Stochastic Modelling and Random Processes

Class test

The class test counts $80/100$ module marks, [x] indicates weight of each question.
Attempt all 5 questions.

1. (a) State the weak law of large numbers and the central limit theorem.

   (b) Consider a continuous-time Markov chain (CTMC) $(X_t : t \geq 0)$ with state space S.
   Define what it means for $(X_t : t \geq 0)$ to be ergodic.
   State the ergodic theorem for $(X_t : t \geq 0)$.
   Define what it means for $(X_t : t \geq 0)$ to be irreducible.
   Give an example of a CTMC that is not ergodic (specify S and transition rates).
   Give an example of a CTMC that is not irreducible but ergodic.

   (c) Give the definition of a diffusion process $(X_t : t \geq 0)$ on $\mathbb{R}$ and write down its generator and the corresponding stochastic differential equation (SDE).
   State Itô's formula for $(X_t : t \geq 0)$ and a smooth function $f : \mathbb{R} \to \mathbb{R}$.

   (d) Define the Erdős-Rényi random graph model $\mathcal{G}_{N,p}$, including the set of all possible graphs and the corresponding probability distribution.
   What is the distribution of the total number of undirected edges K?
   Use the law of large numbers to show that the average degree $\frac{1}{N}\langle k\rangle \to p$ in distribution. [20]

2. A general birth-death process $(X_t : t \geq 0)$ is a continuous-time Markov chain with state space $S = \mathbb{N}_0 = \{0, 1, \ldots\}$ and jump rates

$$x \xrightarrow{\alpha_x} x+1 \quad \text{for all } x \in S\ , \quad x \xrightarrow{\beta_x} x-1 \quad \text{for all } x \geq 1\ .$$

   (a) Sketch the generator as a matrix G and write it as an operator $\mathcal{L}f(x)$.
   Write the master equation in explicit form, i.e. $\frac{d}{dt}\pi_t(x) = \ldots$ for all $x \in S$.
   Under which conditions on the jump rates is the process irreducible?

   (b) Using detailed balance, find a formula for the stationary probabilities $\pi(x)$ in terms of the jump rates and $\pi(0)$, normalization is not required.

   (c) Consider the pure birth process with $\alpha_x = 2^x$ and $\beta_x = 0$.
   Give all communicating classes in S and identify transience or null/positive recurrence.
   Compute the expected explosion time $\mathbb{E}[J_\infty]$ and its variance $\mathrm{Var}[J_\infty]$.
   (Hint: the variance of an $\mathrm{Exp}(\lambda)$ random variable is $1/\lambda^2$.)

   From now on consider $\alpha_x = \beta_x = 2^x$ for $x \geq 1$ and $\alpha_0 = 1$.

   (d) Compute the unique stationary distribution $\pi(x)$ for the process (e.g. using (b)).
   Give all communicating classes in S and identify transience or null/positive recurrence.

   (e) Give the transition probabilities for the corresponding jump chain $(Y_n : n \in \mathbb{N}_0)$.
   Give all communicating classes in S for the jump chain and identify transience or null/positive recurrence.
   Give all stationary distributions for the jump chain.

   [20]

3. Consider an undirected graph G with degree sequence $D = (1, 2, 2, 3, 4)$.

   (a) Draw a graph G with this degree sequence (the vertices should be numbered 1 to 5) and write its adjacency matrix.

   (b) Is it a planar graph? If yes, draw the dual graph.

   (c) Identify all cliques of vertices in the graph.

   (d) Give the matrix of vertex distances $\big(d_{ij} : i, j = 1, \ldots, 5\big)$ and compute the characteristic path length $L(G)$ and the diameter $\mathrm{diam}(G)$ of G.

   (e) Compute the degree distribution $p(k)$ and the average degree $\langle k\rangle$ of G.

   (f) Compute the local clustering coefficients C_i and their average $\langle C_i\rangle$.
   Compute the global clustering coefficient C.

   (g) Give all non-zero entries of the joint degree distribution $q(k, k')$.
   (You can write this directly going through all the edges of the graph).
   Compute the marginal distribution $q(k) = \sum_{k'} q(k, k')$.

[20]

4. **Two-species Moran model with fitness and mutation**
   Consider a fixed population of L individuals in continuous time $t \geq 0$, each individuum i has a type $X_t(i) \in \{A, B\}$. The dynamics consists of two independent processes:

   - **selection:** individuals of types A and B independently produce one offspring of the same type with rate $r_A > 0$ and $r_B > 0$, respectively, which replaces one of the L existing individuals chosen uniformly at random (including the parent);
   - **mutation:** each individuum independently flips its type ($A \to B$ or $B \to A$) with rate $\mu \geq 0$ (which does not depend on the type), for $\mu = 0$ there is no mutation.

   (a) Give the state space of the process $(X_t : t \geq 0)$.
   Is the process irreducible? Give all absorbing states. (May depend on parameter values.)

   (b) Let $N_t = \sum_{i=1}^{L} \delta_{X_t(i),A}$ be the number of individuals of type A at time t.
   Show that $(N_t : t \geq 0)$ is a Markov process by writing down its generator $\mathcal{L}f(n)$.
   Give the state space of the process.

   Consider the rescaled process $(M_t^L : t \geq 0)$ with $M_t^L := \frac{1}{L} N_{tL^\gamma} \in [0, 1]$ with $\gamma \geq 0$.

   (c) Show that for $\gamma = 0$ the limit process $(M_t : t \geq 0)$ as $L \to \infty$ has generator

   $$\mathcal{L}_M f(m) = a(m) f'(m) \quad \text{with drift} \quad a(m) = (r_A - r_B)m(1 - m) + \mu(1 - 2m) \; .$$

   Write down the corresponding SDE (which is deterministic).
   Assume $r_A > r_B$ and sketch the drift $a(m)$, $m \in [0, 1]$ for $\mu = 0$ and for $\mu > 0$.
   In both cases, discuss the limiting behaviour of X_t as $t \to \infty$, with $X_0 \in (0, m)$.

   From now on consider the neutral case $r_A = r_B = 1$ with weak mutation $\mu = \hat{\mu}/L$ with $\hat{\mu} > 0$.

   (d) For which value of $\gamma > 0$ does $M_t^L = \frac{1}{L} N_{tL^\gamma}$ now have a (non-trivial) scaling limit?
   Compute the generator of the limit $(M_t : t \geq 0)$ and write the corresponding SDE.

   (e) For the limit process $(M_t : t \geq 0)$ in (d) with $M_0 = 1$, compute $\mathbb{E}[M_t]$ for all $t \geq 0$.

[20]

5. (a) Define standard Brownian motion $(B_t : t \geq 0)$ as a Gaussian process.
Give the generator $\mathcal{L}_B f(x)$ for $(B_t : t \geq 0)$.

From now on consider the process $(X_t : t \geq 0)$ with $X_t := e^{-ct} B_{e^{2ct}}$ for some fixed $c > 0$.

(b) Is $(X_t : t \geq 0)$ a Gaussian process? Justify your answer.
Compute its mean and covariance function.

(c) Give the distribution of X_t for all $t \geq 0$, and the stationary distribution of the process.

(d) Now use the notation $\tau(t) := e^{2ct}$ and the chain rule to compute

$$\frac{d}{dt}\mathbb{E}\big[f(X_t)\big] = \frac{d}{d\tau}\mathbb{E}\Big[f\Big(\frac{B_\tau}{\sqrt{\tau}}\Big)\Big]\frac{d\tau}{dt} = \mathbb{E}\Big[(\partial_\tau + \mathcal{L}_B) f\Big(\frac{B_\tau}{\sqrt{\tau}}\Big)\Big]\frac{d\tau}{dt} .$$

Here $\partial_\tau f(b/\sqrt{\tau})$ is the partial derivative w.r.t. τ, and $\mathcal{L}_B f(b/\sqrt{\tau}) = \frac{1}{2}\partial_b^2 f(b/\sqrt{\tau})$ is the generator of Brownian motion acting on the variable b.
Write both contributions in terms of f' and f'' using the chain rule, and use this to show that the process $(X_t : t \geq 0)$ has generator

$$\mathcal{L}_X f(x) = -cxf'(x) + cf''(x) ,$$

i.e. it is an Ornstein-Uhlenbeck process.

(e) Write down the SDE and the Fokker-Planck equation for $(X_t : t \geq 0)$.
Give the definition of a martingale.
Is $(X_t : t \geq 0)$ a martingale? Justify your answer.

[20]