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Stochastic Modelling and Random Processes

Hand-out 1
Linear Algebra

Consider a square matrix $A \in \mathbb{R}^{n\times n}$ with elements a_{ij}. The **determinant** of the matrix is given by

$$\det(A) = \sum_{\pi\in S_n} \mathrm{sgn}(\pi) \prod_{i=1}^n a_{i\pi(i)} ,$$

where the first sum is over all permutations π of the indices $1,\ldots,n$ with associated signature $\mathrm{sgn}(\pi) \in \{-1,1\}$. A has n complex **eigenvalues** $\lambda_i,\ldots,\lambda_n \in \mathbb{C}$ which are the roots of the **characteristic polynomial**

$$\chi_A(\lambda) = \det(A - \lambda\mathbb{I}_n) = \prod_{i=1}^n (\lambda_i - \lambda) ,$$

which is a polynomial of degree n. If λ_i is an eigenvalue, so is the complex conjugate $\bar{\lambda}_i$, since χ_A has real coefficients. Furthermore

$$\det A = \prod_{i=1}^n \lambda_i \quad \text{and} \quad \mathrm{Tr}(A) = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i ,$$

where the trace Tr is defined as the sum of the diagonal elements of A.

$|v\rangle \in \mathbb{C}^n$ is **right (column) eigenvector** with eigenvalue $\lambda \in \mathbb{C}$ and $\langle u|$ **left (row) eigenvector** if

$$A|v\rangle = \lambda|v\rangle , \quad \langle u|A = \lambda\langle u| .$$

From now we assume that all eigenvalues are distinct (see overleaf for other cases). Then A has a complete basis of eigenvectors, which can be normalized and are orthogonal in the sense that

$$\langle u_i|v_j\rangle = \delta_{ij} \quad \text{and} \quad \sum_{i=1}^n |v_i\rangle\langle u_i| = \mathbb{I} .$$

Gershgorin theorem. Every eigenvalue of A lies in at least one **Gershgorin disc**

$$D(a_{ii}, R_i) \subseteq \mathbb{C},\ i = 1,\ldots,n , \quad \text{where} \quad R_i = \sum_{j\neq i} |a_{ij}| .$$

Further remarks, including diagonalization

- If all eigenvalues are distinct, the matrix $|v_i\rangle\langle u_i| \in \mathbb{C}^{n\times n}$ projects a vector $\langle x|$ onto the eigenspace of the corresponding eigenvalue λ_i

  $$\langle x|v_i\rangle\langle u_i| = a_i\langle u_i| \quad \text{with coefficient} \quad a_i = \langle x|v_i\rangle .$$

  A itself can be decomposed as a linear combination of such **projectors**, $A = \sum_{i=1}^n \lambda_i|v_i\rangle\langle u_i|$
  For projections we have $|v_i\rangle\langle u_i|\,|v_j\rangle\langle u_j| = \delta_{ij}|v_i\rangle\langle u_i|$ so for powers of A we simply get

  $$A^k = \sum_{i=1}^n \lambda_i^k |v_i\rangle\langle u_i| \quad \text{for all } k \geq 1 .$$

- Alternatively, one often considers the **similarity transformation matrix**

$$Q = \sum_{j=1}^{n} |v_j\rangle\langle e_j| = \big[|v_1\rangle, \ldots, |v_n\rangle\big] \in \mathbb{C}^{n\times n} \qquad \big(\langle e_j| \text{ is } j\text{-th basis vector}\big) ,$$

  built from writing the right column eigenvectors into a square matrix. Then

$$AQ = \sum_{i,j=1}^{n} \lambda_i |v_i\rangle\langle u_i|v_j\rangle\langle e_j| = \sum_{i=1}^{n} \lambda_i |v_i\rangle\langle e_i| = \sum_{i,j=1}^{n} |v_i\rangle\langle e_i|e_j\rangle\langle e_j|\lambda_j = Q\Lambda$$

  where $\Lambda = \sum_{j=1}^{n} \lambda_j |e_j\rangle\langle e_j|$ is the diagonal matrix of eigenvalues, and we have

$$A = Q\Lambda Q^{-1} \quad \text{where the inverse of } Q \text{ is} \quad Q^{-1} = \sum_{j=1}^{n} |e_j\rangle\langle u_j| .$$

- If $A = A^T$ is **symmetric**, then all eigenvalues $\lambda_i \in \mathbb{R}$ are real and the eigenvectors have real entries, 'are equal' in the sense that $\langle u_i|^T = |v_i\rangle$, and form an orthonormal basis of $\mathbb{R}^n$. In this case $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, i.e. $Q^{-1} = Q^T$.

- A matrix is **diagonalizable** with diagonal form Λ as given above if and only if the eigenvectors form a basis. If this is not the case, the matrix is called **defective**, and the **Jordan normal form** is not diagonal.

Justification of projector represenation

Recall for $A \in \mathbb{R}^{n\times n}$ we denote by $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ its eigenvalues, and by $\langle u_i|$ and $|v_i\rangle$ the corresponding left and right eigenvectors. We assume that they both form a basis of $\mathbb{C}^n$ and the normal form of the matrix A is a diagonal matrix. One simple sufficient condition for this is all eigenvalues to be distinct. Let $\lambda_i \neq \lambda_j$ be two distinct eigenvalues. Then

$$\lambda_i \langle u_i|v_j\rangle = \langle u_i|A|v_j\rangle = \lambda_j \langle u_i|v_j\rangle ,$$

which implies $\langle u_i|v_j\rangle = 0$, so corresponding left and right eigenvectors are orthogonal. Even if not all eigenvalues are distinct, as long as eigenvectors form a basis, they can be chosen such that

$$\langle u_i|v_j\rangle = \delta_{ij} \quad \text{for all } i, j = 1, \ldots, n . \tag{1}$$

The projector matrices form a partition of unity (the identity matrix) in the sense that

$$\sum_{i=1}^{n} |v_i\rangle\langle u_i| = \mathbb{I} .$$

This can be seen from (1) and the fact that eigenvectors form a basis, since for any k, l we have

$$\langle u_k|\Big(\sum_{i=1}^{n} |v_i\rangle\langle u_i|\Big)|v_l\rangle = \sum_{i=1}^{n} \langle u_k|v_i\rangle\langle u_i|v_l\rangle = \sum_{i=1}^{n} \delta_{ki}\delta_{li} = \delta_{kl} = \langle u_k|v_l\rangle = \langle u_k|\mathbb{I}|v_l\rangle .$$

This implies of course $\quad A = A\sum_{i=1}^{n} |v_i\rangle\langle u_i| = \sum_{i=1}^{n} \lambda_i |v_i\rangle\langle u_i| .$