Networks and Random Processes

Class test

The class test counts $25/100$ module marks, [x] indicates weight of each question.
Attempt all 5 questions.

1. (a) State the weak law of large numbers and the central limit theorem.

   (b) For a discrete-time, time-homogeneous Markov chain $(X_n : n \in \mathbb{N}_0)$ with discrete state space S, define the transition function and state the Chapman Kolmogorov equations.

   (c) Define what it means for a discrete-time Markov chain to be ergodic, and state the ergodic theorem for time averages of observables.

   (d) Define the spectral density of a matrix $A \in \mathbb{R}^{N\times N}$, and state the Wigner semi-circle law.

   [20]

2. (a) Define the configuration model with N vertices and degree sequence D.

   From now on consider the degree sequence $D = (2, 2, 3, 3, 4)$.

   (b) Draw a graph G with this degree sequence and give its adjacency matrix A.

   (c) Identify a clique of vertices and draw a spanning tree of G.

   (d) Give the matrix of vertex distances d_{ij} and compute the characteristic path length $L(G)$ and the diameter $\mathrm{diam}(G)$ of G.

   (e) Compute the average $\langle C_i\rangle$ of the local clustering coefficients C_i.

   (f) Is the graph G planar? If not draw another planar graph with degree sequence D.
   Is the planar graph a triangulation? If not, add edges to turn it into a triangulation and draw the graph.

   [20]

3. (a) State two equivalent definitions of standard Brownian motion.

   (b) Let $(B_t : t \geq 0)$ be a standard Brownian motion. Show that the process $(X_t : t \geq 0)$ is also a standard Brownian motion, where $X_t = \begin{cases} t\, B_{1/t}\,, & \text{for } t > 0 \\ 0 \,, & \text{for } t = 0 \end{cases}$.

   (c) State the definition of a diffusion process on $\mathbb{R}$.

   (d) Let $(X_t : t \geq 0)$ be a jump process on $S = \mathbb{R}$ with translation invariant rates $r(x, y) = q(y - x)$. Write down the generator of this process.
   Now consider rates which have

   $$\text{mean} \quad \int_{\mathbb{R}} q(z)\, z\, dz = a\epsilon \quad \text{with } a \neq 0 \quad \text{and}$$

   $$\text{finite second moment} \quad \sigma^2 := \int_{\mathbb{R}} q(z)\, z^2\, dz < \infty\, .$$

   By Taylor expansion of the generator, show that the rescaled process $(\epsilon X_{t/\epsilon^2} : t \in [0, T])$ converges to a diffusion process as $\epsilon \to 0$ on the compact time interval $[0, T]$, and give the generator of this diffusion process.
   (You do not have to show continuity of paths of the limiting process.)

   [20]

4. **Birth-death processes**
A birth-death process $(X_t : t \geq 0)$ is a continuous-time Markov chain with state space $S = \mathbb{N}_0 = \{0, 1, \ldots\}$ and jump rates

$$x \xrightarrow{\alpha_x} x+1 \quad \text{for all } x \in S \;, \qquad x \xrightarrow{\beta_x} x-1 \quad \text{for all } x \geq 1 \;.$$

   (a) Give the generator G (as a matrix or as an operator), and the master equation in explicit form.
   Under which conditions on the jump rates is the process irreducible?

   (b) Using detailed balance, find a formula for the stationary probablities $\pi(x)$ in terms of the jump rates and $\pi(0)$, normalization is not required.

   (c) Suppose $\alpha_x = \alpha > 0$ for $x \geq 0$ and $\beta_x = \beta > 0$ for $x > 0$.
   Under which conditions on α and β can the stationary probabilities $\pi(x)$ you found in (b) be normalized? Give a formula for $\pi(x)$ in that case.

   (d) Suppose $\alpha_x = x\alpha$, $\beta_x = x\beta$ for $x \geq 0$ with $\alpha, \beta > 0$ and $X_0 = 1$.
   Derive an explicit formula for the expectation $E[X_t]$, e.g. using the equation

$$\frac{d}{dt}\mathbb{E}[f(X_t)] = \mathbb{E}[(Gf)(X_t)] \quad \text{with} \quad f(x) = x \;.$$

   Qualitatively discuss the behaviour of X_t as $t \to \infty$, depending on the values of α and β. (One sentence for each case is sufficient.)

[20]

5. **Branching process**

   (a) Define a branching process $(Z_n : n \in \mathbb{N}_0)$ on the state space $S = \mathbb{N}_0 = \{0, 1, \ldots\}$ with initial condition $Z_0 = 1$.
   Is this process irreducible?

   (b) Using generating functions or otherwise, show that $\mathbb{E}[Z_n] = \mu^n$,
   where $\mu > 0$ is the expected number of offspring in a single reproduction event.

From now on consider a branching process with offspring distribution $p_k = p^k(1-p)$ for $k \in \mathbb{N}_0$ with $p \in (0, 1)$.

   (c) Compute the probability of extinction for all parameter values $p \in (0, 1)$. (In your answer you may have to distinguish several cases.)
   You may use that the extinction probability is given by the smallest fixed point of the generating function, $G(s) = s$, without proof.

   (d) How does the probability of extinction change, if you start with the initial distribution $Z_0 = 2$?

Consider the (undirected) tree generated by the branching process with $Z_0 = 1$, where each individuum in generation $n \geq 1$ is connected to its parent.

   (e) Compute the expected number of vertices in the tree, i.e. total population size $\mathbb{E}\big[\sum_{n=0}^{\infty} Z_n\big]$, depending on the parameter value $p \in (0, 1)$.

   (f) What is the expected degree distribution for vertices in generations $n \geq 1$, and for the vertex in generation $n = 0$?

[20]