Networks and Random Processes

Class test

The class test counts $25/100$ module marks, [x] indicates weight of each question.

1. (a) State the weak law of large numbers and the central limit theorem.
   (b) Define the Erdős-Rényi random graph model $\mathcal{G}_{N,p}$, including the set of all possible graphs and the corresponding probability distribution.
   Compute the expected degree distribution and the expected number of complete subgraphs of size k for $k \in \{1, \ldots, N\}$.
   (c) State the Wigner semi-circle law.

[12]

2. Consider the undirected graph G with adjacency matrix $A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$.
   (a) Draw the graph G. Identify a clique of vertices and draw a spanning tree of G.
   (b) Give the matrix of vertex distances d_{ij} and compute the characteristic path length $L(G)$ and the diameter $\mathrm{diam}(G)$ of G.
   (c) Give the degree sequence $(k_1, \ldots, k_5)$ and compute the degree distribution $p(k)$ and the average degree $\langle k \rangle$ of G.
   (d) Compute the average $\langle C_i \rangle$ of the local clustering coefficients C_i.
   (e) Is the graph planar? Is it triangulation?
   If not, add edges to turn it into a triangulation and draw the graph.

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3. (a) Define the configuration model with N vertices and degree sequence D.
   (b) Consider the following degree sequences
   $$(1, 2, 3, 4)\ , \quad (3, 3, 3, 3)\ , \quad (1, 3, 3, 1)\ , \quad (2, 3, 3, 2)\ , \quad (1, 1, 1, 1)\ .$$
   Decide whether the sequence is graphical. If yes, draw a planar (simple) graph with that sequence.
   If the graph is connected, draw the dual (multi-)graph.
   (c) Which of the graphs you drew in (b) is a triangulation?
   Give an example of a non-planar graph by drawing it and giving its adjacency matrix.

[12]

4. **Kingman's coalescent**
Consider a system of well mixed, coalescing particles and let N_t be the number of particles at time t with $N_0 = N$. At time $t \geq 0$, each of the $\binom{N_t}{2}$ pairs of particles coalesces independently with rate 1. This can be interpreted as generating an ancestral tree of individuals in a population model, tracing back to a single common ancestor.

   (a) Give the state space and the transition rates of the process $(N_t : t \geq 0)$ and write down the master equation.
   Is the process ergodic? Does it have absorbing states? Give all stationary distributions.

   (b) Show that the mean time to reach the state 1 is given by $\mathbb{E}(T) = 2\big(1 - \frac{1}{N}\big)$.

   (c) Assume that each particle $i \in \{1, \ldots, N\}$ initially has a type $b(i) = i$ (each type is different), and that after coalescing the new particle assumes one of the two types with equal probability.
   What is the distribution of the type of the last individual once $N_t = 1$?

   (d) Assume now that initially $b(i) = A$ with probability $p \in (0, 1)$ and $b(i) = B$ with probability $1 - p$ independently across all $i \in \{1, \ldots, N\}$, with dynamics as in (c).
   What is the distribution of the type of the last individual once $N_t = 1$?

[12]

5. Consider the CTMC $(X_t : t \geq 0)$ with state space $S = \{0, 1\}$, $X_0 = 0$ and generator matrix $G = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$.

   (a) Give the stationary distribution $\boldsymbol{\pi}$. Is it reversible?

   (b) Give the transition matrix P^Y for the corresponding jump chain $(Y_n : n \in \mathbb{N}_0)$ and its stationary distribution $\boldsymbol{\pi}^Y$.

   (c) Compute the eigenvalues of G, and the transition matrix $P(t) = \exp(tG)$ for all $t > 0$.

   (d) For a **discrete-time sampling** of $(X_t : t \geq 0)$ with fixed time interval $\Delta t > 0$ let $(Z_n : n \in \mathbb{N}_0)$ be the DTMC with $Z_n = X_{n\delta t}$.
   Give the transition matrix P^Z of this chain and its stationary distribution $\boldsymbol{\pi}^Y$.
   What is the difference between the DTMCs $(Y_n : n \in \mathbb{N}_0)$ and $(Z_n : n \in \mathbb{N}_0)$?
   (Write one or two sentences.)

[12]

6. Consider the linear voter model $(\eta_t : t \geq 0)$ where individual j influences the opinion of individual i with rate $q(j, i) \geq 0$.

   (a) Give the state space of the model and a formula of the transition rates $c(\eta, \eta^i)$ using the standard notation

   $$\eta^i(k) = \begin{cases} \eta(k) & , k \neq i \\ 1 - \eta(k) \, , & k = i \end{cases} \quad \text{for configurations where the opinion of individual } i \text{ is flipped.}$$

   (b) Is the process ergodic (justify your answer)?
   Give a formula for all stationary distributions of the process, assuming that $q(j, i)$ is irreducible.

   (c) For $q(j, i) = \lambda(\delta_{i+1,j} + \delta_{i-1,j})$, $\lambda > 0$ and $i, j \in \{1, \ldots, 5\}$ (5 individuals) sketch the graphical representation of the model using independent Poisson processes.

   (d) Describe the steps of the random sequential update algorithm to simulate this process.

[12]