Fundamentals of Magnetohydrodynamics (MHD)

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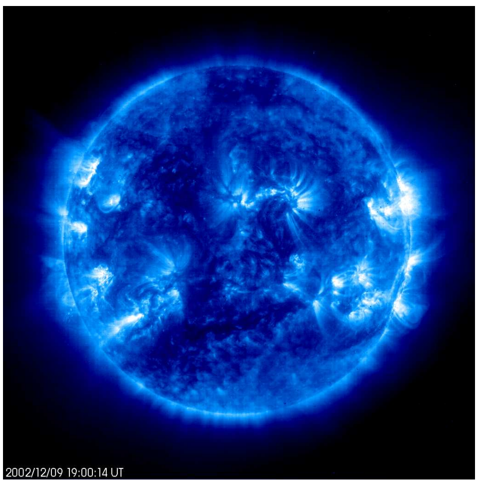
Motivation



Solar Corona in EUV

- Want to understand physical processes in plasmas (ionised conducting fluids)
- Applications: Magnetospheres, Sun and stars, accretion disks, jets etc, laboratory plasmas (e.g. fusion experiments)

Phenomena

- MHD equilibria (e.g. current sheets, flux tubes, loops, etc)
- MHD waves (lecture by V Nakariakov)
- MHD shocks and discontinuities
- Instabilities (lecture by P Browning)
- Magnetic reconnection (lecture by P Browning)
- MHD turbulence
- Magnetic field generation (dynamo processes; lecture by E Kersalé)
- ...

"Derivation" of MHD in a Nutshell I

Plasma at most fundamental level: N particle problem
N particle equations plus Maxwell equations ($N \gg 1$)

$$
\begin{aligned}
m_i \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} &= q_i[\mathbf{E}(\mathbf{x}_i,t) + \mathbf{v}_i \times \mathbf{B}(\mathbf{x}_i,t)] \\
\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} &= \mathbf{v}_i(t) \\
\nabla \cdot \mathbf{E} &= \frac{1}{\epsilon_0} \sum_{i=1}^{N} q_i \delta[\mathbf{x} - \mathbf{x}_i(t)] \\
\nabla \times \mathbf{B} &= \mu_0 \sum_{i=1}^{N} q_i \mathbf{v}_i(t) \delta[\mathbf{x} - \mathbf{x}_i(t)] + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}
\end{aligned}
$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad , \qquad \nabla \cdot \mathbf{B} = 0$$

"Derivation" of MHD in a Nutshell II

- N particle problem: untractable!
- Introduce N particle distribution function $\Gamma(\mathbf{x}_1, \mathbf{v}_1; \ldots ; \mathbf{x}_N, \mathbf{v}_N; t)$
- Liouville equation for Γ, still too nasty

$$\frac{\partial \Gamma}{\partial t} + \sum_{i=1}^{N} \left[\mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} \Gamma + \frac{q_i}{m_i} (\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) \cdot \nabla_{\mathbf{v}_i} \Gamma \right] = 0$$

"Derivation" of MHD in a Nutshell III

- BBGKY hierarchy: Reduce problem to one-particle problem by integrating over $N-1$ particle variables $\mathbf{x}_i$, $\mathbf{v}_i$ (I am glossing over a lot of maths here)
- Leads to equation for one-particle distribution function $f_s(\mathbf{x}, \mathbf{v}, t)$ equation (for species s of n in total)

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s} [\mathbf{E}(\mathbf{x}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{x}, t)] \cdot \nabla_{\mathbf{v}} f_s = C[f_1, \ldots, f_n]$$

- $C_s[f_1, \ldots, f_n]$ = "collision term"
- $C_s = 0$: Vlasov equation for collisionless plasmas

"Derivation" of MHD in a Nutshell IV

- Take velocity moments $\int v_x^k v_y^m v_z^n f_s \, d^3v$ of equation for f_s to derive multifluid equations (k, m, n integers)
- Examples:
  - particle density $n_s = \int f_s d^3v$
  - average velocity $\mathbf{u}_s = \int \mathbf{v} f_s d^3v / n_s$
  - etc
- Results in an infinite hierarchy of equations: n^{th} moment equation depends on terms with $(n+1)^{\text{th}}$ moment

"Derivation" of MHD in a Nutshell V

- See first two resulting equations

$$\frac{\partial n_s}{\partial t} + \boldsymbol{\nabla} \cdot (n_s \mathbf{u}_s) = 0$$

$$m_s n_s \left[\frac{\partial \mathbf{u}_s}{\partial t} + (\mathbf{u}_s \cdot \nabla)\mathbf{u}_s \right] + \nabla \cdot \mathbf{P} - q_s n_s [\mathbf{E}(\mathbf{x}, t) + \mathbf{u}_s \times \mathbf{B}(\mathbf{x}, t)] = \mathbf{F}$$

- Need closure condition to truncate moment hierarchy
- Usually closure condition is some assumption regarding third or fourth order moments

"Derivation" of MHD in a Nutshell VI

- From now assume only two fluids: electrons and protons (Remark: $m_p \approx 1836\, m_e$)
- Define:
  - charge density: $\rho_c = e(n_p - n_e) \approx 0$, so $n_e \approx n_p = n$ (quasi-neutrality)
  - mass density: $\rho = m_p n_p + m_e n_e = (m_p + m_e)n$ ($\approx m_p n$)
  - velocity: $\mathbf{v} = \frac{m_p n_p \mathbf{v}_p + m_e n_e \mathbf{v}_e}{m_p n_p + m_e n_e} = \frac{m_p \mathbf{v}_p + m_e \mathbf{v}_e}{m_p + m_e}$ ($\approx \mathbf{v}_p$)
  - current density: $\mathbf{j} = e(n_p \mathbf{v}_p - n_e \mathbf{v}_e) = en(\mathbf{v}_p - \mathbf{v}_e)$
  - (total) pressure: $p = p_p + p_e$

Assumptions

- Plasma quasi-neutral (see above)
- Pressure scalar (see above)
- Typical length scales much larger than kinetic length scales, e.g. gyro radii, skin depth etc
- Typical time scales much slower than kinetic time scales, e.g. gyro frequencies
- Velocity much smaller than speed of light

MHD is a theory describing large-scale and slow phenomena compared to kinetic theory

MHD Equations: Fluid Equations

Mass Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Equation of Motion (Momentum equation)

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{j} \times \mathbf{B} - \nabla p + \mathbf{F}$$

Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{R}$$

Also needed: Energy equation and Equation of State (will be discussed later)

MHD Equations: Maxwell Equations

Ampère's law (displacement current neglected)

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Solenoidal condition

$$\nabla \cdot \mathbf{B} = 0$$

Poisson equation for $\mathbf{E}$: "solved" by quasi-neutrality assumption

Mass Conservation

- Integrate continuity equation over a volume V:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int\limits_V \frac{\partial \rho}{\partial t}\, dV = -\int\limits_V \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) dV = -\int\limits_S (\rho \mathbf{v}) \cdot \mathbf{n} dS$$

- Mass M inside volume V changes if there is net mass in- or outflow through the boundary S
- Without flow through boundaries, M in V is conserved.

Momentum Conservation

- Rewrite momentum equation in conservation form:

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \boldsymbol{\nabla} \cdot \left[\rho \mathbf{v}\mathbf{v} + \left(p + \frac{\mathbf{B}^2}{2\mu_0} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{\mu_0} \right] = \mathbf{F}$$

- Integrate momentum equation over a volume V:

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \int\limits_V \frac{\partial (\rho \mathbf{v})}{\partial t}\, dV = -\int\limits_S \mathbf{T} \cdot \mathbf{n} dS + \int\limits_V \mathbf{F} dV$$

  where $\mathbf{T} = \rho \mathbf{v}\mathbf{v} + \left(p + \frac{\mathbf{B}^2}{2\mu_0} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{\mu_0}$.

- Total momentum $\mathbf{P}$ inside volume V changes due to stresses on boundary and external forces.

Ohm's Law

- Ohm's Law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{R}$$

  can be regarded as the leading order terms of the electron fluid equation of motion.
- $\mathbf{R}$ represents different forms of Ohm's law:
  - ideal: $\mathbf{R} = \mathbf{0}$
  - resistive: $\mathbf{R} = \eta \mathbf{j}$ ($\eta =$ resistivity)
  - more general forms could include: Hall term $\mathbf{j} \times \mathbf{B}/en$, (electron) pressure term, inertial terms etc

The Induction Equation

- The electric field can be completely eliminated from the MHD equations
- Combine Faraday's law and Ohm's law to obtain the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B} - \mathbf{R})$$

- Ideal form

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Resistive Induction Equation

- Resistive MHD: $\mathbf{R} = \eta \mathbf{j}$
- Assume $\eta =$ constant for simplicity
- Then

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{\eta}{\mu_0} \nabla \times [\nabla \times \mathbf{B}] \\ &= \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta}{\mu_0} \Delta \mathbf{B}\end{aligned}$$

Magnetic Reynolds Number

- Non-dimensionalise equation ($\mathbf{B} = B_0 \tilde{\mathbf{B}}$ etc)

$$\frac{\partial \tilde{\mathbf{B}}}{\partial \tilde{t}} = \nabla \times (\tilde{\mathbf{v}} \times \tilde{\mathbf{B}}) + \frac{1}{R_m} \Delta \tilde{\mathbf{B}}$$

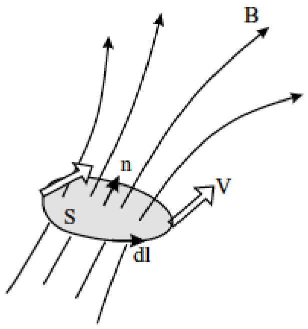
with

$$R_m = \frac{\mu_0 L_0 v_0}{\eta}, \quad \text{magnetic Reynolds number}$$

- Usually $R_m \gg 1$ for the applications we consider (order $10^6 - 10^{12}$)
- Non-ideal term only important if second derivatives of B large $\Longrightarrow$ strong current density!

Magnetic Flux and Line Conservation

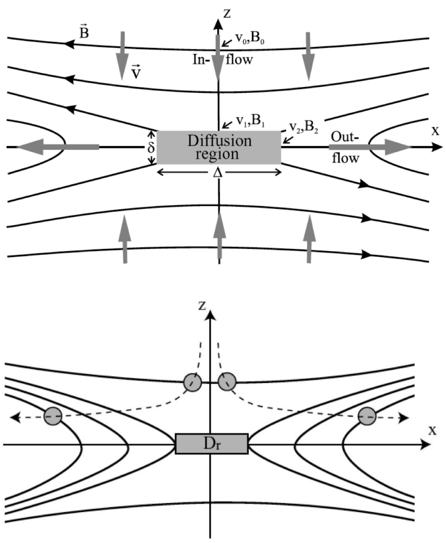
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{n}\cdot\mathbf{B}dS = \int_S \mathbf{n}\cdot\frac{\partial\mathbf{B}}{\partial t}dS - \oint_l \mathbf{V}\times\mathbf{B}\cdot d\mathbf{l}$$
$$= -\int_S \nabla\times(\mathbf{E}+\mathbf{V}\times\mathbf{B})\mathbf{n}dS$$



so magnetic flux conserved if ideal Ohm's law applies ($\mathbf{V}=\mathbf{v}$)

Line conservation (without proof): for ideal MHD plasma elements stay on the same field line!
(for detailed discussion, see e.g. Schindler, 2007)

Resistive MHD: A few remarks



- Usually $R_m \gg 1$ in solar applications, i.e. solar plasma ideal
- Violated in localized regions of strong current density (large derivatives of B-field)
- Localized non-ideal regions can have global effects!
- Important: Current sheets, magnetic null points, separators etc

Energy Equation

- Can be written in different forms depending on thermodynamic variables used
- E.g. using the equation of state for an ideal gas and internal energy $e = p/(\gamma - 1)\rho$

$$\rho \frac{\partial e}{\partial t} + \rho(\mathbf{v} \cdot \nabla) e + (\gamma - 1)\rho e \boldsymbol{\nabla} \cdot \mathbf{v} = -\mathcal{L}$$

where

$$\mathcal{L} = \underbrace{\nabla \cdot \mathbf{q}}_{\text{heat flux}} + \overbrace{L_r}^{\text{radiative losses}} - \underbrace{\eta \mathbf{j}^2}_{\text{Ohmic heating}} - \overbrace{H}^{\text{everything else}}.$$

Energy Equation: Another Form

Using pressure p, we get for ideal MHD ($\eta = 0$, no heat flux etc)

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

or for resistive MHD ($\eta \neq 0$)

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \boldsymbol{\nabla} \cdot \mathbf{v} = (\gamma - 1)\eta |\mathbf{j}|^2$$

Term on right hand side: Ohmic heating

Energy Conservation

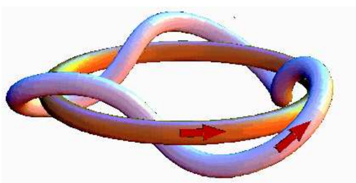
- Energy equations presented above are not in conservative form!
- Have to use momentum equation, multiply by $\mathbf{v}$ and combine with energy equation to get

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e + \frac{B^2}{2\mu_0}\right) + \nabla \cdot \left[\frac{\rho v^2}{2}\mathbf{v} + (\rho e + p)\mathbf{v} + \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}\right] = 0$$

for ideal and resistive MHD!

- More terms necessary if e.g. external forces are present in the momentum equation

Magnetic Helicity



- Vector potential $\mathbf{A}$:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

- Magnetic Helicity:

$$H = \int_V \mathbf{A} \cdot \mathbf{B}\, dV$$

- H is a measure of how much a magnetic field are interlinked, twisted etc.
- Remark: H is only one of infinitely many "invariants" of ideal MHD

Gauge Invariance

- H is not gauge invariant in general:
  Let $\mathbf{A}' = \mathbf{A} + \nabla\psi$ (same $\mathbf{B}$ obviously)

$$H' = H + \int_V \mathbf{B} \cdot \nabla\psi \, dV = H + \int_S \psi \mathbf{B} \cdot d\mathbf{S}$$

- The surface integral only vanishes if $B_n = 0$, i.e no field lines cross boundary
- In many practical situations gauge invariant forms of magnetic helicity have to be used, e.g.

$$H_{rel} = \int_V (\mathbf{A} + \mathbf{A}_0) \cdot (\mathbf{B} - \mathbf{B}_0) \, dV$$

Magnetic Helicity Conservation I

- In general one finds that (without proof):

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -2 \int_V \mathbf{E} \cdot \mathbf{B} \, dV$$

  (see e.g. Biskamp, 1993, or Biskamp, 2000)
- H is conserved in ideal MHD, i.e.

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0,$$

  because

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}.$$

Magnetic Helicity Conservation II

- Even in non-ideal cases the integral on right hand side is small, so magnetic helicity is at least approximately conserved
- "Small" here means that other quantities (e.g. magnetic energy) change much more rapidly than H (see e.g. Schindler, 2007, for a detailed calculation).
- A general remark: Helicity conservation means the value of the *total* helicity in a volume does not change!
- However, within the volume helicity density ($\mathbf{A} \cdot \mathbf{B}$ or equivalent) will generally be redistributed!
- Analogy: Conservation of total mass, but mass density changes in space and time

Magnetic pressure and tension

- Important for MHD equilibria, waves etc
- Lorentz force

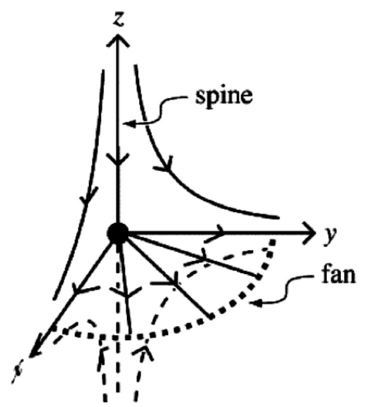
$$\begin{aligned} \mathbf{j} \times \mathbf{B} &= \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \\ &= \underbrace{\frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B}}_{\text{magnetic tension}} - \underbrace{\nabla \left(\frac{B^2}{2\mu_0}\right)}_{\text{magnetic pressure}} . \end{aligned}$$

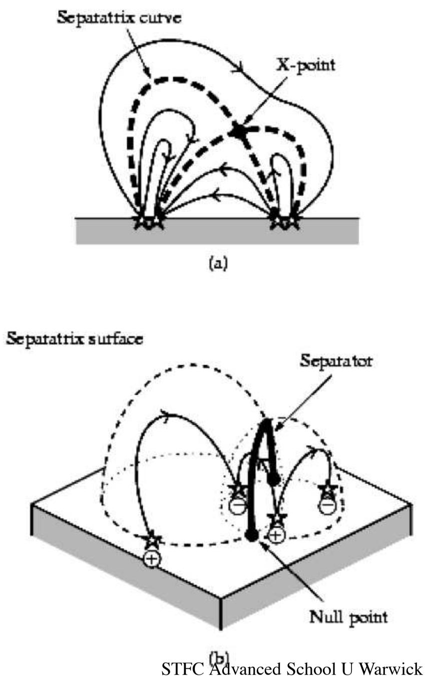
- Plasma beta: ratio of plasma pressure and magnetic pressure:

$$\beta_p = \frac{2\mu_0 p}{B^2}$$

Magnetic Null Points

- Points in space where $\mathbf{B} = 0$
- Important for defining the connectivity and topology of magnetic field configurations





Current Sheets

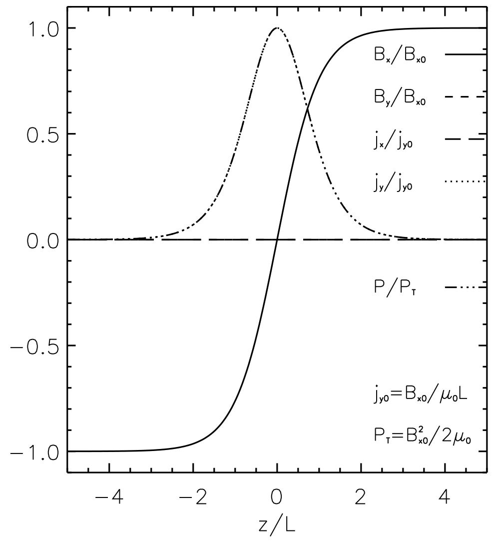
- Current sheets: can be singular MHD structures (discontinuities) or finite (e.g. neutral sheets)
- Here: non-singular current sheets in 1D (justified by ratio of length scales)
- Equilibrium structures: Total pressure across sheet is constant

$$\frac{B^2(z)}{2\mu_0} + p(z) = p_T = \text{constant}$$

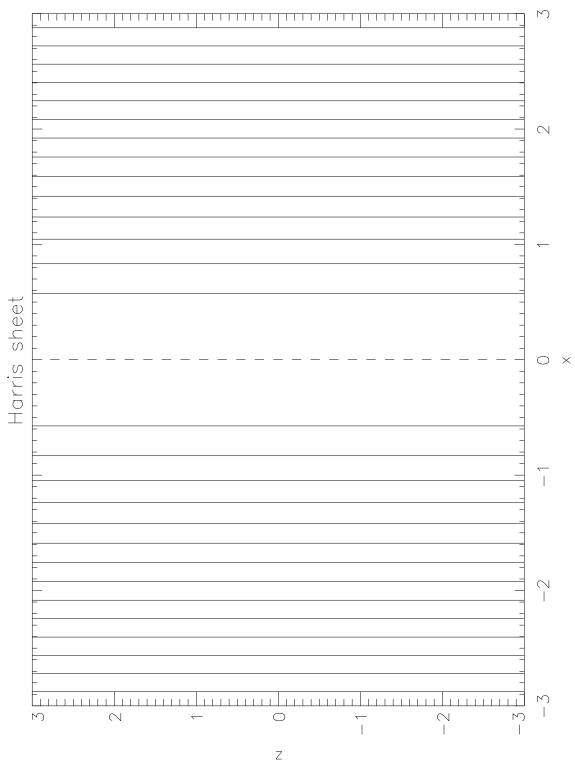
- Often used: Harris Sheet (E. Harris, 1962)
  Originally a kinetic equilibrium, but is also an MHD equilibrium

Harris Sheet

- $\mathbf{B} = B_0 \tanh(z/L)\hat{\mathbf{x}}$
- $p(z) = p_0/\cosh^2(z/L) + p_b$
- $B_0^2/(2\mu_0) = p_0$



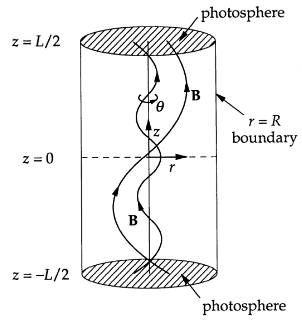
Field Lines



Flux tubes

- Simplest case: 1D equilibria in cylindrical geometry (use r, ϕ, z as cylindrical coordinates)
- Can be used as models for coronal loops, also for magnetic structures in solar interior
- Equilibrium ($\mathbf{B} = (0,\, B_\phi(r),\, B_z(r))$):

$$\frac{d}{dr}\left(\frac{B_\phi^2(r) + B_z^2(r)}{2\mu_0} + p(r)\right) + \frac{B_\phi^2}{\mu_0 r} = 0$$



Flux tubes: Examples

- Bennett pinch (Bennett 1934) – only $B_\phi(r)$ and $p(r)$:

$$B_\phi(r) = \frac{\mu_0 I_0}{2\pi}\frac{r}{r^2 + a^2}, \quad p(r) = \frac{\mu_0 I_0^2}{8\pi^2}\frac{a^2}{(r^2 + a^2)^2}$$

- Gold-Hoyle tube (Gold and Hoyle, 1960) – 1D force-fee flux tube with $B_\phi(r)$ and $B_z(r)$ non-zero

$$B_\phi(r) = \frac{B_0 a r}{r^2 + a^2}, \quad B_z(r) = \frac{B_0 a^2}{r^2 + a^2}$$

MHD equilibria: Symmetric Systems

Translational, rotational or helical symmetry:

MHD can be reduced to a single nonlinear elliptic second-order PDE

Here just a quick reminder how to do that for translational invariance without external forces

$$\mathbf{j} \times \mathbf{B} - \nabla p = \mathbf{0}$$

For more details (also on the other cases and with external forces) : see lecture notes on my web page [a]

[a] http://www-solar.mcs.st-and.ac.uk/~thomas/teaching/mhdlect.pdf

Translational Invariance 1

Assume $\frac{\partial}{\partial y} = 0 \Longrightarrow$ Invariance in y-direction

Satisfy $\nabla \cdot \mathbf{B} = 0$ by $\mathbf{B} = \nabla A \times \mathbf{e}_y + B_y \mathbf{e}_y$

Then

$$\mathbf{B} \cdot \nabla A = \underbrace{(\nabla A \times \mathbf{e}_y) \cdot \nabla A}_{=0} + B_y \underbrace{\mathbf{e}_y \cdot \nabla A}_{=0 \text{ since } \frac{\partial A}{\partial y}=0} = 0.$$

A is constant along magnetic field lines!

Translational Invariance 2

Translational Invariance 3

$$-\left(\nabla B_y \times \mathbf{e}_y\right) \cdot \nabla A = \left(\nabla A \times \mathbf{e}_y\right) \cdot \nabla B_y = \mathbf{0}$$

B_y is constant along field lines $\Longrightarrow$ can take $B_y = g(A)$

So

$$\nabla B_y = \frac{dg}{dA} \nabla A$$

and

$$\mathbf{j} \times \mathbf{B} - \nabla p = \frac{1}{\mu_0} \left(-\Delta A - \mu_0 \frac{df}{dA} - g(A) \frac{dg}{dA} \right) \nabla A = \mathbf{0}$$

Translational Invariance 4

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)A=\mu_0\frac{d}{dA}\left(p(A)+\frac{B_y^2}{2\mu_0}\right)=F(A)$$

Grad-Shafranov(-Schlüter) equation for translational invariance

Single nonlinear 2nd order elliptic partial differential equation:
boundary conditions for A needed (e.g. Dirichlet or von Neumann)

Some analytical solutions known
(for special choices of $F(A)$)

3D MHS

- Representation of $\mathbf{B}$ to guarantee $\nabla\cdot\mathbf{B}=0$ much more difficult
- Euler Potentials (Clebsch representation):

$$\mathbf{B}=\nabla\alpha\times\nabla\beta$$

intrinsically nonlinear
existence of global α and β not guaranteed
(could use four potentials instead).

- Vector potential

$$\mathbf{B}=\nabla\times\mathbf{A}$$

Which gauge for $\mathbf{A}$? Boundary conditions for $\mathbf{A}$?

- Use $\mathbf{B}$ directly, ensure $\mathbf{B}$ solenoidal by numerical means

Euler Potential Equations

$$\begin{aligned}\nabla\beta\cdot\nabla\times(\nabla\alpha\times\nabla\beta) &= \mu_0\frac{\partial p}{\partial\alpha}\\ \nabla\alpha\cdot\nabla\times(\nabla\beta\times\nabla\alpha) &= \mu_0\frac{\partial p}{\partial\beta}\end{aligned}$$

Further difficulty: these equations are of mixed type!

What are the appropriate boundary conditions for solving them?

Force-free Fields 1

For the rest of this lecture I shall focus on force-free fields, because they are most relevant for the solar corona, e.g. for extrapolation of the coronal magnetic field from photospheric measurements

For the corona the plasma beta $\beta_p = 2\mu_0 p/B^2 \ll 1$ is usually much smaller than unity, so

$$\mathbf{j}\times\mathbf{B} = \mathbf{0}$$

Current density field-aligned/parallel to $\mathbf{B}$ everywhere, i.e.

$$\mu_0\mathbf{j} = \alpha(\mathbf{r})\,\mathbf{B}$$

Force-free Fields 2

Since $\nabla \cdot \mathbf{j} = 0$ and $\nabla \cdot \mathbf{B} = 0$ we get

$$\mathbf{B} \cdot \nabla \alpha = 0$$

i.e. α is constant along magnetic field lines.

Basic equations to solve:

$$\begin{aligned} \nabla \times \mathbf{B} &= \alpha(\mathbf{r})\,\mathbf{B} \\ \mathbf{B} \cdot \nabla \alpha &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

Force-free Fields 3

- Potential fields : $\mathbf{j} = \mathbf{0}$, $\alpha = 0$
- Linear force-free fields: $\mathbf{j} = \alpha \mathbf{B}$, $\alpha =$ constant $\neq 0$
- Nonlinear force-free fields $\mathbf{j} = \alpha(\mathbf{r})\mathbf{B}$, $\mathbf{B} \cdot \nabla \alpha = 0$

All three classes are used for extrapolation of coronal magnetic fields, but the last one is the most important class (but also most difficult to calculate !)

Further Reading

- Biskamp, *Nonlinear Magnetohydrodynamics*, Cambridge UP, 1993
- Biskamp, *Magnetic Reconnection in Plasmas*, Cambridge UP, 2000
- Boyd and Sanderson, *The Physics of Plasmas*, Cambridge UP, 2003
- Freidberg, *Ideal Magnetohydrodynamics*, Plenum Press, 1987
- Goedbloed and Poedts, *Principles of Magnetohydrodynamics*, Cambridge UP, 2004
- Priest, *Solar Magnetohydrodynamics*, Reidel, 1982
- Schindler, *Physics of Space Plasma Activity*, Cambridge UP, 2007