Thermal Physics II – Solutions for Problem Sheet 8

1. Number of States for ideal Bosons and Fermions

- a) Each particle has 29 choices. Thus we have

  $$\Omega = 29^{12}$$

  if the particle are distinguishable.

- b) For the first particle, we have 29 choices; 28 choices for the second paticle; ... ; 18 choices for the last. Moreover, we have to take into account that the particles are indistinguishable having 12! similar states:

  $$\Omega = \frac{S!}{(N!(S-N)!)} = \frac{29 \times 28 \times ... \times 18}{12!} \,.$$

  $S = 29$ is here the number of available energy states and $N = 12$ is the number of particles.

- c) For Bosons, the Pauli exclusion principle does not apply. Thus, we can order $N + S - 1$ objects (N particles and $S - 1$ boundaries between the energy states). However, the particles as well as the boundaries are indistinguishable. Hence, we have

  $$\Omega = \frac{(N + S - 1)!}{N!\,(S-1)!} = \frac{(12 + 29 - 1)!}{12!\,(29-1)!} = \frac{40!}{12!\,28!} \,.$$

2. Conduction Electrons as ideal Fermi Gas

- The electro density is $n_e = 8.1 \times 10^{23}\,\mathrm{cm}^{-3} = 8.1 \times 10^{29}\,\mathrm{m}^{-3}$.
- The electron Fermi energy is

$$E_F = \frac{h^2}{8m_e}\left(\frac{3n_e}{\pi}\right)^{2/3} = 5.07 \times 10^{-18}\,\mathrm{J}\,.$$

- The temperature is $T = 300\,\mathrm{K}$. Thus the thermal energy is

$$k_B T = 4.14 \times 10^{-21}\,\mathrm{J}\,.$$

- As the Fermi energy is much larger than the thermal energy (three orders of magnitude), a $T = 0$ treatment is possible.
- The highest occupies momentum is the Fermi momentum connected to the Fermi energy by $E_F = p_F^2/(2m_e)$. This relations yields

$$p_F = \sqrt{2m_e E_F} = 3.04 \times 10^{-24}\,\mathrm{kg\,m\,s^{-1}}\,.$$

- The pressure of the electrons is the Fermi pressure which is given by

$$p_F = \frac{2}{5} n E_F = 1.64 \times 10^{12}\,\mathrm{N\,m^2} = 1.64 \times 10^{12}\,\mathrm{Pa}\,.$$

  This pressure is balanced by the attraction to the ions.

3. Critical Values for the Bose-Einstein-Condensation

- The paticles must be Bosons to BEC to occur. As the number of proton is equal to the number of electrons, we must have an even number of neutron in oder to have an even number of Fermions in the composite atoms. We have 37 protons and 37 electrons. The average atomic weight is 85.47. Thus, stable rubidium with 49 and 48 neutrons exist. For BEC, we need the isotope with 48 neutrons and an atomic weight of 85.
- BEC occurs when the cooling reached a stage where $n\Lambda^3 = 2.612$
- At the given temperature, the thermal wave length is

$$\Lambda = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2} \approx 7 \times 10^{-7}\,\mathrm{m}\,.$$

- With these numbers, we find for the critical density that just fulfills the BEC-condition

$$n_c = 10^{19}\,\mathrm{m}^3 = 10^{13}\,\mathrm{cm}^3\,.$$

- This corresponds to a volume of

$$V = \frac{N}{n} = \frac{10^4}{10^{19}\,\mathrm{m}^3} = 10^{-15}\,\mathrm{m}^{-3} = 10^3\,\mu\mathrm{m}^3\,.$$

  Thus, the traps had a linear dimension of roughly $10\,\mu$m.

4. Comic Microwave Background as Blackbody Radiation

- The VERY isotropic cosmic background radiation can be excellently fit by a temperature of $T = 2.735\,\mathrm{K}$.
- The Planck spectrum peaks at

$$\lambda_{max} = \frac{hc}{4.96\,k_B T}\,.$$

  This corresponds to a maximum of $\lambda = 1.9\,\mathrm{mm}$ ($160\,\mathrm{GHz}$).
- The whole cosmic microwave background shifts due to the expansion of the universe. As a result, all photons get 'streched' and the related blackbody temperature decreases.