Negative Immersions for One-relator Groups

Joint w/ Lars Louder.

Let $G = F/\langle\langle w \rangle\rangle$ be a one-relator group.

Magnus '32. The word problem is solvable in G.

Question 1 (a) ~~The~~ What about the conjugacy problem?
(b) ——— the isomorphism problem?

Question 2: When is G hyperbolic?

The answer to q. 2 "should" be related to the subgroup structure of G.

Theorem (Newman): If G has torsion ($\Leftrightarrow$ w is a proper power) then G is hyperbolic.

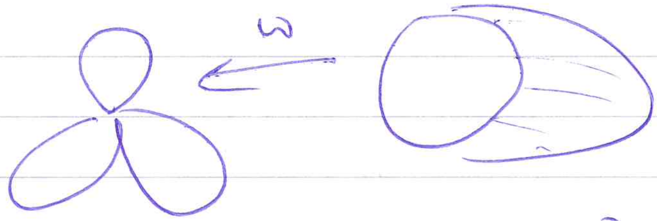
Question 2': Is G hyperbolic iff G doesn't contain a Baumslag-Solitar subgroup? $BS(m,n) = \langle a, b \mid b a^m b^{-1} = a^n \rangle$.

Question 3 (Baumslag): Is G coherent? i.e. is every f.g. subgroup of G f.p.?

For convenience, henceforth we'll assume G is torsion-free.

1. Non-positive Immersions

Think of G as $\pi_1 X$:

$X =$ 

Def^n (Wise): An immersion of 2-complexes

$$Y \looparrowright X$$

is a locally injective combinatorial map.

X has non-positive immersions (NPI) if, whenever $Y \looparrowright X$ is an immersion ~~of~~ and ~~compact~~ Y compact, either $\chi(Y) \le 0$ or Y is contractible.

Theorem A (Louder–W., Helfer–Wise 2014):

~~If $G = \pi_1 X$ is one-relator X has~~

If G is one-relator & X is as above, X has NPI.

Corollary B: If G is ~~[illegible]~~ one-relator & $H \le G$ is f.g.,

$\Rightarrow \dim H_2(H, \mathbb{Q}) < \infty$.

Sketch proof of Corollary 8: Wlog $H \neq 1$.

Let $X^H \hookrightarrow X$ be the cover of X corresponding to H.

~~We can~~ Adding relators one at a time, we build a sequence of π_1-surjective maps of 2-complexes

$$X_1 \to X_2 \to \cdots \to X_i \to \cdots \to X^H \to X$$

'compact connected'

s.t. $H = \varinjlim \pi_1 X_i$.

After "folding", we may promote this to a sequence of π_1-surjective <u>inclusions</u>

$$Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_i \hookrightarrow \cdots \to X^H \to X$$

s.t. $H = \varinjlim \pi_1 Y_i$.

Since $\pi_1 Y_i \twoheadrightarrow H$, Y_i is not contractible, so $\chi(Y_i) \leq 0$

$$1 - b_1(Y_i) + b_2(Y_i) \leq 0.$$

But $b_1(Y_i) \leq b_1(Y_1)$ bounded, so ~~$b_2(Y_i)$~~ $b_2(Y_i) \leq b_1(Y_i) - 1$ also bounded.

Since homology commutes with direct limits

$b_2(\cancel{\pi} X^H)$ is also bounded.

But Lyndon proved that X is aspherical, so

$\cancel{b}\ H_2(H;\mathbb{Q}) \cong H_2(X^H;\mathbb{Q})$. $\square$

2. Negative Immersions

Defn: X has <u>negative immersions</u> (NI) if, whenever $Y \looparrowright X$ is an immersion of compact, connected 2-complexes, either

$$\chi(Y) < 0$$

or $Y \simeq$ a graph.

Question 4: ~~Which~~ For which one-relator G does X have NI?

Example: If G is 2-generator and non-free (eg $G \cong \mathbb{Z}^2$, $BS(m,n)$, ...)

then $X \xrightarrow{id} X$ is an immersion, and $\chi(X) = 0$, but not $\simeq$ a graph.

$\therefore$ need $\mathrm{rk}\, G > 2$.

Def^n (Puder, 2014):

Let $w \in F$. The primitivity rank of w is

$$\pi(w) = \min\{\operatorname{rk} H : w \in H \leq F,\ w \text{ not primitive in } H\}.$$

~~Theorem B (Louder–W, 2018):~~

eg $\pi(w) = 0 \iff w = 1$
$\pi(w) = 1 \iff w$ proper power
$\pi(w) = \infty \iff w$ primitive.

If $\operatorname{rk} F = 2$, $\pi(w) \leq 2$.

~~Theor~~

Remark: $\pi(w)$ is computable, using Whitehead's algorithm.

Theorem B: ~~There is a set finite set of~~ Let $G = \pi_1 X = F/\langle\langle w \rangle\rangle$ be a one-relator group. There is a finite set of immersions of one-relator complexes $Z_i \looparrowright X$ s.t., for any immersion of a compact, connected complex $Y \looparrowright X$ s.t. $\chi(Y) \geq 2 - \pi(w)$

(i) if $\chi(Y) > 2 - \pi(w)$, $Y \simeq$ a graph, with no free faces

(ii) if $\chi(Y) = 2 - \pi(w)$, $Y \looparrowright X$ factors through some $Z_i \looparrowright X$.

Corollary: X has NI iff $\pi(w) > 2$.

Corollary: ~~X~~ G is $(\pi(w)-1)$-free, ie

every subgp H ~~with~~ generated by $< \pi(w)$ elts is free.

~~Corollary~~ In particular, if $\pi(w) > 2$ then G has no B.S. subgps.

This motivates:

Conjecture: If $\pi(w) > 2$ then G is hyperbolic.

This leaves the case $\pi(w) = 2$ (including all 2-generator G.)

Theorem C (C(w)): Let G, X as above. If $\pi(w) = 2$ then there is a unique

$Z \hookrightarrow X$ s.t.

whenever $Y \hookrightarrow X$ and $\chi(Y) \geq 0$, $= 0$, & Y has no free faces, $Y \hookrightarrow X$ factors through X.

~~We call Y~~

Conjecture: If $\pi(w) = 2$, G is hyperbolic relative to $\pi_1 Z \leq$??? G.