Jt w/ S. Dowdall & I Kapovich

1

G a group, $u: G \twoheadrightarrow \mathbb{Z} \Rightarrow G \cong G_{\varphi_u} = \ker(u) \rtimes_{\varphi_u} \mathbb{Z}$, $\varphi_u \in \mathrm{Out}(\ker(u))$

$\varphi_u(x) = t^{-1}xt$.

[different t give conjugate autos → Out]

G is a (f.g.) free-by-cyclic group if it admits such a splitting as a semidirect product w/ ker(u) (f.g.) free group, $\ker(u) \cong F_N$, $N < \infty$.

[In this case it often happens that G splits like this in many distinct ways, giving rise to families of automorphisms.... Out wkshp]

Organize distinct splittings via the BNS-invariant:

Write $H^1(G) = \mathrm{Hom}(G;\mathbb{R}) \supset H^1_{\mathbb{Z}}(G) = \{u \in H^1(G) \mid u(G) = \mathbb{Z}\}$ [homs surjecting $\mathbb{Z}$]

BNS-invt is an $\mathbb{R}_+$-inv't subset $\Sigma G \subset H^1(G) \setminus \{0\}$ [usually think of sphere of directions...]

[We require a more general type of splitting to explain properties of ΣG:]

Say $u \in H^1_{\mathbb{Z}}G$ is dual to an (f.g.) ascending HNN extension if $\exists$ finitely gen $Q_u \subset \ker(u)$ s.t. $\varphi_u(Q_u) \subset Q_u$ and $G = \bigcup_{k<0} \varphi_u^k Q_u$. Then

$G \cong Q*_{\varphi_u} = \langle Q_u, t \mid t^{-1}xt = \varphi_u x \ \forall x \in Q_u \rangle$ — special case ker u f.g. $\Rightarrow Q_u = \ker u$ $\Rightarrow$ semidirect product

Theorem (Bieri-Neumann-Strebel, Geoghegan-Mihalik-Sapir-Wise) Suppose G is a free-by-cyclic group. Then ΣG is open and $u \in H^1_{\mathbb{Z}}G$ is dual to an ascending HNN ext. iff $u \in \Sigma G$. In this case Q_u is free. Furthermore, ker(u) is finitely gen (and so get semidirect prod. struct) iff $u \in \Sigma G \cap -\Sigma G$.

(*) Ex $G = \langle x_1, x_2, x_3, t \mid t^{-1}x_i t = \varphi(x_i), i = 1,2,3 \rangle$

$\varphi: x_1 \mapsto x_2$

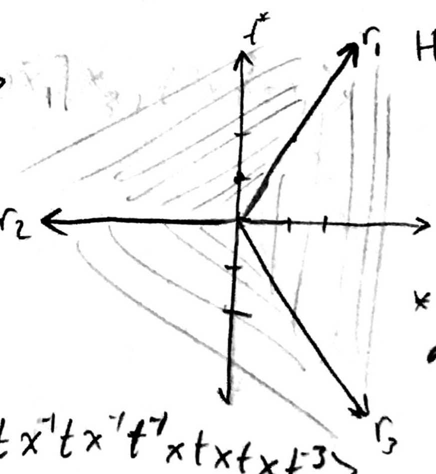
$x_2 \mapsto [\bar{x}_2, \bar{x}_1] x_3$

$x_3 \mapsto x_1$

RIGHT BOARD

$\Sigma G = H^1G \setminus r_1 \cup r_2 \cup r_3$

$\Sigma G \cup -\Sigma G$



Brown's algorithm

$= \langle x, t \mid x^{-1}tx^{-1}tx^{-1}t^{-1}xtxtxt^{-3} \rangle$

Every integral pt in ΣG gives a endo/auto.

B-N-S showed for 3-mfd groups, ΣG is precisely cone on fibered faces of Thurston norm ball + Harkens back to result of Stallings: $G \to \mathbb{Z}$ f.g. ker iff defined by a fibration over S^1.
Further conn. to 3-mfd via Alex. poly & norm – Dunfield, McMullen, Button. Tomorrow D. Kielak will describe an analogue of Thurston norm for f.b.c. groups. Discussion today also follows analogies w/ fibered 3-mfd setting, via work of Fried & McMullen on dyn. of pA's.

Every integral pt in ΣG gives an endomorphism/automorphism.

[Since this is a workshop on Outer space and outer autos] [We'll describe] relationships among these and their actions on (compactified) outer space.

Theorem 1: G a free-by-cyclic group, $\Sigma_0 \subset \Sigma G$ a compt, integral $u, u' \in \Sigma_0 \cap -\Sigma G$ w/ monodromies $\varphi_u, \varphi_{u'}$. Then φ_u is fully irred. and atoroidal iff $\varphi_{u'}$ is.

Recall: $\varphi \in \mathrm{Out}(F_n)$ is fully irred iff φ has no nontrivial periodic conj. classes of free factors. φ is atoroidal if no nontrivial periodic conj. classes of elts.

(Brinkmann, Bestvina-Feighn) G_φ is δ-hyp. iff φ is atoroidal

– Thm 1 is thus about fully irreducibles [Rk: False w/o atoroidal..]

How do we relate monodromies? Semiflows on a 2-complex:

(Bestvina-Handel) If $\varphi \in \mathrm{Out}(F_N)$ is fully irreducible, then φ admits an expanding irreducible train track rep'n.

Γ a graph w/ no val. 1 verts, $\pi_1\Gamma \cong F_N$, $f:\Gamma \to \Gamma$ a graph map [edges $\to$ edge paths], $f_* = \varphi$, and $\forall$ edges e:

① $f^k|_e$ is locally injective $\forall k \geq 1$, and
(no back tracking)

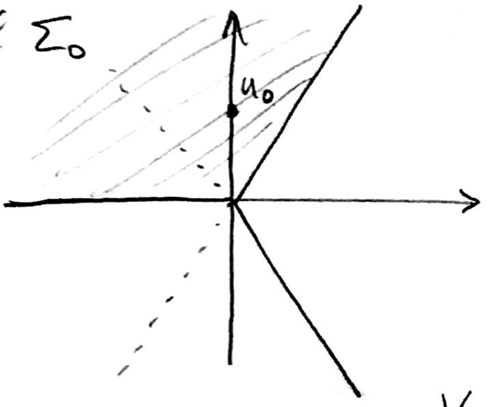
② $\forall$ $I \subset e$ open interval, $f^k(I) = \Gamma$ for suff. large k.

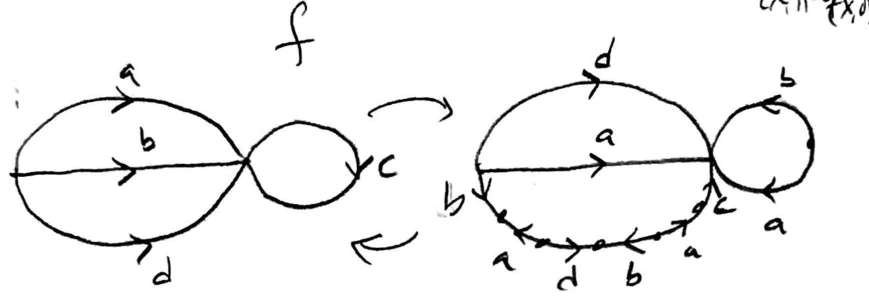
[Note: can have exp. irred tt w/o. being fully irred]

Suppose $\Sigma_0 \subset \Sigma G$, $u_0 \in \Sigma_0 \cap \Sigma G$ integral w/ fully irred ator. monodromy $\varphi_{u_0} = \varphi_0 \in \mathrm{Out}(F_N)$, $f:\Gamma \to \Gamma$ exp. irred tt. rep., let $X = X_f = \Gamma \times [0,1] / (x,1) \sim (fx, 0)$

mapping torus:

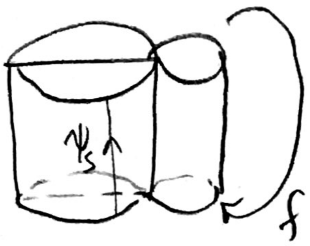
Ex Σ_0





$x_1 = \bar{b}a$
$x_2 = \bar{a}d$
$x_3 = c$

$X = X_f$



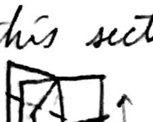
$\psi_s : X \circlearrowleft$ suspension semiflow (action of $\mathbb{R}_+$: $\psi_s \psi_{s'} = \psi_{s+s'}$ $\forall s, s' \geq 0$)
gen. by $\psi_s(x,t) = (x, t+s)$

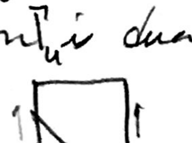
$\Gamma = \Gamma \times \{0\}$ is a cross section of ψ_s, 1^{st} return map is f.

Theorem 2 With notation above, $\Sigma_0 \cap H^1_{\mathbb{Z}}G = \{u \in H^1_{\mathbb{Z}}G \mid u$ dual to a x-section of ψ_s & first-return to section $\Gamma_u \circlearrowleft f_u$ is an exp irred. tt map weakly rep'in $\varphi_u\}$

x-section is $\eta^{-1}(*)$, $\eta : X \to S^1$ s.t. $\forall x \in X$, $s \mapsto \eta(\psi_s(x))$ is monotone

this section Γ_u is dual to $\eta_* : \pi_1 X = G \to \pi_1 S^1 = \mathbb{Z} \in H^1(G)$





Weak representative: $\mu = f_{u*} : \pi_1 \Gamma_u = H$ ↺ endomorphism (not nec. inj)

$G \underset{VK}{\cong} \langle H, r \mid r^{-1} x r = \mu x \ \forall x \in H \rangle \underset{\text{Tietze}}{\cong} \langle Q_u, r \mid r^{-1} x r = \varphi_u x \ \forall x \in Q_u \rangle$

$K_\mu = \bigcup_{j=1}^{\infty} \ker(\mu^j)$, $Q_u = H / K_\mu$ ↺ φ_u descent of

What is Q_u? why free? $H \xrightarrow{\mu} \mu H \xrightarrow{\mu} \mu^2 H \xrightarrow{\mu} \cdots$

nested in $H \cong F_n$, rank goes down (Hopfian) or μ restricts to injection b/,

$\exists J$ so $\mu^J H \overset{\mu|}{\longrightarrow} \mu^J H$, $\uparrow \mu^J$, $\uparrow \mu^J$

$\Rightarrow K_\mu = \ker \mu^J$ & $Q_u \xrightarrow{\varphi_u} Q_u$

Two types of 1-cells

vertical [from vertex of Γ]

☆ ψ_s-inv't

x-section

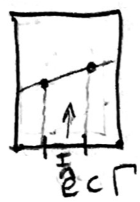
horizontal [from edges of Γ]

x-section

vertices of $\Gamma_u = \Gamma_u \cap X^1$ [1-skeleton ... perturb to flow into vertical]

+ valence 2 ... [flow into vertical]

$\Rightarrow f_u$ exp inv'd tt map:



all (intervals in) edges of Γ_u are flowed-images of intervals in edges in Γ,

Geometric construction of $\varphi_u : Q_u$ ↺ w/ exp inv. tt. rep.

cover for $\mu^J H$ $\to \widetilde{\Gamma}_u \supset \overline{\Gamma}_u$ Stallings core

$$\begin{array}{ccc} & & \widetilde{\Gamma}_u \\ & \overset{F}{\nearrow} & \downarrow \\ \Gamma_u & \xrightarrow{f_u^J} & \Gamma_u \end{array}$$

In fact, $F(\Gamma_u) = \overline{\Gamma}_u$ idea: Γ_u = image of immersed loops (no back-tracks)

Now lift: $\bar{f}_u : \overline{\Gamma}_u \to \overline{\Gamma}_u$,

For Thm 1: reparam. semiflow, get flow equiv. h.e.'s.
(φ_0, φ_u autos)

$$\begin{array}{ccc} & X_{f_u} & \\ \swarrow & & \searrow \\ X_f & & X_{\bar{f}_u} \end{array}$$

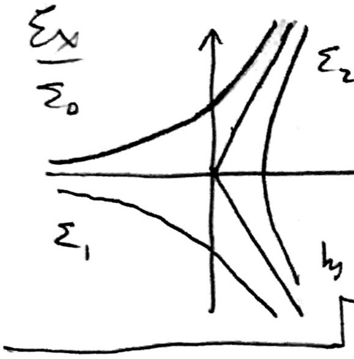
inv't free factor detectable via semiflow

For remainder, assume G f.b.c. (hyp), $\Sigma_0 \subset \Sigma G$ a "fully irred. compt". ⑤

Theorem 3 $\exists\ h : \Sigma_0 \to \mathbb{R}_+$ anal., convex, (-1)-homogeneous s.t. $\forall\, u \in \Sigma_0 \cap H^1_{\mathbb{Z}} G$,

$$\log(\lambda(\varphi_u)) = \log(\lambda(f_u)) = \log(\lambda(\bar{f}_u)) = h(u)$$

and $h(u) \to \infty$ as $u \to \partial \Sigma_0$



all f.i. compts! ($\Sigma G \cup -\Sigma G$ connected!)

level curves of h;

$\log(\lambda(\varphi_u)) =$ trans. dist on outer Voglmann's outer space w/ cross sections by Handel-Mosher, Algom-Kfir

Levitt & Lustig showed fully irred. have N-S dyn. on compt. outer Vogt outer space $\overline{CV_N}$; stable/unstable trees

Theorem 4 $\exists$ top $\mathbb{R}$-tree T w/ $G \curvearrowright T$ and a function

$$d_\cdot : \Sigma_0/\mathbb{R}_+ \longrightarrow \text{Metrics}(T)$$

varying continuously in finite subtrees, so that $\forall\, u \in \Sigma_0$, $g \in G$, $x, y \in T$

$$d_u(g \cdot x, g \cdot y) = \lambda^{u(g)}\, d_u(x, y)$$

where $\lambda = e^{h(u)}$

For each $u \in \Sigma_0 \cap H^1_{\mathbb{Z}} G$, (T, d_u) is stable tree

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Tree is also being studied from another perspective by Algom-Kfir, Hilion, Mj.

What is it? $\tilde{X}$ univ. cover of lifted semi flow $\tilde{\Psi}$,

defines an equiv. rel'n via closure of trans closure of equiv by flow lines.

metrics: $H = G^{ab}/\text{torsion}$, $\cong \mathbb{Z}^b$

$$\hat{X} \downarrow X$$

H-covering space, construct $\mathbb{Z}[H]$-module of transversals to Ψ. $\mathcal{T}(\Psi)$

(free $\mathbb{Z}$ module / flow & subdivide

f.p. as $\mathbb{Z}[H]$-module (using $\hat{f} < \ell \ldots$)

Given $u \in \Sigma_0$, $e^u : H \to \mathbb{R}_+ \Rightarrow \mathbb{Z}[H]$-module $\mathbb{R}_u$

Construct a! $\mathbb{Z}[H]$-mod. hom. $\mu_u : \mathcal{T}(\Psi) \to \mathbb{R}_u$

$\mu_u(\tau) > 0\ \forall$ trans. $\tau \Rightarrow$ metric on T