Thermal Physics II – Solutions for Problem Sheet 3

1. Limits of Derived Results

In the lecture, the heat capacity of an Einstein solid was derived to be

$$c_N = \frac{3N\hbar^2\omega^2}{k_B T^2} \frac{\exp(-\beta\hbar\omega)}{[1-\exp(-\beta\hbar\omega)]^2} ,$$

where $\beta = 1/(k_B T)$ as usual.

Low temperature limit:

- For $T \to 0$ ($\beta \to \infty$), we have $\exp(-\beta\hbar\omega) \to 0$ or better $\exp(\beta\hbar\omega) \to \infty$.
- Thus, we find

$$\lim_{T\to 0} c_N \propto \lim_{T\to 0} \frac{\beta^2}{\exp(\beta\hbar\omega)} = \lim_{T\to 0} \frac{2\beta}{\beta \exp(\beta\hbar\omega)} \propto \lim_{T\to 0} \exp(-\beta\hbar\omega)$$
$$= 0 .$$

  In the second step, l'Hospital's rule was used.

- We find an exponential decay of the heat capacity as $T \to 0$ which has, of course, the limit $\lim_{T\to 0} c_N = 0$.

High temperature limit:

- For $T \to \infty$, we have $\beta \to 0$. In this case, the exponential can be expanded for small arguments:

$$\exp(-\beta\hbar\omega) \approx 1 - \beta\hbar\omega .$$

- Using this expansion in the expression for the heat capacity, we find

$$\lim_{T\to\infty} c_N = \frac{3N\hbar^2\omega^2}{k_B T^2} \frac{1-\beta\hbar\omega}{[1-1+\beta\hbar\omega]^2} \approx \frac{3N\hbar^2\omega^2}{k_B T^2} \frac{1}{[\beta\hbar\omega]^2} = 3Nk_B ,$$

  as $\beta\hbar\omega$ is small compared to unity (second step).

- As a result, we find a heat capacity of k_B per oscillator and direction.

2. Counting Microstates

- If we would have no restrictions, each quark can have one of the 6 possible colours **independently** of the other quarks. Thus, we have 6 possibilities per quark which is multiplied with the possibilities of the other quarks.

  $\Longrightarrow$ baryons: $6^3 = 216$ possibilities;

  $\Longrightarrow$ mesons: $6^2 = 36$ possibilities.

- However, this ignores the fact that quarks are micro-particles which cannot be distinguished. Thus, we have to divide by the number of identical configurations

  $\Longrightarrow$ baryons: $6^3/3! = 36$ possibilities;

  $\Longrightarrow$ mesons: $6^2\ 2! = 18$ possibilities.

- If we take the "white restriction" and the indistinguishability of the quarks into account, we have

  - two choices for baryons:
    $\langle$red, blue, yellow$\rangle$ and $\langle$anti-red, anti-blue, anti-yellow$\rangle$;
  - three choices for mesons:
    $\langle$red, anti-red$\rangle$, $\langle$blue, anti-blue$\rangle$ and $\langle$yellow, anti-yellow$\rangle$.

  Swapping the colours for barions or the colour with the anti-colour for the mesons does not make a new particle.