New Approaches to Loopy Random Graph Ensembles

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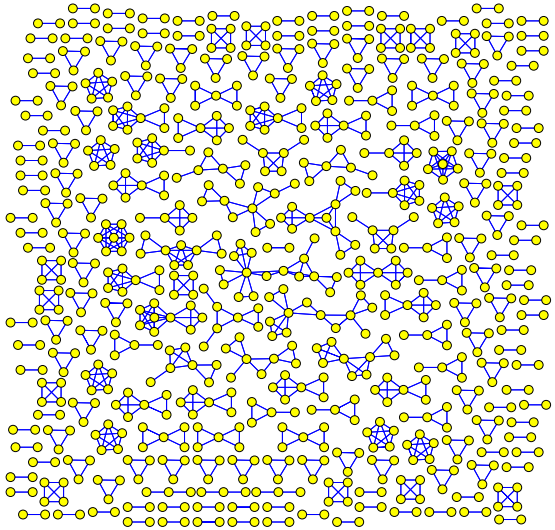


Outline

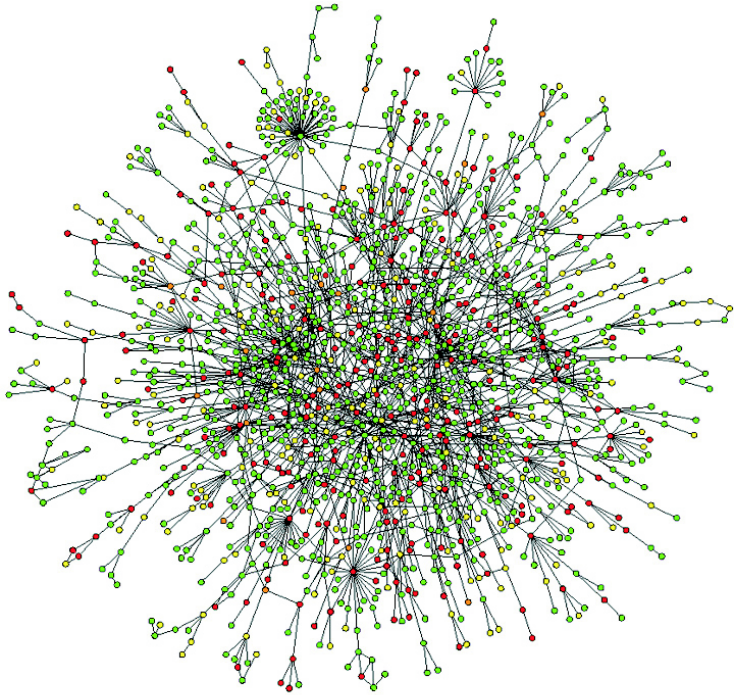
Stochastic processes on networks

- **protein interaction networks**

  dynamics: chemical reaction eqns

  nodes: proteins $i, j = 1 \ldots N$
  links: $c_{ij} = c_{ji} = 1$ if i can bind to j
  $c_{ij} = c_{ji} = 0$ otherwise

  nondirected graphs,
  $N \sim 10^4$, links/node ~ 7

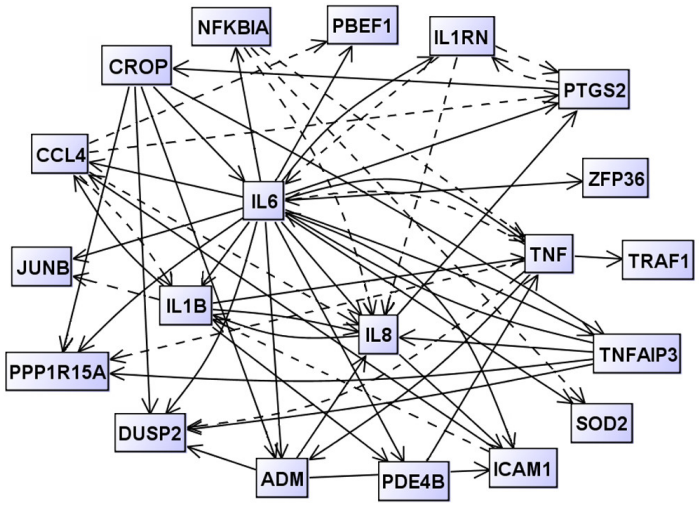


- **gene regulation networks**

  dynamics: gene transcription/expression

  nodes: genes $i, j = 1 \ldots N$
  links: $c_{ij} = 1$ if j is transcription factor of i
  $c_{ij} = 0$ otherwise

  directed graphs,
  $N \sim 10^4$, links/node ~ 5



Tailoring random graphs

stat mech of processes on network $\mathbf{c}^\star$,
use ***random* graph c** as proxy

- tailored random graph ensemble Ω_L:

  maximum entropy ensemble, constrained by values of $\boldsymbol{\omega}(\mathbf{c}) = \{\omega_1(\mathbf{c}), \ldots, \omega_L(\mathbf{c})\}$

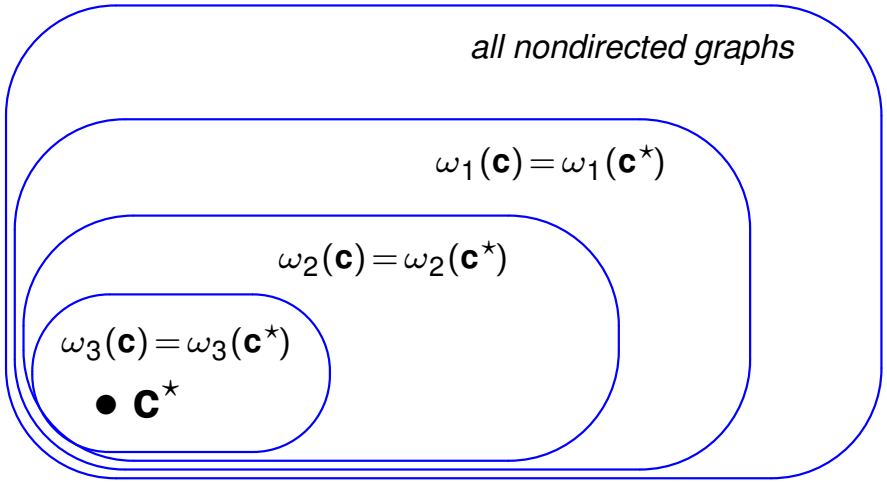
$$\Omega_L^{\rm hard}: \qquad p(\mathbf{c}) \propto \prod_{\ell \leq L} \delta_{\omega_\ell(\mathbf{c}),\omega_\ell(\mathbf{c}^\star)}$$

$$\Omega_L^{\rm soft}: \qquad p(\mathbf{c}) \propto {\rm e}^{\sum_{\ell=1}^L \hat{\omega}_\ell \omega_\ell(\mathbf{c})}, \qquad \sum_{\mathbf{c}} p(\mathbf{c})\omega_\ell(\mathbf{c}) = \omega_\ell(\mathbf{c}^\star) \;\; \forall \ell$$

- approx model solution:

  average generating functions of process over $\mathbf{c}$ in Ω_L

  larger $L \to$ better approx



How to choose observables $\boldsymbol{\omega}(\mathbf{c}) = \{\omega_1(\mathbf{c}), \ldots, \omega_L(\mathbf{c})\}$
to carry over from $\mathbf{c}^\star$ to the ensemble?

e.g. spin models $H(\boldsymbol{\sigma}) = -\sum_{i<j} c_{ij} J_{ij} \sigma_i \sigma_j$

- statics: replica method

$$\overline{\mathrm{e}^{-\beta \sum_{\alpha=1}^{n} H(\boldsymbol{\sigma}^\alpha)}} = \frac{\sum_{\mathbf{c}} \delta_{\boldsymbol{\omega},\boldsymbol{\omega}(\mathbf{c})} \mathrm{e}^{\sum_{i<j} c_{ij} A_{ij}}}{\sum_{\mathbf{c}} \delta_{\boldsymbol{\omega},\boldsymbol{\omega}(\mathbf{c})}}, \qquad A_{ij} = \beta J_{ij} \sum_{\alpha=1}^{n} \sigma_i^\alpha \sigma_j^\alpha$$

- dynamics: generating functional analysis

$$\overline{\mathrm{e}^{-\mathrm{i} \sum_{it} \hat{h}_i(t) \sum_j c_{ij} J_{ij} \sigma_j(t)}} = \frac{\sum_{\mathbf{c}} \delta_{\boldsymbol{\omega},\boldsymbol{\omega}(\mathbf{c})} \mathrm{e}^{\sum_{i<j} c_{ij} A_{ij}}}{\sum_{\mathbf{c}} \delta_{\boldsymbol{\omega},\boldsymbol{\omega}(\mathbf{c})}}, \qquad A_{ij} = -\mathrm{i} J_{ij} \sum_t [\hat{h}_i(t)\sigma_j(t) + \hat{h}_j(t)\sigma_i(t)]$$

in both cases
to be done *analytically*:

$$\sum_{\mathbf{c}} \underbrace{\delta_{\boldsymbol{\omega},\boldsymbol{\omega}(\mathbf{c})}}_{hard} \underbrace{\mathrm{e}^{\sum_{i<j} c_{ij} A_{ij}}}_{easy}$$

seems to boil down to this:
can we calculate ensemble entropy?

calculations feasible for:

$$p(k|\mathbf{c}) = \frac{1}{N}\sum_i \delta_{k,\sum_j c_{ij}}, \qquad W(k,k'|\mathbf{c}) = \frac{1}{\bar{k}N}\sum_{ij} c_{ij}\ \delta_{k,\sum_r c_{ir}}\delta_{k',\sum_r c_{jr}}$$

Shannon entropy

$S = N^{-1}\sum_{\mathbf{c}} p(\mathbf{c})\log p(\mathbf{c})$

$$S = \underbrace{\frac{1}{2}[1+\log(\frac{N}{\langle k\rangle})]}_{Erdos-Renyi\ \ entropy} - \Big\{ \underbrace{\frac{1}{\langle k\rangle}\sum_k p(k)\log[\frac{p(k)}{\pi(k)}]}_{degree\ \ complexity} + \underbrace{\frac{1}{2}\sum_{k,k'} W(k,k')\log\Big[\frac{W(k,k')}{W(k)W(k')}\Big]}_{wiring\ \ complexity} \Big\} + \epsilon_N$$

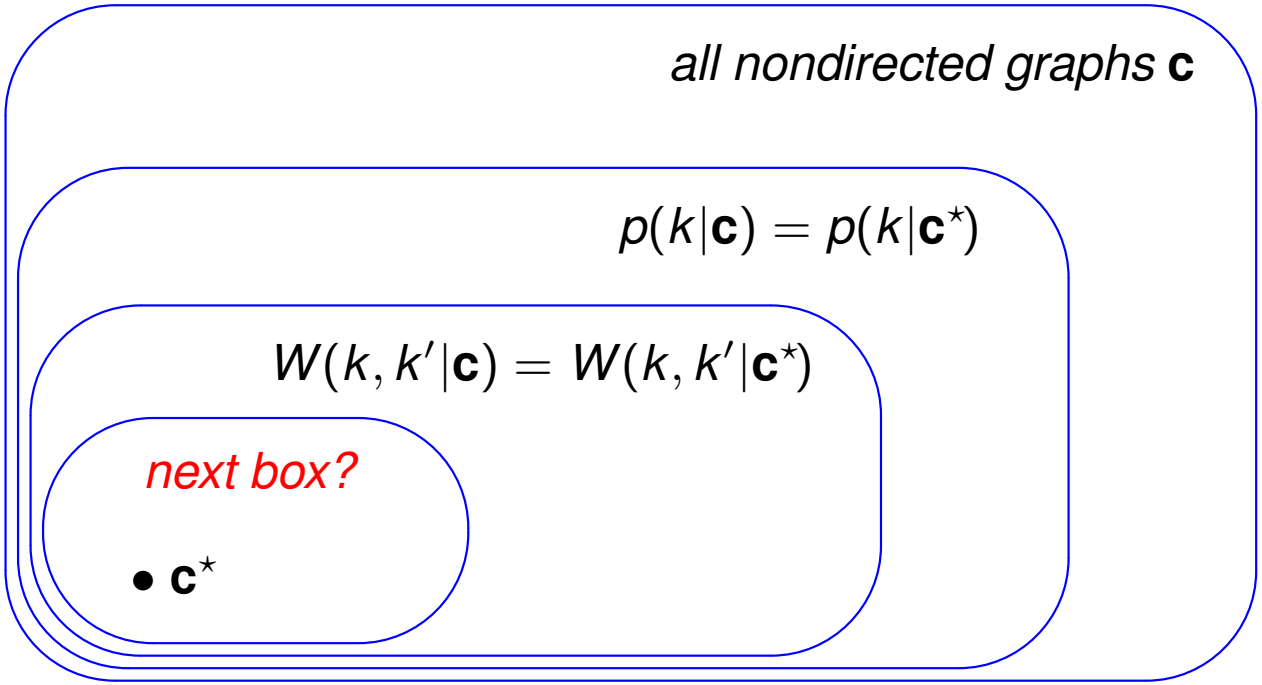
$\lim_{N\to\infty}\epsilon_N = 0$

$\pi(k) = \mathrm{e}^{-\langle k\rangle}\langle k\rangle^k/k!$

similar for directed graphs, formulae in terms of $\vec{k} = (k_{\rm in}, k_{\rm out})$, $p(\vec{k})$ *and* $W(\vec{k},\vec{k}')$

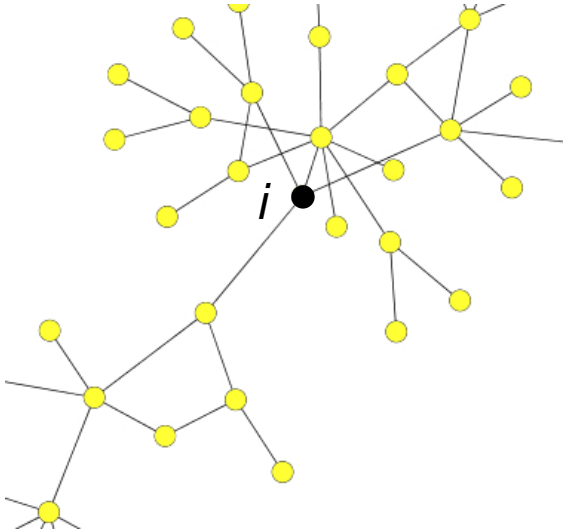
(e.g. Annibale, Coolen, Roberts et al, 2009-2011)

Tailoring random graphs further ...



obvious candidates:

generalised degrees, node neighbourhoods, ...



$k_i = \sum_j c_{ij} = 4$

$m_i = \sum_{jk} c_{ij} c_{jk} = 20$

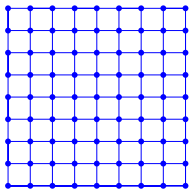
$n_i = (k_i;\ \{\xi_i^s\}) = (4;\ 3,4,6,7)$

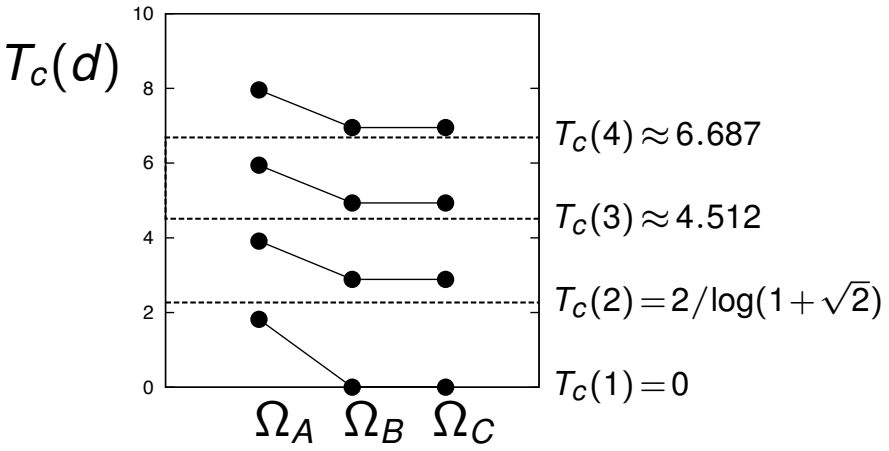
Graphs with many short loops

Ising spin models on tailored random graphs

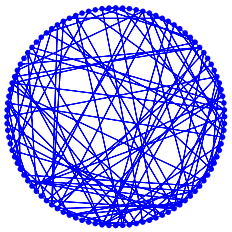
Ω_A: graphs with imposed $\bar{k}$
Ω_B: graphs with imposed $p(k)$
Ω_C: imposed $p(k)$ and $W(k,k')$

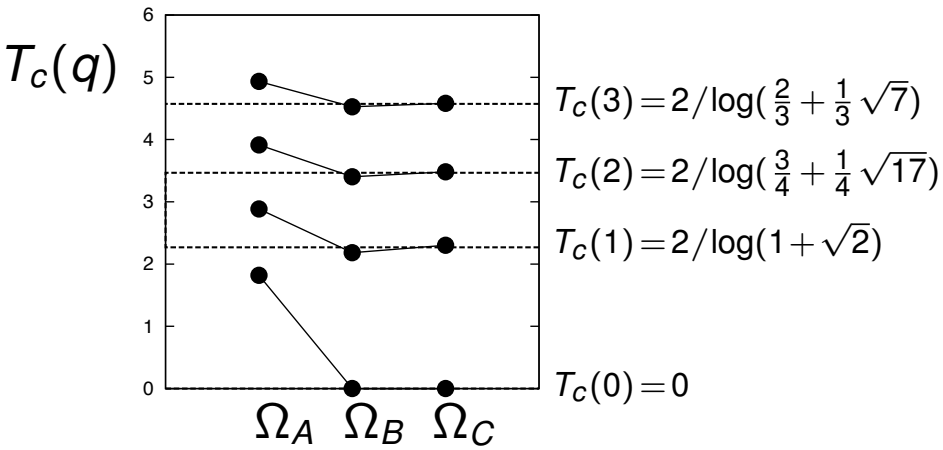
- $\mathbf{c}^\star$ = d-dim cubic lattice
  $p(k) = \delta_{k,2^d}$





- $\mathbf{c}^\star$ = 'small world' lattice
  $p(k \geq 2) = \mathrm{e}^{-q} q^{k-2}/(k-2)!$





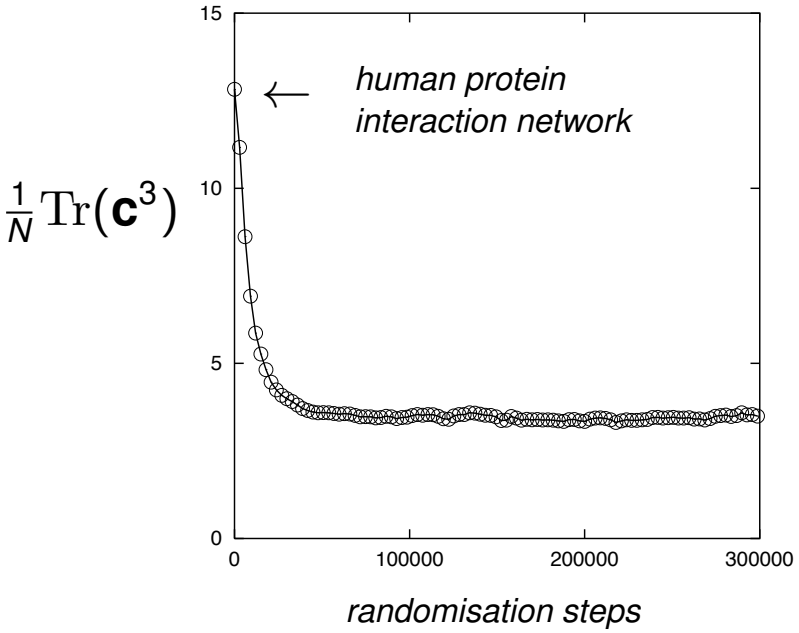
most informative
next observable $\omega(\mathbf{c})$?

- random graphs with prescribed $p(k)$ and $W(k,k')$: *locally tree-like* ...

  in contrast:

  protein interaction networks $\mathbf{c}^\star$: *many short loops* ...

  lattice-like networks $\mathbf{c}^\star$: *many short loops* ...



- so $\omega(\mathbf{c})$ must count short loops,

  but most of our analysis methods (replicas, GFA, cavity, belief prop) tend to require locally tree-like graphs ...

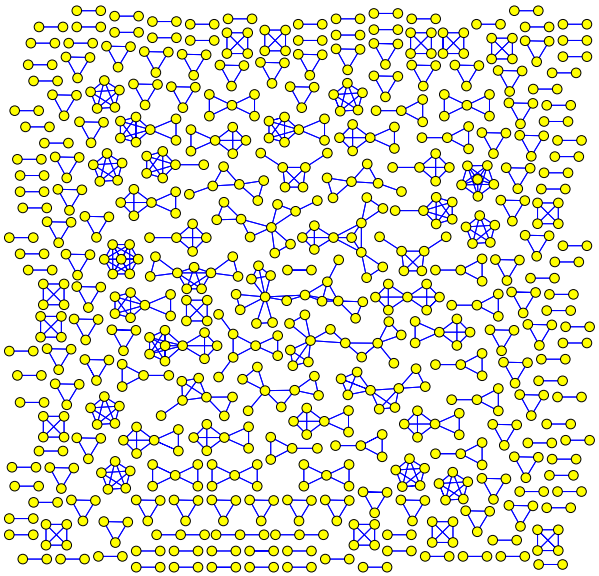
Immune model of Agliari and Barra (2013)

interaction between B-cells and T-cells

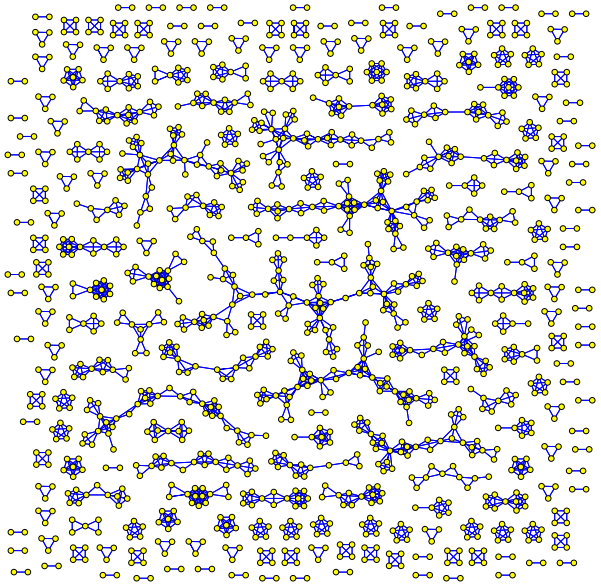
$$H(\boldsymbol{\sigma}) = -\frac{1}{2}\sum_{ij=1}^{N} J_{ij}\sigma_i\sigma_j - \sum_{\mu=1}^{\alpha N} h_\mu \sum_{i=1}^{N} \sigma_i \xi_i^\mu$$

$$J_{ij} = \sum_{\mu=1}^{\alpha N} \xi_i^\mu \xi_j^\mu, \qquad p(\xi_i^\mu) = \frac{c}{2N}\left[\delta_{\xi_i^\mu,1} + \delta_{\xi_i^\mu,-1}\right] + (1-\frac{c}{N})\delta_{\xi_i^\mu,0}$$

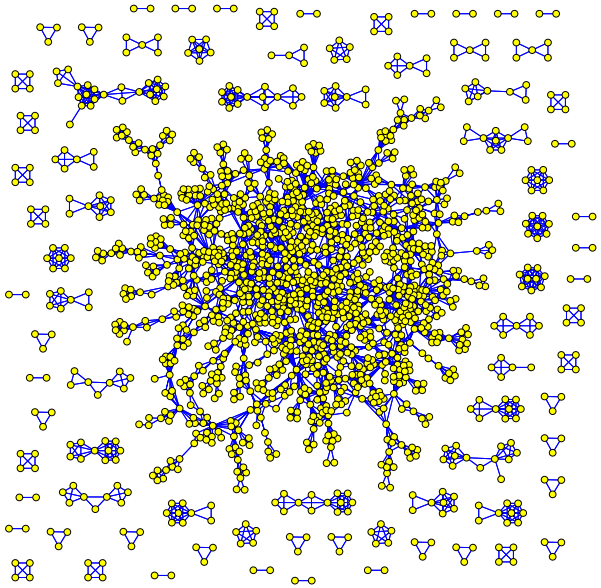
$\alpha c^2 < 1$



$\alpha c^2 = 1$



$\alpha c^2 > 1$



(see also Newman, 2003)

Immune versus neural network models

mathematically very similar ...
both store and recall information ...

$$p(\boldsymbol{\sigma}) \propto \mathrm{e}^{-\beta H(\boldsymbol{\sigma})} \qquad H(\boldsymbol{\sigma}) = -\frac{1}{2}\sum_{ij=1}^{N} J_{ij}\sigma_i\sigma_j - \sum_{\mu=1}^{\alpha N} h_\mu \sum_{i=1}^{N} \sigma_i \xi_i^\mu$$

- Hopfield model: bond dilution
  c_{ij}: finitely connected tree-like graph

$$J_{ij} = c_{ij}\sum_{\mu=1}^{\alpha N} \xi_i^\mu \xi_j^\mu, , \qquad p(\xi_i^\mu) = \frac{1}{2}\left[\delta_{\xi_i^\mu,1} + \delta_{\xi_i^\mu,-1}\right] \qquad h_\mu = \mathcal{O}(\frac{1}{N})$$

recall of one N-bit pattern at a time

- Immune model: pattern dilution

$$J_{ij} = \sum_{\mu=1}^{\alpha N} \xi_i^\mu \xi_j^\mu, \qquad p(\xi_i^\mu) = \frac{c}{2N}\left[\delta_{\xi_i^\mu,1} + \delta_{\xi_i^\mu,-1}\right] + (1-\frac{c}{N})\delta_{\xi_i^\mu,0}, \qquad h_\mu = \mathcal{O}(1)$$

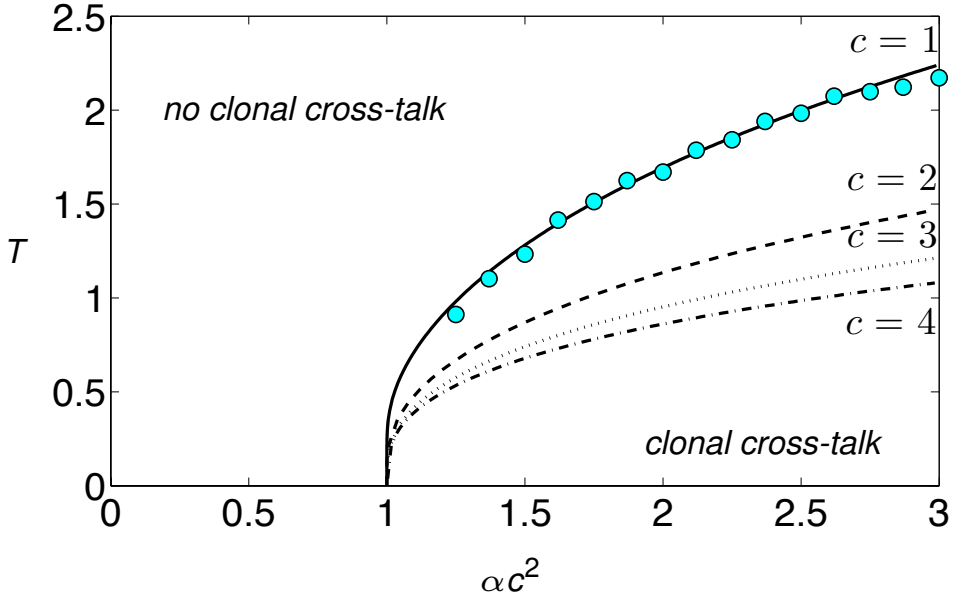
simultaneous recall of $\mathcal{O}(N)$ c-bit patterns

analysis *very* different!!

Model exactly solvable

using replica techniques,
in spite of the many short loops ...

$$f = -\frac{1}{\beta N} \log \sum_{\boldsymbol{\sigma}} \mathrm{e}^{\beta \sum_{i<j} J_{ij}\sigma_i\sigma_j}$$



here: $\mathbf{J} = \boldsymbol{\xi}^\dagger \boldsymbol{\xi}$
$\boldsymbol{\xi}$: sparse $p \times N$ matrix with indep distributed entries

Hubbard-Stratonovich type mapping
to model with spins + Gaussian fields,
on tree-like bipartite graph $\boldsymbol{\xi}$

$$\sum_{\boldsymbol{\sigma}} \mathrm{e}^{\beta \sum_{i<j} J_{ij}\sigma_i\sigma_j} = \int \frac{\mathrm{d}\mathbf{z}}{(2\pi)^{p/2}} \sum_{\boldsymbol{\sigma}} \mathrm{e}^{\sqrt{\beta} \sum_{\mu i} z_\mu \xi_{\mu i}\sigma_i - \frac{1}{2}\sum_\mu z_\mu^2}$$

solvable because it is a special case!

Harmonic oscillator of loopy graphs

Simplest nontrivial graph ensemble

controlled average connectivity,
controlled nr of triangles
(Strauss, 1986)

$$p(\mathbf{c}) \propto \mathrm{e}^{u \sum_{ij} c_{ij} + v \sum_{ijk} c_{ij} c_{jk} c_{ki}}$$

- quantities of interest:

$$\langle k \rangle = \Big\langle \frac{1}{N} \sum_{ij} c_{ij} \Big\rangle, \qquad \langle m \rangle = \Big\langle \frac{1}{N} \sum_{ijk} c_{ij} c_{jk} c_{ki} \Big\rangle, \qquad S = -\frac{1}{N} \sum_{\mathbf{c}} p(\mathbf{c}) \log p(\mathbf{c})$$

- generating function

$$\phi = \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{u \sum_{ij} c_{ij} + v \sum_{ijk} c_{ij} c_{jk} c_{ki}}$$

$$\langle k \rangle = \partial \phi / \partial u$$
$$\langle m \rangle = \partial \phi / \partial v$$
$$S = \phi - u \langle k \rangle - v \langle m \rangle$$

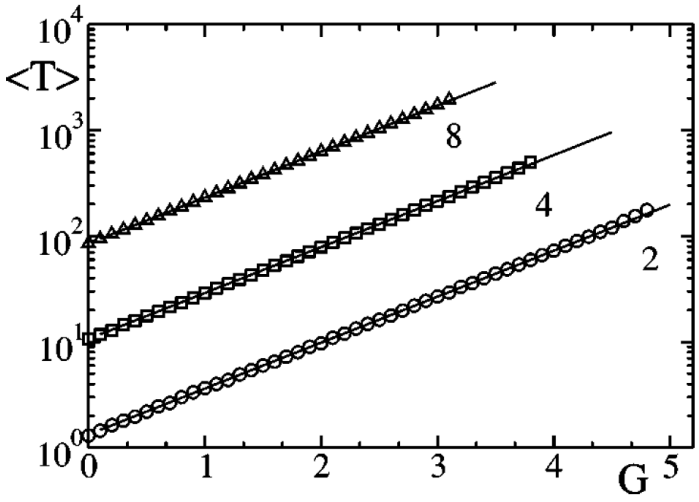
the challenge:
how to do the sum over graphs

Early results

- **Strauss** (1986)
  - — no theory
  - — triangles 'clump together'
- **Burda et al** (2004)
  - — $u = -\frac{1}{2}\log(N/\bar{k}-1)$, $v = \mathcal{O}(1)$
    if $v=0$: ER ensemble with $\langle k\rangle = \bar{k}$
  - — diagrammatic perturbation theory in v, formula for nr of triangles:
    $\lim_{N\to\infty}\langle T\rangle = \bar{k}^3 \mathrm{e}^v$
  - — regular regime, 'clumped' regime
  - — unresolved subtleties in expansion, expect series to explode for $v \sim \log N$
    i.e. when $\langle T\rangle = \mathcal{O}(N)$

Strauss ensemble:

$$p(\mathbf{c}) \propto \mathrm{e}^{u\sum_{ij} c_{ij} + v\sum_{ijk} c_{ij}c_{jk}c_{ki}}$$



$G = 6v$

intuition behind Burda's results

- exact identities:

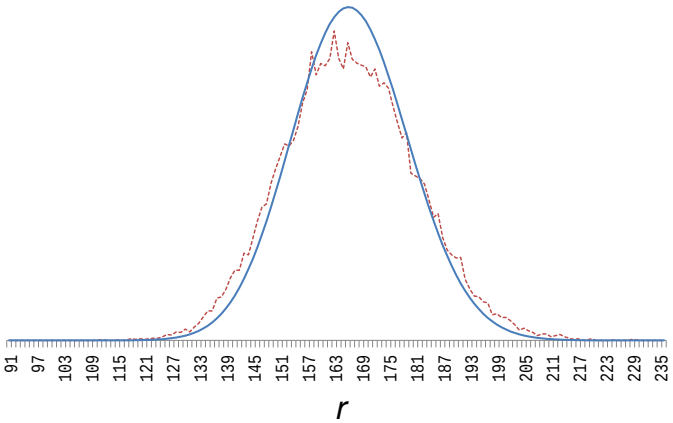
$$\phi = \frac{1}{2}(N-1)\log(\mathrm{e}^{2u}+1) + \frac{1}{N}\log\sum_{r\geq 0} p(r|u)\mathrm{e}^{vr}$$

$$p(r|u) = \sum_{\mathbf{c}} p_{\mathrm{ER}}(\mathbf{c}|u)\ \delta_{r,\sum_{ijk} c_{ij}c_{jk}c_{ki}} \qquad (\textit{ER triangle distr})$$

$$\bar{r}(u) = \frac{N(N-1)(N-2)}{(1+\mathrm{e}^{-2u})^3}$$

- assume $p(r|u)$ is Poissonnian:
  $p(r|u) = \mathrm{e}^{-\bar{r}(u)}\bar{r}(u)^r/r!$

$$\phi = \frac{1}{2}(N-1)\log(\mathrm{e}^{2u}+1) + (\mathrm{e}^{v}-1)\frac{(N-1)(N-2)}{(1+\mathrm{e}^{-2u})^3}$$



gives: $\langle k\rangle = \bar{k} + \mathcal{O}(\frac{1}{N}), \qquad N\langle m\rangle = \bar{k}^3\mathrm{e}^{v} + \mathcal{O}(\frac{1}{N})$

- **Park and Newman** (2005)

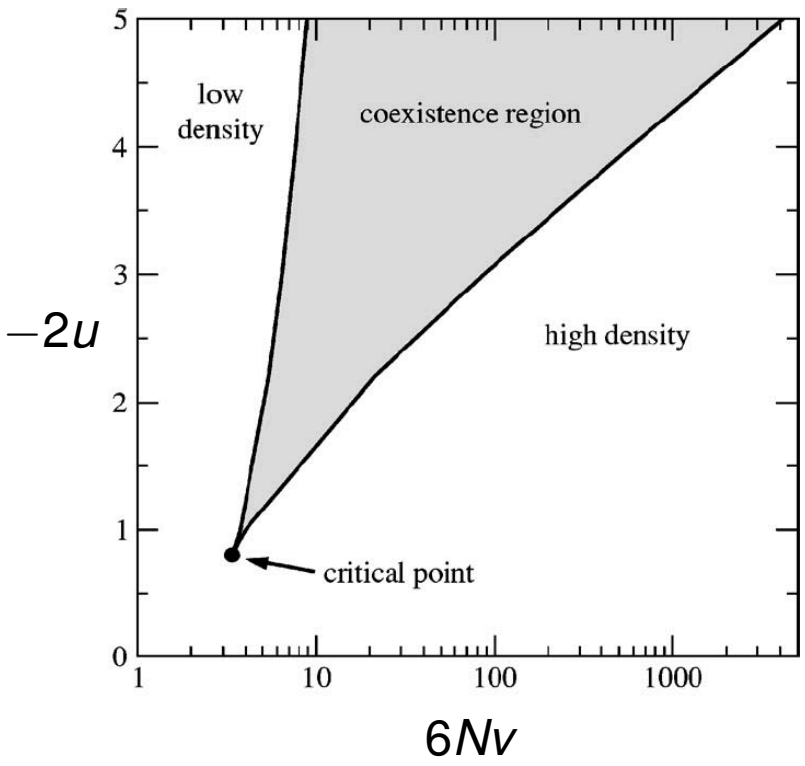
— $u = \mathcal{O}(1)$, $v = \mathcal{O}(N^{-1})$
if $v = 0$: ER ensemble with $\langle k \rangle = \mathcal{O}(N)$

— mean-field approx:

$$p(\mathbf{c}) \propto \mathrm{e}^{u \sum_{ij} c_{ij} + v \sum_{ijk} c_{ij} \langle c_{jk} c_{ki} \rangle}$$

— coupled eqns for
$m = \langle c_{ij} \rangle$ and $q = \langle c_{ik} c_{kj} \rangle$

result: phase diagram



Generalisations

- **control closed paths of all lengths** $\ell \leq L$ (Strauss: $L = 3$)

$$p(\mathbf{c}) \propto \mathrm{e}^{u \sum_{ij} c_{ij} + \sum_{\ell=3}^{L} v_\ell \sum_{i_1 \ldots i_\ell} c_{i_1 i_2} c_{i_2 i_3} \ldots c_{i_\ell i_1}}$$

generating function:
(use $\sum_{ij} c_{ij} = \sum_{ij} c_{ij} c_{ji}$)

$$\phi = \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{u \mathrm{Tr}(\mathbf{c}^2) + \sum_{\ell=3}^{L} v_\ell \mathrm{Tr}(\mathbf{c}^\ell)}$$

$$\langle k \rangle = \frac{\partial \phi}{\partial u}, \qquad \langle m_\ell \rangle = \frac{1}{N} \langle \mathrm{Tr}(\mathbf{c}^\ell) \rangle = \frac{\partial \phi}{\partial v_\ell}, \qquad S = \phi - u \langle k \rangle - \sum_{\ell=3}^{L} v_\ell \langle m_\ell \rangle$$

- control closed paths of all lengths, since $\mathrm{Tr}(\mathbf{c}^\ell) = N \int \mathrm{d}\mu \, \mu^\ell \varrho(\mu|\mathbf{c})$:

control eigenvalue density $\varrho(\mu)$

$$p(\mathbf{c}) \propto \mathrm{e}^{N \int \mathrm{d}\mu \; \hat{\varrho}(\mu) \varrho(\mu|\mathbf{c})}$$

$$(\textit{above}: \; \hat{\varrho}(\mu) = \sum_{\ell} v_\ell \mu^\ell)$$

generating function:

$$\phi = \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{N \int \mathrm{d}\mu \; \hat{\varrho}(\mu) \varrho(\mu|\mathbf{c})}$$

$$\varrho(\mu) = \frac{\delta \phi}{\delta \hat{\varrho}(\mu)}, \qquad S = \phi - \int \mathrm{d}\mu \; \hat{\varrho}(\mu) \varrho(\mu)$$

Calculation road map

target: $\phi = \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{u \sum_{ij} c_{ij} + N \int \mathrm{d}\mu \; \hat{\varrho}(\mu) \varrho(\mu|\mathbf{c})}$ *Strauss* : $\hat{\varrho}(\mu) = v\mu^3$

Burda's scaling regime : $u = \frac{1}{2} \log(\bar{k}/N), \quad N \to \infty$

Shannon entropy? *observables*? *phase transitions*?

- derive expression in which sum over graphs can be done:

$$\phi = \lim_{\varepsilon, \Delta \downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{u \sum_{ij} c_{ij}} \prod_{\mu} \left[Z(\mu + \mathrm{i}\varepsilon|\mathbf{c})^{\mathrm{i}\Delta\lambda(\mu)} \; \overline{Z(\mu + \mathrm{i}\varepsilon|\mathbf{c})}^{\,-\mathrm{i}\Delta\lambda(\mu)} \right]$$

$$Z(\mu|\mathbf{c}) = \int \mathrm{d}\boldsymbol{\phi} \; \mathrm{e}^{-\frac{1}{2}\mathrm{i}\boldsymbol{\phi} \cdot [\mathbf{c} - \mu \mathbf{I}] \boldsymbol{\phi}}, \qquad \lambda(\mu) = \frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}\mu} \hat{\varrho}(\mu)$$

- replica analysis, saddle-point eqns for $N \to \infty$, analytical continuation to *imaginary* dimension, limits $\epsilon \downarrow 0$ and $\Delta \downarrow 0$

- replica symmetry, bifurcation analysis, phase transitions and entropy

feasible?

Conversion into a replica formulation

ensemble constraints written in terms of spectrum,
use Edwards-Jones formula (1976):

$$\varrho(\mu|\mathbf{c}) = \frac{2}{N\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Im} \frac{\partial}{\partial\mu} \log Z(\mu+\mathrm{i}\varepsilon|\mathbf{c}), \qquad Z(\mu|\mathbf{c}) = \int \mathrm{d}\boldsymbol{\phi}\, \mathrm{e}^{-\frac{1}{2}\mathrm{i}\boldsymbol{\phi}\cdot[\mathbf{c}-\mu\mathbf{I}]\boldsymbol{\phi}}$$

insert into ϕ,
integrate by parts,
discretise μ-integral:

$$\begin{aligned}
\phi &= \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{u\sum_{ij} c_{ij} + N\int \mathrm{d}\mu\ \hat{\varrho}(\mu) \frac{2}{N\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Im} \frac{\partial}{\partial\mu} \log Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})} \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{u\sum_{ij} c_{ij} - \frac{2}{\pi}\int \mathrm{d}\mu\ \mathrm{Im}\log Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})\ \frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)} \\
&= \lim_{\varepsilon,\Delta\downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{u\sum_{ij} c_{ij} - \frac{2\Delta}{\pi}\sum_{\mu} \mathrm{Im}\log Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})\ \frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)} \\
&= \lim_{\varepsilon,\Delta\downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{u\sum_{ij} c_{ij}} \prod_{\mu} \mathrm{e}^{-2\mathrm{Im}\log Z(\mu+\mathrm{i}\varepsilon|\mathbf{c}).\ \frac{\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)}
\end{aligned}$$

$$\mathrm{e}^{-2\ \mathrm{Im}\log z} = z^{\mathrm{i}}.\overline{z}^{\ -\mathrm{i}}$$

$$\phi = \lim_{\varepsilon,\Delta\downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{u\sum_{ij} c_{ij}} \prod_{\mu} \Big[Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})^{\mathrm{i}}\ \overline{Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})}^{-\mathrm{i}} \Big]^{\frac{\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)}$$

Flavours of the replica method

the replica dimension n ...

- $n=0$: *Kac (1968), Sherrington, Kirkpatrick (1975), Parisi (1979)*
  stat mech of disordered spin systems

$$\log Z = \lim_{n\to 0} \frac{1}{n}(Z^n - 1), \qquad \overline{\log Z} = \lim_{n\to 0} \frac{1}{n} \log \overline{Z^n}$$

- $n\in\mathbb{R},\ >0$: *Sherrington (1980), Coolen, Penney, Sherrington (1993)*
  'slow' dynamics of parameters in 'fast' spin system
  (partial annealing, $n = T/T'$)

- $n\in\mathbb{R},\ <0$: *Dotsenko, Franz, Mezard (1994)*
  slow dynamics evolves to *maximise* free energy of fast system

many applications of finite n replica method,
neural networks, protein folding, ...

here: $n \notin \mathbb{R}$ ...

Derivation of order parameter eqns

replica dimensions:
$n_\mu = -m_\mu = \frac{\mathrm{i}\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)$

- prepare summation over graphs:

$$\phi = \lim_{\epsilon,\Delta\downarrow 0}\frac{1}{N}\log\sum_{\mathbf{c}}\mathrm{e}^{u\sum_{ij}c_{ij}}\prod_{\mu}\Big[Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})^{n_\mu}\,\overline{Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})}^{m_\mu}\Big]$$

use $u = -\frac{1}{2}\log(N/\bar{k})$, and

$$p_{\mathrm{ER}}(\mathbf{c}|\bar{k}) = \prod_{i<j}\Big[\frac{\bar{k}}{N}\delta_{c_{ij},1} + (1-\frac{\bar{k}}{N})\delta_{c_{ij},0}\Big]$$

to get

$$\phi = \frac{1}{2}\bar{k} + \lim_{\epsilon,\Delta\downarrow 0}\frac{1}{N}\log\Big\langle\prod_{\mu}\Big[Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})^{n_\mu}\,\overline{Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})}^{m_\mu}\Big\rangle_{\mathrm{ER}} + \mathcal{O}(\frac{1}{N})$$

$$Z(\mu|\mathbf{c}) = \int\mathrm{d}\boldsymbol{\phi}\;\mathrm{e}^{-\frac{1}{2}\mathrm{i}\boldsymbol{\phi}\cdot[\mathbf{c}-\mu\mathbf{I}]\boldsymbol{\phi}}$$

- evaluate $Z(\mu+\mathrm{i}\epsilon|\mathbf{c})^{n_\mu}$ and $\overline{Z(\mu+\mathrm{i}\epsilon|\mathbf{c})}^{m_\mu}$ for integer n_μ and m_μ,

  and do average over graphs

$$\left\langle \prod_\mu \left\{ \Big[\prod_{\alpha_\mu=1}^{n_\mu} \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{\phi}\; \mathrm{e}^{-\frac{1}{2}\varepsilon\boldsymbol{\phi}^2-\frac{1}{2}\mathrm{i}\boldsymbol{\phi}\cdot(\mathbf{c}-\mu\mathbb{1})\boldsymbol{\phi}} \Big] \Big[\prod_{\beta_\mu=1}^{m_\mu} \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{\psi}\; \mathrm{e}^{-\frac{1}{2}\varepsilon\boldsymbol{\psi}^2+\frac{1}{2}\mathrm{i}\boldsymbol{\psi}\cdot(\mathbf{c}-\mu\mathbb{1})\boldsymbol{\psi}} \Big] \right\} \right\rangle_{\mathrm{ER}}$$

$$= \int \prod_i \Big[\mathrm{d}\boldsymbol{\phi}^i \mathrm{d}\boldsymbol{\psi}^i\; \mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)(\boldsymbol{\phi}^i)^2-\frac{1}{2}(\varepsilon+\mathrm{i}\mu)(\boldsymbol{\psi}^i)^2} \Big] \mathrm{e}^{\frac{1}{2}\sum_{i\neq j}\log\left\{1+\frac{\bar{k}}{N}\left[\exp[\mathrm{i}(\boldsymbol{\psi}^i\cdot\boldsymbol{\psi}^j-\boldsymbol{\phi}^i\cdot\boldsymbol{\phi}^j)]-1\right]\right\}}$$

$$\boldsymbol{\phi}^i = \{\phi^i_{\mu,\alpha_\mu\leq n_\mu}\}, \quad \boldsymbol{\psi}^i = \{\psi^i_{\mu,\beta_\mu\leq m_\mu}\}$$

- introduce functional order parameter

$$\begin{array}{l}\forall\boldsymbol{\phi} = \{\phi_{\mu,\alpha_\mu\leq n_\mu}\} \\ \forall\boldsymbol{\psi} = \{\psi_{\mu,\beta_\mu\leq m_\mu}\}\end{array} : \qquad 1 = \int \mathrm{d}\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi})\; \delta\Big[\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi}) - \frac{1}{N}\sum_i \delta(\boldsymbol{\phi}-\boldsymbol{\phi}^i)\delta(\boldsymbol{\psi}-\boldsymbol{\psi}^i)\Big]$$

theory in terms of
$\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi})$ *and* $\hat{\mathcal{P}}(\boldsymbol{\phi},\boldsymbol{\psi})$

- path integral form:

$$\phi = \lim_{N\to\infty}\lim_{\epsilon,\Delta\downarrow 0}\frac{1}{N}\log\int\{\mathrm{d}\mathcal{P}\mathrm{d}\hat{\mathcal{P}}\}\mathrm{e}^{N\Psi[\{\mathcal{P},\hat{\mathcal{P}}\}]} = \lim_{\epsilon,\Delta\downarrow 0}\mathrm{extr}_{\{\mathcal{P},\hat{\mathcal{P}}\}}\ \Psi[\{\mathcal{P},\hat{\mathcal{P}}\}]$$

$$\begin{aligned}\Psi[\{\mathcal{P},\hat{\mathcal{P}}\}] &= \mathrm{i}\int\mathrm{d}\boldsymbol{\phi}\mathrm{d}\boldsymbol{\psi}\ \hat{\mathcal{P}}(\boldsymbol{\phi},\boldsymbol{\psi})\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi})\\ &\quad+\frac{1}{2}\bar{k}\int\mathrm{d}\boldsymbol{\phi}\mathrm{d}\boldsymbol{\psi}\mathrm{d}\boldsymbol{\phi}'\mathrm{d}\boldsymbol{\psi}'\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi})\mathcal{P}(\boldsymbol{\phi}',\boldsymbol{\psi}')\mathrm{e}^{\mathrm{i}(\boldsymbol{\psi}\cdot\boldsymbol{\psi}'-\boldsymbol{\phi}\cdot\boldsymbol{\phi}')}\\ &\quad+\log\int\mathrm{d}\boldsymbol{\phi}\mathrm{d}\boldsymbol{\psi}\ \mathrm{e}^{-\frac{1}{2}\boldsymbol{\phi}\cdot(\varepsilon\mathbf{I}-\mathrm{i}\mathbf{M})\boldsymbol{\phi}-\frac{1}{2}\boldsymbol{\psi}\cdot(\varepsilon\mathbf{I}+\mathrm{i}\mathbf{M})\boldsymbol{\psi}-\mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\phi},\boldsymbol{\psi})}\end{aligned}$$

$$M_{\mu,\alpha;\mu',\alpha'} = \mu\delta_{\mu\mu'}\delta_{\alpha\alpha'}$$

$$\boldsymbol{\phi} = \{\phi_{\mu,\alpha_\mu\le n_\mu}\},\qquad n_\mu = \frac{\mathrm{i}\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)$$
$$\boldsymbol{\psi} = \{\psi_{\mu,\beta_\mu\le m_\mu}\},\quad m_\mu = -\frac{\mathrm{i}\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)$$

- saddle-point eqns, $\mathcal{Q} = \exp[-\mathrm{i}\hat{\mathcal{P}}]$:

$$\begin{aligned}\mathcal{Q}(\boldsymbol{\phi},\boldsymbol{\psi}) &= \exp\left[\bar{k}\int\mathrm{d}\boldsymbol{\phi}'\mathrm{d}\boldsymbol{\psi}'\ \mathcal{P}(\boldsymbol{\phi}',\boldsymbol{\psi}')\mathrm{e}^{\mathrm{i}(\boldsymbol{\psi}\cdot\boldsymbol{\psi}'-\boldsymbol{\phi}\cdot\boldsymbol{\phi}')}\right]\\ \mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi}) &= \frac{\mathcal{Q}(\boldsymbol{\phi},\boldsymbol{\psi})\mathrm{e}^{-\frac{1}{2}\boldsymbol{\phi}\cdot(\varepsilon\mathbf{I}-\mathrm{i}\mathbf{M})\boldsymbol{\phi}-\frac{1}{2}\boldsymbol{\psi}\cdot(\varepsilon\mathbf{I}+\mathrm{i}\mathbf{M})\boldsymbol{\psi}}}{\int\mathrm{d}\boldsymbol{\phi}'\mathrm{d}\boldsymbol{\psi}'\ \mathcal{Q}(\boldsymbol{\phi}',\boldsymbol{\psi}')\mathrm{e}^{-\frac{1}{2}\boldsymbol{\phi}'\cdot(\varepsilon\mathbf{I}-\mathrm{i}\mathbf{M})\boldsymbol{\phi}'-\frac{1}{2}\boldsymbol{\psi}'\cdot(\varepsilon\mathbf{I}+\mathrm{i}\mathbf{M})\boldsymbol{\psi}'}}\end{aligned}$$

Replica symmetric ansatz

- de Finetti's theorem, combined with $\mathcal{P}(\boldsymbol{\psi},\boldsymbol{\phi}) = \overline{\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi})}$:

$$\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi}) = \int\{\mathrm{d}\pi\}\mathcal{W}[\{\pi\}]\Big[\prod_{\mu}\prod_{\alpha_\mu=1}^{n_\mu}\pi(\phi_{\mu,\alpha_\mu}|\mu)\Big]\Big[\prod_{\mu}\prod_{\beta_\mu=1}^{m_\mu}\overline{\pi(\psi_{\mu,\beta_\mu}|\mu)}\Big]$$

$$\int\{\mathrm{d}\pi\}\mathcal{W}[\{\pi\}] = 1, \qquad \int\mathrm{d}\phi\;\pi(\phi|\mu) = 1 \;\; \textit{for all } \mu$$

- insert into saddle-point eqns,
  $\Delta\downarrow 0$: $\Delta\sum_\mu \to \int\mathrm{d}\mu$

$$\mathcal{W}[\{\pi\}] = \frac{\sum_{\ell\geq 0}\mathrm{e}^{-\bar{k}}\frac{\bar{k}^\ell}{\ell!}\int\Big(\prod_{r\leq\ell}\{\mathrm{d}\pi_r\}\mathcal{W}[\{\pi_r\}]\Big)\mathcal{D}[\{\pi_1,\ldots,\pi_\ell\}]\delta\Big[\pi-\mathcal{F}[\{\pi_1,\ldots,\pi_\ell\}]\Big]}{\sum_{\ell\geq 0}\mathrm{e}^{-\bar{k}}\frac{\bar{k}^\ell}{\ell!}\int\Big(\prod_{r\leq\ell}\{\mathrm{d}\pi_r\}\mathcal{W}[\{\pi_r\}]\Big)\mathcal{D}[\{\pi_1,\ldots,\pi_\ell\}]}$$

$$\mathcal{F}(\phi|\mu;\pi_1,\ldots,\pi_\ell) = \frac{\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)\phi^2}\prod_{r\leq\ell}\hat{\pi}_r(\phi|\mu)}{\int\mathrm{d}\phi'\;\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)\phi'^2}\prod_{r\leq\ell}\hat{\pi}_r(\phi'|\mu)}, \qquad \hat{\pi}(\phi|\mu) = \int\mathrm{d}x\;\pi(x|\mu)\mathrm{e}^{-\mathrm{i}x\phi}$$

$$\mathcal{D}[\{\pi_1,\ldots,\pi_\ell\}] = \mathrm{e}^{-\frac{2}{\pi}\mathrm{Im}\int\mathrm{d}\mu\;\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)\;\log\int\mathrm{d}s\;\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)s^2}\prod_{r\leq\ell}\hat{\pi}_r(s|\mu)}, \qquad \textit{no loops}:\;\mathcal{D}[..] = 1$$

- nature of RS solutions $\mathcal{W}[\{\pi\}]$
  (similar to spectrum calculations)

$$\pi(\phi|\mu) = \frac{\mathrm{e}^{-\frac{1}{2}\epsilon\phi^2 - \frac{1}{2}\mathrm{i}x(\mu)\phi^2 + y(\mu)\phi}}{\int \mathrm{d}\phi' \; \mathrm{e}^{-\frac{1}{2}\epsilon\phi'^2 - \frac{1}{2}\mathrm{i}x(\mu)\phi'^2 + y(\mu)\phi'}} : \qquad \mathcal{W}[\{\pi\}] \;\rightarrow\; W[\{x,y\}]$$

- insert into RS eqns,
  take $\epsilon \downarrow 0$

$$W[\{x,y\}] = \frac{\sum_{\ell\geq 0} p_\ell \int \left(\prod_{r\leq\ell}\{\mathrm{d}x_r \mathrm{d}y_r\} W[\{x_r,y_r\}]\right)\tilde{\mathcal{D}}[\ldots]\;\delta\big[x - F_x[\ldots]\big]\delta\big[y - F_y[\ldots]\big]}{\sum_{\ell\geq 0} p_\ell \int \left(\prod_{r\leq\ell}\{\mathrm{d}x_r \mathrm{d}y_r\} W[\{x_r,y_r\}]\right)\tilde{\mathcal{D}}[\ldots]}$$

$$F_x[\mu|\{x_1,\ldots,x_\ell\}] = -\mu - \sum_{r\leq\ell}\frac{1}{x_r(\mu)}$$

$$F_y[\mu|\{x_1,y_1,\ldots,x_\ell,y_\ell\}] = -\sum_{r\leq\ell}\frac{y_r(\mu)}{x_r(\mu)} \qquad\qquad p_\ell = \mathrm{e}^{-\bar{k}}\bar{k}^\ell/\ell!$$

$$\tilde{\mathcal{D}}[\{x_1,y_1,\ldots,x_\ell,y_\ell\}] = \mathrm{e}^{\int \mathrm{d}\mu\;\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)\;\left[\frac{1}{2}\mathrm{sgn}\left(F_x[\mu|x_1,\ldots,x_\ell]\right) + \frac{1}{\pi}\frac{F_y^2[\mu|x_1,y_1,\ldots,x_\ell,y_\ell]}{F_x[\mu|x_1,\ldots,x_\ell]}\right]}$$

everything now real-valued!

- alternative form

$$W[\{x,y\}] = \frac{\tilde{\mathcal{D}}[\{x,y\}] \sum_{\ell\geq 0} p_\ell \int \big(\prod_{r\leq\ell}\{\mathrm{d}x_r \mathrm{d}y_r\} W[\{x_r,y_r\}]\big)\delta\big[x - F_x[\ldots]\big]\delta\big[y - F_y[\ldots]\big]}{\sum_{\ell\geq 0} p_\ell \int \big(\prod_{r\leq\ell}\{\mathrm{d}x_r \mathrm{d}y_r\} W[\{x_r,y_r\}]\big)\tilde{\mathcal{D}}[F_x[\ldots], F_y[\ldots]]}$$

$$F_x[\mu|\{x_1,\ldots,x_\ell\}] = -\mu - \sum_{r\leq\ell} 1/x_r(\mu)$$

$$F_y[\mu|\{x_1,y_1,\ldots,x_\ell,y_\ell\}] = -\sum_{r\leq\ell} y_r(\mu)/x_r(\mu) \qquad p_\ell = \mathrm{e}^{-\bar{k}}\bar{k}^\ell/\ell!$$

$$\tilde{\mathcal{D}}[\{x,y\}] = \exp\Big[\int \mathrm{d}\mu\, \frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)\, \big[\frac{1}{2}\mathrm{sgn}[x(\mu)] + \frac{1}{\pi}\frac{y^2(\mu)}{x(\mu)}\big]\Big]$$

Phase transition formulae

- Remaining symmetry:

  invariance under $\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi}) \to \mathcal{P}(-\boldsymbol{\phi},-\boldsymbol{\psi})$
  here: $\{y\} \to \{-y\}$

$$\begin{aligned} \textit{weakly symmetric solns}: &\quad W[\{x,-y\}] = W[\{x,y\}] \\ \textit{strongly symmetric solns}: &\quad W[\{x,y\}] = W[\{x\}]\,\delta[\{y\}] \end{aligned}$$

$$W[\{x\}] = \frac{\tilde{\mathcal{D}}[\{x,0\}]\sum_{\ell\geq 0} p_\ell \int\big(\prod_{r\leq\ell}\{\mathrm{d}x_r\mathrm{d}\}W[\{x_r\}]\big)\delta\big[x-F_x[\ldots]\big]}{\sum_{\ell\geq 0} p_\ell \int\big(\prod_{r\leq\ell}\{\mathrm{d}x_r\}W[\{x_r\}]\big)\tilde{\mathcal{D}}[F_x[\ldots],0]}$$

- symmetry-breaking:

$$\begin{aligned} \textit{SG type}: &\quad W[\{x,y\}] = W[\{x\}]\delta[\{y\}] \;\to\; W[\{x,-y\}] = W[\{x,y\}] \\ \textit{F type}: &\quad W[\{x,y\}] = W[\{x\}]\delta[\{y\}] \;\to\; W[\{x,-y\}] \neq W[\{x,y\}] \end{aligned}$$

  continuous transitions:

  functional moment expansion

$$\begin{aligned} \psi(\mu_1,\ldots,\mu_n|\{x\}) &= \int\{\mathrm{d}y\}W[\{y|x\}]\; y(\mu_1)\ldots y(\mu_n) \\ &= \mathcal{O}(\varepsilon^n) \quad \textit{close to transition} \end{aligned}$$

lowest orders:

- F-type bifurcation, $\mathcal{O}(\epsilon)$:

$$\psi(\mu|\{x\}) = \int \{\mathrm{d}x'\}\, \mathcal{B}_1[\mu;\{x,x'\}]\psi(\mu|\{x'\}) + \mathcal{O}(\epsilon^2)$$

$$\mathcal{B}_1[\mu;\{x,x'\}] = -\bar{k}\frac{W[\{x'\}]}{x'(\mu)}\frac{\sum_{\ell\geq 0} p_\ell \int\big(\prod_{r\leq\ell}\{\mathrm{d}x_r\}W[\{x_r\}]\big)\delta\big[x-F_x[\{x_1,\ldots,x_\ell,x'\}]\big]}{\sum_{\ell\geq 0} p_\ell \int\big(\prod_{r\leq\ell}\{\mathrm{d}x_r\}W[\{x_r\}]\big)\delta\big[x-F_x[\{x_1,\ldots,x_\ell\}]\big]}$$

$F_x[\{x_1,\ldots,x_\ell,x'\}] = F_x[\{x_1,\ldots,x_\ell\}] - \frac{1}{x'}$:

$$\mathcal{B}_1[\mu;\{x,x'\}] = -\bar{k}\frac{W[\{x'\}]}{x'(\mu)}\frac{\sum_{\ell\geq 0} p_\ell \int\big(\prod_{r\leq\ell}\{\mathrm{d}x_r\}W[\{x_r\}]\big)\delta\big[x+\frac{1}{x'}-F_x[\{x_1,\ldots,x_\ell\}]\big]}{\sum_{\ell\geq 0} p_\ell \int\big(\prod_{r\leq\ell}\{\mathrm{d}x_r\}W[\{x_r\}]\big)\delta\big[x-F_x[\{x_1,\ldots,x_\ell\}]\big]}$$

$$= -\bar{k}\frac{W[\{x'\}]}{x'(\mu)}\frac{W[\{x+\frac{1}{x'}\}]}{\tilde{\mathcal{D}}[\{x+\frac{1}{x'},0\}]}\frac{\tilde{\mathcal{D}}[\{x,0\}]}{W[\{x\}]}$$

hence

$$\int \{\mathrm{d}x'\}\mathcal{B}[\{x,x'\}]\zeta(\mu|\{x'\}) = -x(\mu)\zeta(\mu|\{x\})$$

$$\mathcal{B}[\{x,x'\}] = \bar{k}\frac{W[\{x+\frac{1}{x'}\}]\tilde{\mathcal{D}}[\{x',0\}]}{\tilde{\mathcal{D}}[\{x+\frac{1}{x'},0\}]}$$

- SG-type bifurcation, $\mathcal{O}(\epsilon^2)$:

$$\psi(\mu,\mu'|\{x\}) = \int \{\mathrm{d}x'\}\mathcal{B}_2(\mu,\mu';\{x,x'\}]\psi(\mu,\mu'|\{x'\})$$

$$\mathcal{B}_2(\mu,\mu';\{x,x'\}] = \frac{\bar{k}W[\{x'\}]}{x'(\mu)x'(\mu')}\frac{\sum_{\ell\geq 0}p_\ell\int\big(\prod_{r\leq\ell}\{\mathrm{d}x_r\}W[\{x_r\}]\big)\delta\big[x+\frac{1}{x'}-F_x[\{x_1,\ldots,x_\ell\}]\big]}{\sum_{\ell\geq 0}p_\ell\int\big(\prod_{r\leq\ell}\{\mathrm{d}x_r\}W[\{x_r\}]\big)\delta\big[x-F_x[\ldots]\big]}$$

$$= \frac{\bar{k}W[\{x'\}]}{x'(\mu)x'(\mu')}\frac{W[\{x+\frac{1}{x'}\}]}{\tilde{\mathcal{D}}[\{x+\frac{1}{x'},0\}]}\frac{\tilde{\mathcal{D}}[\{x,0\}]}{W[\{x\}]}$$

hence

$$\int \{\mathrm{d}x'\}\mathcal{B}[\{x,x'\}]\zeta(\mu,\mu'|\{x'\}) = x(\mu)x(\mu')\zeta(\mu|\{x\})$$

$$\mathcal{B}[\{x,x'\}] = \bar{k}\frac{W[\{x+\frac{1}{x'}\}]\tilde{\mathcal{D}}[\{x',0\}]}{\tilde{\mathcal{D}}[\{x+\frac{1}{x'},0\}]}$$

RS Entropy

follows directly from

$$\phi_{\rm RS} = \log \sum_{\ell\geq 0} {\rm e}^{-\bar{k}} \frac{\bar{k}^\ell}{\ell!} \int \Big(\prod_{r\leq \ell} \{{\rm d}\pi_r\} \mathcal{W}[\{\pi_r\}] \Big) \mathcal{D}[\{\pi_1,\ldots,\pi_\ell\}] \Big\}$$
$$+\bar{k} - \frac{1}{2}\bar{k} \int \{{\rm d}\pi {\rm d}\pi'\} \mathcal{W}[\{\pi\}] \mathcal{W}[\{\pi'\}] \mathcal{E}[\{\pi,\pi'\}]$$

with

$$\mathcal{E}[\{\pi,\pi'\}] = {\rm e}^{\frac{2}{\pi}{\rm Im}\int {\rm d}\mu\ \hat{\varrho}(\mu)\frac{\rm d}{{\rm d}\mu}\log\int {\rm d}x{\rm d}x'{\rm e}^{-{\rm i}xx'}\pi(x|\mu)\pi'(x'|\mu)}$$

Extremisation:

$$\int \{{\rm d}\pi'\} \mathcal{W}[\{\pi'\}] \mathcal{E}[\{\pi,\pi'\}] = \frac{\sum_{\ell\geq 0} {\rm e}^{-\bar{k}} \frac{\bar{k}^\ell}{\ell!} \int \Big(\prod_{r\leq \ell} \{{\rm d}\pi_r\} \mathcal{W}[\{\pi_r\}] \Big) \mathcal{D}[\{\pi_1,\ldots,\pi_\ell,\pi\}]}{\sum_{\ell\geq 0} {\rm e}^{-\bar{k}} \frac{\bar{k}^\ell}{\ell!} \int \Big(\prod_{r\leq \ell} \{{\rm d}\pi_r\} \mathcal{W}[\{\pi_r\}] \Big) \mathcal{D}[\{\pi_1,\ldots,\pi_\ell\}]}$$

(equivalent with earlier RS eqn)

Summary

- new analytical approach to loopy random graph ensembles, developed for generalised Strauss model

- based on replica form for generating function, in which sum over graphs can be done

$$\phi = \lim_{\varepsilon,\Delta\downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{u\sum_{ij} c_{ij}} \prod_{\mu} \Big[Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})^{\mathrm{i}\Delta\lambda(\mu)}\, \overline{Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})}^{\,-\mathrm{i}\Delta\lambda(\mu)} \Big]$$

$$Z(\mu|\mathbf{c}) = \int \mathrm{d}\boldsymbol{\phi}\; \mathrm{e}^{-\frac{1}{2}\mathrm{i}\boldsymbol{\phi}\cdot[\mathbf{c}-\mu\mathbf{1}]\boldsymbol{\phi}}$$

- intuitive RS order parameter eqns

$$\mathcal{W}[\{\pi\}] = \frac{\sum_{\ell\geq 0} \mathrm{e}^{-\bar{k}}\frac{\bar{k}^{\ell}}{\ell!} \int \Big(\prod_{r\leq\ell}\{\mathrm{d}\pi_r\}\mathcal{W}[\{\pi_r\}]\Big)\mathcal{D}[\{\pi_1,\ldots,\pi_\ell\}]\delta\Big[\pi-\mathcal{F}[\{\pi_1,\ldots,\pi_\ell\}]\Big]}{\sum_{\ell\geq 0} \mathrm{e}^{-\bar{k}}\frac{\bar{k}^{\ell}}{\ell!} \int \Big(\prod_{r\leq\ell}\{\mathrm{d}\pi_r\}\mathcal{W}[\{\pi_r\}]\Big)\mathcal{D}[\{\pi_1,\ldots,\pi_\ell\}]}$$

$\mathcal{D}[\{\pi_1,\ldots,\pi_\ell\}]$ indep of $\{\pi_1,\ldots,\pi_\ell\}$ only when loops are absent

- potential for symmetry-breaking phase transitions

to be done – short term

- solve RS phase transition equations,
  analytically or numerically (population dynamics)
- generate phase diagram,
  characterise phases (symmetries, entropies)
- physical meaning of $W[\{x, y\}]$
- numerical simulations

to be done – longer term

- integrate with earlier tailored random graph ensembles
- beyond RS ...
- extend formalism to $\hat{\varrho}(\mu) = \mathcal{O}(\log N)$,
  (where numbers of short loops scale as N)