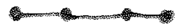
I. RAAGs and their automorphisms

$\Gamma = (V, E)$ 

$A_\Gamma = \langle V \mid [v,w] \text{ if } \underset{v\quad w}{\bullet\!\!-\!\!\bullet} \subseteq \Gamma \rangle$.

__Ex:__

		
F_4	$\mathbb{Z}^3$	$A_\Gamma = \pi_1(S^3 \setminus$  $)$
$Out(F_4)$	$GL_3\mathbb{Z}$	$Out(A_\Gamma) = ?$
CV_4	$Q_3 = SL_3\mathbb{R}/SO_3\mathbb{R}$	? ? ? ? Linked ?

__Generators for Aut/Out(A_Γ)__

Laurence-Servatius generators

$lk(v) = \{w \in \Gamma \mid d(v,w) = 1\}$

$st(v) = lk(v) \cup \{v\}$.

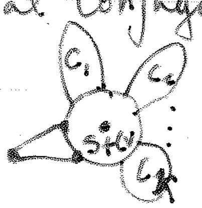
1. Inversions, graph automorphisms
2. Transvections

$v \mapsto vw$

(a) __Fold:__ v not connected to w

(b) __Twist:__ v connected to w.

3. Partial Conjugations:



$C_i \mapsto v C_i v^{-1}$

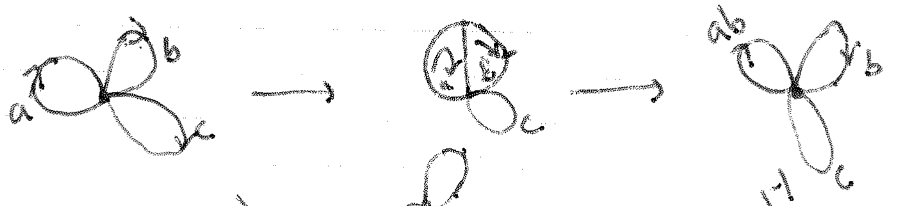
$\boxed{F_n}$ $U(A_\Gamma) = \langle 1, 3, 2a \rangle \leq Out$

$\boxed{\mathbb{Z}^n}$ $T(A_\Gamma) = \langle 2b \rangle$

II. __Outer space and blowups:__

Two basic moves on CV_n

1) __Fold__



$a \mapsto ab$
$b \mapsto b$
$c \mapsto c$

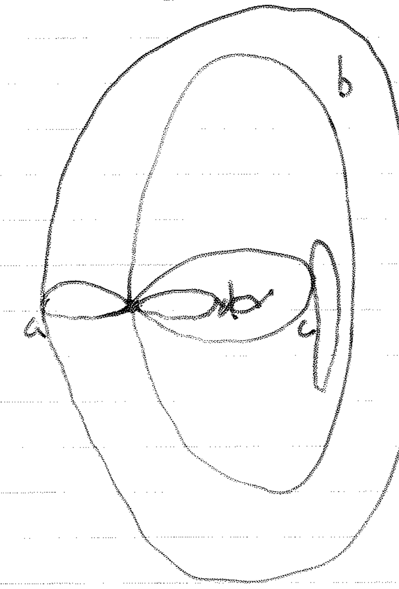
2) Partial Conjugation

$a \mapsto bab^{-1}$
$b \mapsto b$
$c \mapsto c$

Analogue of the rose: Salvetti Complex S_Γ

$$S_\Gamma^{(1)} = \bigvee_{i=1}^{|V|} S^1$$

$\Gamma =$ 



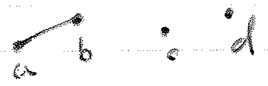
$S_\Gamma^{(2)} =$ Fill in T^2 along $[v,w]$

⋮

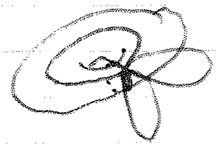
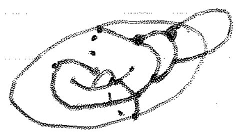
$S_\Gamma^{(k)} =$ Fill T^k along k-clique

$\pi_1(S_\Gamma) = A_\Gamma$, S_Γ aspherical, NPC.

Outer space for A_Γ:

$\Gamma =$ 

Consider marked CAT(0) metric spaces

$S_\Gamma =$  

$SL_2\mathbb{Z} \times SL_2\mathbb{Z} \leq Out(A_\Gamma)$

$\curvearrowright \mathbb{H}^2 \times \mathbb{H}^2 = \mathbb{H}^2 \times CV_2^{red}$ ← folds

↖ Dehn twists.

Charney-Stambaugh-Vogtmann: Construct untwisted outer space:

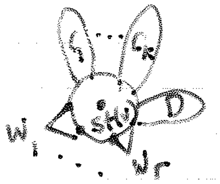
- finite-dim'l, contractible, simplicial complex
- $U(A_\Gamma) \curvearrowright \Sigma_\Gamma$ prop. disc. cocompactly by automorphisms

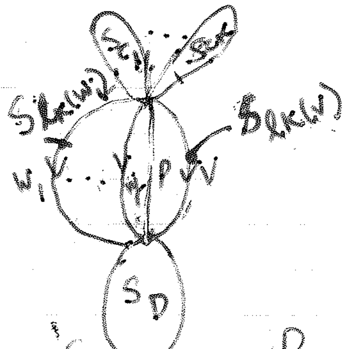
1-skeleton:

Choose $v \in V$

Folds: $w_1, \dots, w_r$

PCs: $C_1, \dots C_k$



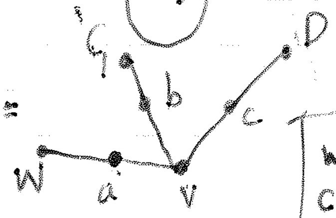


Fill in $st(v)$

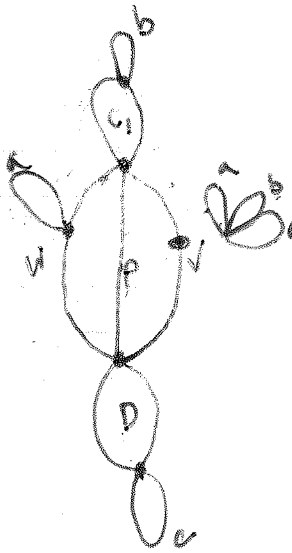
$e_v \cup e_p \rightsquigarrow S^1 \times S_{lk(v)}$

$e_{w_i} \rightsquigarrow [0,1] \times S_{lk(w_i)}$

Example: $\Gamma =$



$\boxed{\begin{array}{l} w \mapsto wv \\ c_1 \mapsto v c_1 v^{-1} \end{array}}$



Properties of blowups:

1. Special
2. Hyperplanes don't separate
3. Collapse hyperplanes. $\overline{N}(H) = H \times [0,1] \to H$
   $\Rightarrow c: X \to S_\Gamma$

[1 + 2 H hyperplane $\rightsquigarrow \chi_H \in H^1(X)$: counts algebraic intersection with H.]

Points of Σ_Γ: (X, α) $\alpha: X \to S_\Gamma$ homotopy equivalence s.t. $\alpha \circ c^{-1}: S_\Gamma \to S_\Gamma$ in $U(A_\Gamma)$.

Eq. relation:

$f: X \to X'$ cubical isomorphism

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & S_\Gamma \\ X' \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

Collapsing gives a poset structure

Σ_Γ = simplicial realization.

Action:

$\phi \in U(A_\Gamma)$, $\rho : S_\Gamma \to S_\Gamma$ $\qquad \phi.(X,\alpha) = (X, \rho\circ\alpha)$.

The Torelli subgroup: $A_\Gamma \to \mathbb{Z}^{|V|} \rightsquigarrow$
$$\begin{tikzcd} \mathrm{Aut}(A_\Gamma) \arrow[r] \arrow[dr] & GL(\mathbb{Z}^{|V|}) \\ & \mathrm{Out}(A_\Gamma) \arrow[u, "\Psi"'] \end{tikzcd}$$

$\mathcal{I}(A_\Gamma) = \ker \Psi$

Day: $\mathcal{I}(A_\Gamma)$ generated by PCs and commutator transvections $(v \mapsto [w_1, w_2]v) \Rightarrow \mathcal{I}(A_\Gamma) \leq U(A_\Gamma)$

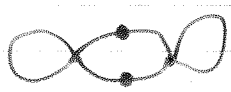
Thm (Wade, Toinet) $\mathcal{I}(A_\Gamma)$ is torsion-free.

Geometric Proof: Suppose $\phi \in \mathcal{I}(A_\Gamma)$, $\phi^n = 1$

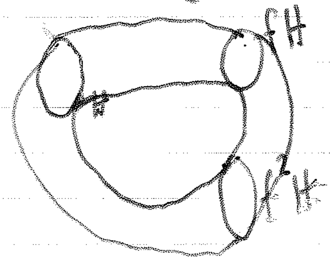
$\phi \curvearrowright \Sigma_\Gamma$ has a fixed pt $\Rightarrow (X,\alpha) \xrightarrow{f} (X,\alpha)$ $\quad [f] = \phi$.

Thm (B) Any automorphism of X acts non-trivially on $H_1(X)$.

$f_*(\chi_H) = \chi_{f^{-1}H}$.



- $H_1 \cap H_2 \neq \emptyset$ $\exists \gamma \in H_1$ $\chi_{H_1}(\gamma) = 1$, $\chi_{H_2}(\gamma) = 0$
- $H_1 \cap H_2 = \emptyset$, Every component of $X \setminus H_1 \cup H_2$ has homology



$\Rightarrow f = id$.