IMC Preparation
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Analysis I

1. Does there exist a bijective map $\pi : \mathbb{N} \to \mathbb{N}$ such that $\sum\limits_{n=1}^{\infty} \frac{\pi(n)}{n^2} < \infty$?

2. Let $0 < a < b$. Prove that
$$\int_a^b (x^2+1)e^{-x^2}dx \geq e^{-a^2} - e^{-b^2}.$$

3. a) Is it true that for every bijection $f : \mathbb{N} \to \mathbb{N}$ the series $\sum\limits_{n=1}^{\infty} \frac{1}{nf(n)}$ is convergent?
   b) Is it true that for every bijection $f : \mathbb{N} \to \mathbb{N}$ the series $\sum\limits_{n=1}^{\infty} \frac{1}{n+f(n)}$ is divergent?

4. Let $f : \mathbb{R} \to [0,1)$ be a continuously differentiable function. Prove that
$$\left| \int_0^1 f^3(x)dx - f^2(0) \int_0^1 f(x)dx \right| \leq \max_{0 \leq x \leq 1} |f'(x)| \left(\int_0^1 f(x)dx \right)^2.$$

5. Let $g : [0,1] \to \mathbb{R}$ be a continuous function and let $f_n : [0,1] \to \mathbb{R}$ be a sequence of functions defined by $f_0(x) = g(x)$ and $f_{n+1}(x) = \frac{1}{x} \int_0^x f_n(t)dt$ $(x \in (0,1], n = 0,1,2,\dots)$. Determine $\lim\limits_{n\to\infty} f_n(x)$ for every $x \in (0,1]$.

6. Let $f : [0;1] \to [0;1]$ be a differentiable function such that $|f'(x)| \neq 1$ for all $x \in [0;1]$. Prove that there exist unique points $\alpha, \beta \in [0,1]$ such that $f(\alpha) = \alpha$ and $f(\beta) = 1 - \beta$.

7. Suppose that f and g are real-valued functions on the real line and $f(r) \leq g(r)$ for every rational r. Does this imply that $f(x) \leq g(x)$ for every real x if
   a) f and g are non-decreasing?
   b) f and g are continuous?

8. Prove or disprove the following statements:
   (a) There exists a monotone function $f : [0,1] \to [0,1]$ such that for each $y \in [0,1]$ the equation $f(x) = y$ has uncountably many solutions x.
   (b) There exists a continuously differentiable function $f : [0,1] \to [0,1]$ such that for each $y \in [0,1]$ the equation $f(x) = y$ has uncountably many solutions x.

9. (HW, due 24 Jan) Let a, b, c be positive reals. Prove that
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

10. (HW, due 24 Jan)
    (a) Let $a_1, a_2, \dots$ be a sequence of real numbers such that $a_1 = 1$ and $a_{n+1} > \frac{3}{2}a_n$ for all n. Prove that the sequence $\frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$ has a finite limit or tends to infinity.
    (b) Prove that for all $\alpha > 1$ there exists a sequence $a_1, a_2, \dots$ with the same properties such that $\lim \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}} = \alpha$.