Polynomials

Problem 1 Let x_1, x_2, x_3 be the roots of $x^3 + ax^2 + bx + c$. Determine

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}.$$

Problem 2 Let $P(x) = x^3 + ax^2 + bx + c$ have roots x_1, x_2, x_3. Find the polynomial $Q(x) = x^3 + \alpha x^2 + \beta x + \gamma$ of degree 3 that has roots x_1x_2, x_1x_3, x_2x_3.

Problem 3 Find all polynomials $P(x)$ such that

$$P(x) = \frac{P(x-1) + P(x+1)}{2}, \quad \text{for all } x \in \mathbb{R}.$$

Problem 4 Determine all polynomials $P(x)$ such that $P(x^2+1) = (P(x))^2 + 1$ and $P(0) = 0$.

Problem 5 Find all polynomials P satisfying $xP(x-1) = (x-2)P(x)$.

Problem 6 Show that if a polynomial $P(x)$ with real coefficients satisfies $P(x) \geq 0$ for all $x \in \mathbb{R}$, then it can be represented as

$$P(x) = Q_1^2(x) + \ldots + Q_n^2(x),$$

for some polynomials $Q_1, \ldots, Q_n$ with real coefficients.

Problem 7 Let polynomial $P(x)$ with real coefficients satisfy inequality $P(x) > 0$ for all $x > 0$. Show that there are polynomials $Q(x)$ and $R(x)$ with non-negative coefficients satisfying

$$P(x) = Q(x)/R(x).$$

Problem 8 Represent

$$S(x_1, \ldots, x_n) = \sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} x_i^3 x_j$$

as a polynomial in the elementary symmetric polynomials.

Problem 9 Can polynomials

$$x^5 - x - 1 \quad \text{and} \quad x^2 + ax + b,$$

with $a, b \in \mathbb{Q}$, have a common complex root?

Problem 10 Find all pairs of integers $m, n \in \mathbb{N}$ such that the polynomial

$$P(x) = 1 + x + x^2 + \ldots + x^m$$

divides the polynomial

$$Q(x) = 1 + x^n + x^{2n} \ldots + x^{mn}.$$