Arnaud Hilion — Word complexity for attracting laminations of free group automorphisms (with Gilbert Levitt)

F free group
∂F Gromov boundary
$\partial^2 F = \partial F \times \partial F \setminus \Delta$, $F \curvearrowright \partial^2 F$

- A lamination L is a subset of $\partial^2 F$ which is closed, F-invariant, flip-invariant (ie $(X,Y) = (Y,X)$)

Example: [Bestvina-Feighn-Handel]
$\phi \in Out(F_n)$ has a finite collection of "attracting laminations". Consider a relative train track map $f: G \to G$ rep. ϕ

exponentially growing (EG) stratum $\longleftrightarrow$ attracting lamination of ϕ
(obtained by iterating an edge)

In particular, if ϕ is fully irreducible, there is exactly one attracting lamination.

- "space of leaves" of "dual space" to L

$$\partial F / L := \partial F / \sim \quad X \sim Y \text{ iff } (X,Y) \in L$$

$\partial F / L$ is compact, $F \curvearrowright \partial F / L$ by homeo

Ex: ϕ iwip, $\partial F / L_\phi^+ = \hat{T}_\phi^+$ compact $\mathbb{R}$-tree [Q-map, Levitt-Lustig, Kapovich, Coulbois, Hilion]

ϕ iwip $\Rightarrow$ $\phi \curvearrowright \overline{CV_n}$ by North-South dynamics with fixed pts T_ϕ^+, T_ϕ^-

$\hat{T}_\phi^+ = \overline{T_\phi^+} \cup \partial T_\phi^+$ with observer's topology
metric completion

$L_\phi^\pm = L_\phi^+ \cup L_\phi^-$
$\hookleftarrow L_{\phi^{-1}}^+$

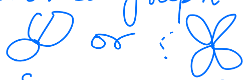
Q: what is $\partial F / L_\phi^\pm$? $\quad \partial F / L_\phi^\pm = \partial G_\phi$ where $G_\phi = F \rtimes_\phi \mathbb{Z}$
[Cannon-Thurston, Mj, Kapovich-Lustig etc...]

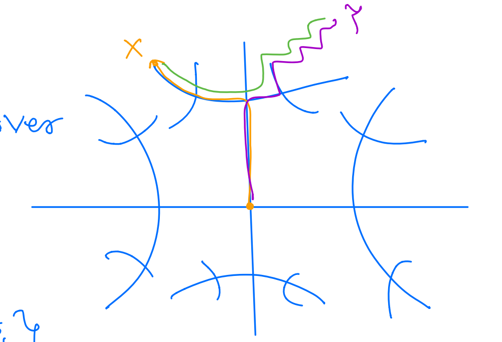
- Q: Characterize topological (dynamical) properties of $\partial F / L$ in terms of combinatorial (dynamical) properties of L.

Q. when is $\partial F / L$ a tree? $\partial F / L$ is a tree $\Leftrightarrow$ L is ?

• Other model for laminations:

fix marked graph G $\rightsquigarrow$ $\tilde{G}$ universal cover





let $L = \{(X,Y) \in \partial^2 F\}$

$(X,Y) \rightsquigarrow$ biinfinite line z in $\tilde{G}$

$\rightsquigarrow$ biinfinite reduced word in $E(G) = \{e_1, \dots e_k, \bar{e}_1, \bar{e}_2, \dots \bar{e}_k\}$

$\Sigma_G = \{ \dots x_i x_{i+1} \dots \mid \text{"reduced"} \}$

L is a subshift of Σ_G $:=$ $L \subset \Sigma_G$ which is closed, $\underbrace{S\text{-invariant}}_{\sim F\text{-invariant (from previous definit}^n)}$ (shift), flip-invariant

"abstract lamination $\rightsquigarrow$ symbolic lamination"

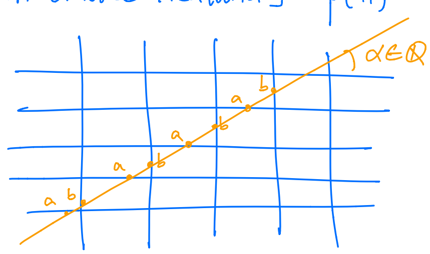
• Tools from symbolic dynamics

L has complexity function $p = p_L : \mathbb{N} \longrightarrow \mathbb{N}$

$n \longmapsto \# \mathcal{L}_n(L) = \#\{\text{subwords of length } n \text{ occurring in a leaf of } L\}$

$$\boxed{\text{topological entropy of } L = \lim_{n \to \infty} \frac{1}{n} \log p_L(n)}$$

- p bounded $\Longleftrightarrow$ ultimately periodic words
- Thm [Morse Hedlund] $p(n) = n+1 \Longleftrightarrow L$ is Sturman



- Thm [Pansiot] φ a non-erasing substitution, u a fixed point of φ

then $p_u(n) \asymp \begin{cases} 1 & \longleftrightarrow u \text{ ultimately periodic} \\ n \\ n \log n \\ n \log \log n \\ n^2 \end{cases}$

$f \asymp g$ if $\exists c$ $f(x) \le c\, g(x)$, $g(x) \le c\, f(x)$

• non-errasing substitution:

$\mathbb{A} = \{a_1, \dots a_k\}$
$\mathbb{A}^* =$ free monoid on $\mathbb{A}$
φ substitution = morphism of $\mathbb{A}^*$ ie $\varphi(a_i) \neq$ empty word

Ex. $\varphi(a_1) = a_1 a_2$
$\vdots$
$\varphi(a_k) = a_1 a_k a_2 a_1$

Ex. $\varphi(a) = ab$
$\varphi(b) = bab$
$u = abbabbababbab\dots\dots$

Choose a_i st $\varphi(a_i) = a_i\dots\dots$. Then iterate a_i by φ to get an ∞ ray.
count # words of length n in u.
Then Pansiot's thm says $P_u(n) \asymp \{1, n, \dots\dots\dots$

" u generates a lamination say L_u. Now $u' \in L_u$, then $P_{u'}(n) = P_u(n)$"
for same φ, can get different fixed points u.

Example: φ: $a \mapsto a$
$b \mapsto ba$
$c \mapsto cb$
$d \mapsto dc$

substitut^n
free group autom
relative train track

polynomial

1. $b \mapsto ba \mapsto ba^2 \mapsto ba^3 \mapsto \dots\dots \longrightarrow ba^\infty$ Example of constant complexity.

2. $c \mapsto cb \mapsto cbba \mapsto cbbaba^2 \longmapsto \dots \longrightarrow cbaba^2ba^3ba^4\dots\dots$

Example of quadratic complexity, fix n. If $k >> n$, then linear many words depending on posit^n of b. If $k > n/2$ again linear. Thus get quadratic complexity.

Example: φ: $a \mapsto b$
$b \mapsto c$
$c \mapsto ca$

iwip
$\lambda > 1$

iwip

$|\varphi^k(e)| \sim \lambda^k$

$\Rightarrow P_u(n) \asymp n$

"To see other complexities, don't want to be polynomial nor iwip, so need at least 2 EG stratum".

Example: φ: $a \mapsto ab$, $b \mapsto bab$ $\}$ λ_1; $c \mapsto d$, $d \mapsto dac$ $\}$ $\lambda_2 < \lambda_1$

Example: φ: $a \mapsto ab$, $b \mapsto bab$ $\}$ λ_2; $c \mapsto cd$, $d \mapsto dacd$ $\}$ λ_1

iterating d:

$d\,ac\,ab\,d\,abbab\,dac\,ab \ldots$

$\varphi^k(d) \simeq \lambda_1^k$

$p(n) \asymp n \log n$

iterating c:

$cd \cdot dacd \cdot dacd \cdot ab \cdot cd \cdot dcad \ldots$

$\varphi^k(c) \simeq k\lambda_1^k$

appears when two stratum with same λ interact.

$p(n) \asymp n \log\log n$

Theorem: (Hilion–Levitt) $\phi \in Out(F_n)$, $\varphi \in Aut(F_n)$
L_ϕ an attracting lamination
$u \in \partial F$ a limit point of φ.

Then:

$$P_{L_\phi}(n) \text{ or } P_u(n) \asymp \begin{cases} 1 \\ n \\ n\log n \\ n \log\log n \\ n^2 \end{cases}$$

complexity invariant under changing basis.

Tools for proof: Completely split train tracks (CT's) [Feighn–Handel]

main property of CT is the following:

$f: G \to G$
γ finite path in G, $\exists\, c$ st $\forall\, k \geq c$, $f_\#^k(\gamma) = \gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_r$

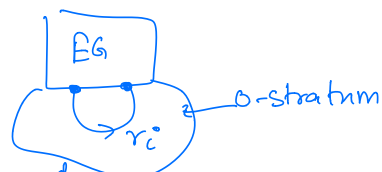
reduced path

splittings

i.e. no cancellation at $\bullet$ under iteration by f.

γ_i are either:

- single edge e in an irreducible stratum
- an INP $f_\#(\gamma_i) = \gamma_i$
- connecting path



- exceptional paths $e u^p \bar{e}$ where $f(e) = eu^d$, $f(e') = e' u^{d'}$, $d \neq d'$, $d, d' > 0$