$$

$$\chi(g,\nu) = (1+z_0)\lim_{N\to\infty}\left(\frac{1}{m^2} + \frac{1}{m^4}\overbrace{L^{-2N}\mu_N}^{\nu_N} + O\left(\frac{1}{m^4}L^{-2N}g_N^2\right)\right)$$

$$= \frac{1+z_0}{m^2}$$

$$\frac{\partial}{\partial\nu}\chi(g,\nu) \overset{m^2\downarrow 0}{\sim} \frac{1}{m^4}L^{-2N}\frac{\partial\mu_N}{\partial\mu_0} \sim \frac{c}{m^4}\left(\frac{g_0}{g_0}\right)$$

5 Susceptibility
$(g, \nu) \longrightarrow (m^2, g_\infty, \nu_\infty, z_0)$
$\chi(g,\nu) = (1+z_0) \lim_{N\to\infty} \left(\frac{1}{m^2} + \frac{1}{m^4} \overbrace{L^{-2N}\mu_N}^{\nu_N} + O\left(\frac{1}{m^4} L^{-2N} g_N^2\right) \right)$
$= \frac{1+z_0}{m^2}$
$\frac{\partial}{\partial \nu} \chi(g,\nu) \overset{m^2 \downarrow 0}{\sim} \frac{1}{m^4} L^{-2N} \frac{\partial \mu_N}{\partial \mu_0} \sim \frac{c}{m^4} \left(\frac{g_\infty}{g_0}\right)^{+\frac{n+2}{n+8}} \sim \frac{c}{m^4 (\log m^{-2})^{\frac{n+2}{n+8}}}$