Networks and Random Processes

Class test

The class test counts $25/100$ module marks, [x] indicates weight of each question.
Attempt all 5 questions.

1. (a) State the weak law of large numbers and the central limit theorem.
   (b) Define the Erdős-Rényi random graph model $\mathcal{G}_{N,p}$, including the set of all possible graphs and the corresponding probability distribution.
   Compute the expected degree distribution.
   (c) Define what it means for a real-valued process $(M_t : t \geq 0)$ to be a martingale.
   State Itô's formula for a process $(X_t : t \geq 0)$ on state space S with generator $\mathcal{L}$ and a function $f : S \to \mathbb{R}$ which does not explicitly depend on time. Include the expression for the quadratic variation of the martingale.
   (d) Give the generator of the Poisson process $(N_t : t \geq 0)$ with rate $\lambda > 0$.
   Use Itô's formula to show that $N_t - \lambda t$ is a martingale and compute its quadratic variation. [20]

2. Consider the undirected graph G with adjacency matrix $A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$.
   (a) Draw the graph G. Identify a clique of vertices and draw a spanning tree of G.
   (b) Give the matrix of vertex distances d_{ij} and compute the characteristic path length $L(G)$ and the diameter $\mathrm{diam}(G)$ of G.
   (c) Give the degree sequence $(k_1, \ldots, k_6)$ and compute the degree distribution $p(k)$ and the average degree $\langle k \rangle$ of G.
   (d) Compute the global clustering coefficient C and the average $\langle C_i \rangle$ of the local clustering coefficients C_i.
   (e) Give all non-zero entries of the joint degree distribution $q(k, k')$.
   Compute the marginal $q(k)$. For all k' with $q(k') > 0$ compute the conditional distribution $q(k|k')$ and the corresponding expectation $k_{nn}(k')$.
   [20]

3. (a) State two equivalent definitions of standard Brownian motion.
   (b) Let $(B_t : t \geq 0)$ be a standard Brownian motion. Prove that for any $\lambda > 0$, the process $(X_t : t \geq 0)$ with $X_t := \frac{1}{\lambda} B_{t\lambda^2}$ is also a standard Brownian motion.
   (c) State the definition of a diffusion process on $\mathbb{R}$.

   From now on, consider the Ornstein-Uhlenbeck process $(X_t : t \geq 0)$ given by the SDE
   $$dX_t = -\alpha X_t dt + \sigma dB_t \quad \text{with } \alpha > 0 \quad \text{and} \quad X_0 = x_0 \text{ (deterministic)} .$$
   (d) Write down the generator of this process.
   Derive equations for the mean $m(t) := \mathbb{E}[X_t]$ and the variance $v(t) := \mathbb{E}[X_t^2] - \mu(t)^2$ and solve them with the above deterministic initial condition $X_0 = x_0$.
   (e) Is $(X_t : t \geq 0)$ a Gaussian process?
   Use the result of (d) to specify the distribution of X_t for all $t \geq 0$, and also give the stationary distribution as $t \to \infty$.
   [20]

4. **Birth-death processes**
A general birth-death process $(X_t : t \geq 0)$ is a continuous-time Markov chain with state space $S = \mathbb{N}_0 = \{0, 1, \ldots\}$ and jump rates

$$x \xrightarrow{\alpha_x} x+1 \quad \text{for all } x \in S \;, \qquad x \xrightarrow{\beta_x} x-1 \quad \text{for all } x \geq 1 \;.$$

(a) Give the generator G as a matrix and as an operator, and write the master equation in explicit form, i.e. $\frac{d}{dt}\pi_t(x) = \ldots$ ($x = 0$ may need special consideration).
Under which conditions on the jump rates is the process irreducible?

(b) Using detailed balance, find a formula for the stationary probabilities $\pi(x)$ in terms of the jump rates and $\pi(0)$, normalization is not required.

(c) Suppose $\alpha_x = \alpha > 0$ for $x \geq 0$ and $\beta_x = x\beta$ for $x \geq 1$ with $\beta > 0$.
Under which conditions on α and β can the stationary probabilities $\pi(x)$ you found in (b) be normalized?
In that case compute the normalization and give a formula for $\pi(x)$.

(d) Suppose $\alpha_x = \beta_x = 2^x$ for $x \geq 1$ and $\alpha_0 = 1$.
Can the stationary probabilities $\pi(x)$ you found in (b) be normalized?
If yes, compute the normalization and give a formula for $\pi(x)$.
Give the transition probabilities of the corresponding jump chain $(Y_n : n \in \mathbb{N}_0)$.
Does it have a stationary distribution? If yes, give a formula.

[20]

5. Consider an even number L of individuals, each having one of two possible types denoted by $X_t(i) \in \{A, B\}$ for all $i = 1, \ldots, L$ and continuous times $t \geq 0$. Each individual changes its type independently of all others at rate 1, in short $A \xrightarrow{1} B$ and $B \xrightarrow{1} A$.

(a) Denoting by $X_t = (X_t(i) : i = 1, \ldots, L)$ the vector of types, give the state space of the process $(X_t : t \geq 0)$. Is this process irreducible? Does it have absorbing states?

From now on consider $N_t := \sum_{i=1}^{L} \delta_{X_t(i),A}$ to be the number individuals of type A at time t.

(b) Give state space and generator of the process $(N_t : t \geq 0)$. Is it irreducible?
Show that the stationary distribution is of binomial form and give the parameters.

(c) Consider the rescaled process $U_t^L := \frac{1}{L} N_t$ on the state space $[0, 1]$.
Write down the generator of $(U_t^L : t \geq 0)$ and compute its limit as $L \to \infty$.
Use this to show that the limit process $U_t := \lim_{L\to\infty} U_t^L$ is deterministic and is given as a solution to the ODE $\frac{d}{dt} U_t = 1 - 2U_t$.
Solve this ODE for general initial condition $U_0 \in [0, 1]$.

(d) Now take $N_0 = L/2$ and consider the 'fluctuation process' $Z_t^L = \frac{N_t - L/2}{\sqrt{L}} \in \mathbb{R}$.
Write down the generator of this process, and use this to show that $(Z_t^L : t \geq 0)$ converges as $L \to \infty$ to an Ornstein-Uhlenbeck process $(Z_t : t \geq 0)$ with generator

$$\mathcal{L}f(z) = -2zf'(z) + \frac{1}{2}f''(z) \;.$$

[20]