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Stochastic Modelling and Random Processes

Hand-out 3
Poisson processes

Poisson random variables.
Let $X \sim \mathrm{Poi}(\lambda)$ be a **Poisson** random variable with **intensity** $\lambda \geq 0$, i.e.

$$\mathbb{P}[X = k] = \frac{\lambda^k}{k!}\, e^{-\lambda} \quad \text{for all } k \in \mathbb{N}_0 \ .$$

We have $\mathbb{E}[X] = \lambda$, $\mathrm{Var}[X] = \lambda$ and the **characteristic function** of X is

$$\Phi_X(t) = \mathbb{E}[e^{itX}] = \sum_{k=0}^{\infty} e^{itk} \frac{\lambda^k}{k!}\, e^{-\lambda} = \exp\big(\lambda(e^{it} - 1)\big) \ .$$

Therefore, if $X_i \sim \mathrm{Poi}(\lambda_i)$, $i = 1, \ldots, n$ are independent Poisson, then the sum is also Poisson,

$$S = \sum_{i=1}^{n} X_i \sim \mathrm{Poi}(\lambda_1 + \ldots + \lambda_n) \ .$$

For $\alpha \in [0, 1]$, an α**-thinning** $\alpha \circ X$ of an integer random variable $X \in \mathbb{N}_0$ is defined as

$$\alpha \circ X = \sum_{k=1}^{X} Z_k \quad \text{with} \quad Z_k \sim \mathrm{Be}(\alpha) \in \{0, 1\} \quad \text{i.i.d. Bernoulli} \ .$$

For Poisson variables we have $X \sim \mathrm{Poi}(\lambda),\ \alpha \in [0, 1] \quad \Rightarrow \quad \alpha \circ X \sim \mathrm{Poi}(\alpha\lambda)$.
This follows directly from computing the characteristic function

$$\Phi_{\alpha\circ X}(t) = \mathbb{E}\big(e^{it\sum_{k=1}^{X} Z_k}\big) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\, e^{-\lambda}\mathbb{E}(e^{itZ_k})^n = e^{\lambda(\Phi_Z(t)-1)} = \exp\big(\lambda\alpha(e^{it} - 1)\big)$$

where we have used $\Phi_Z(t) = 1 - \alpha + \alpha e^{it} = 1 + \alpha(e^{it} - 1)$.

Poisson processes.
A **Poisson process** $N = (N_t : t \geq 0) \sim \mathrm{PP}(\lambda)$ with **rate** $\lambda > 0$ is a Markov chain with independent stationary increments, and $N_t \sim \mathrm{Poi}(\lambda t)$ for all $t \geq 0$. The holding times are independent $\mathrm{Exp}(\lambda)$ variables with mean $1/\lambda$. The above properties for Poisson random variables imply the following:

- **Adding Poisson processes.**
  Let $N^i \sim \mathrm{PP}(\lambda_i)$ be independent Poisson processes, and define their sum $M = (M_t : t \geq 0)$ via $M_t := N_t^1 + \ldots + N_t^n$ for all $t \geq 0$. Then $M \sim \mathrm{PP}(\lambda_1 + \ldots + \lambda_n)$ is a Poisson process.

- Let $\tau_1, \ldots, \tau_n$ be independent $\mathrm{Exp}(\lambda_i)$ **exponential** random variables, corresponding to the holding times of Poisson processes N^i. Then

  $$\min\{\tau_1, \ldots, \tau_n\} \sim Exp(\lambda_1 + \ldots + \lambda_n) \ ,$$

  corresponding to the holding time of the process $N^1 + \ldots + N^n$.

- **Thinning.**
  An α-thinning $\alpha \circ N$ of a Poisson process $N \sim \mathrm{PP}(\lambda)$ is defined via $(\alpha \circ N)_t = \alpha \circ N_t$ for all $t \geq 0$, i.e. independently keep jumps with probability α and erase the others.
  Then $\alpha \circ N \sim \mathrm{PP}(\alpha\lambda)$ is again a Poisson process.