Limit Cycles

Have found that orbits cannot cross, can be attracted to (fixed points), etc. One other possibility is limit cycle.

ODE is 'well behaved' ie: all derivatives exist and are continuous –

Therefore, all orbits smoothly follow neighbours in phase space.

One other possibility only:

\[
\text{limit cycle} \rightarrow
\]

orbits approach closed curve as \( t \to \infty \)

NB – complete description of all details is non trivial – here give the basics.

Limit cycle – an example

Consider

\[
F = x + y - x(x^2 + y^2) \\
G = -(x - y) - y(x^2 + y^2)
\]

Fixed point \( F = G = 0 \) is \( \bar{x} = 0, \bar{y} = 0 \)

Stability analysis

\[
x = \bar{x} + \delta x \hspace{1cm} y = \bar{y} + \delta y
\]

\[
F = \delta x + \delta y = a\delta x + b\delta y \hspace{1cm} p = a + d = 2 \hspace{1cm} q = ad - bc = 2 \hspace{1cm} p^2 < 4q
\]

- unstable spiral

In addition – to look elsewhere in phase plane, rewrite in polars

\[
x = r \cos \theta \hspace{1cm} y = r \sin \theta \hspace{1cm} x^2 + y^2 = r^2
\]

use following identities

\[
x \frac{dx}{dt} + y \frac{dy}{dt} = r \frac{dr}{dt} \hspace{1cm} x \frac{dy}{dt} - y \frac{dx}{dt} = r^2 \frac{d\theta}{dy}
\]
\[
\begin{align*}
\frac{dx}{dt} &= x + y - x(x^2 + y^2) \\
\frac{dy}{dt} &= -(x - y) - y(x^2 + y^2)
\end{align*}
\]

then

\[
\begin{align*}
r \frac{dr}{dt} &= x^2 + xy - x^2(x^2 + y^2) - xy + y^2 - y^2(x^2 + y^2) \\
&= r^2 - r^4
\end{align*}
\]

\[
\begin{align*}
r^2 \frac{d\theta}{dt} &= -x^2 + xy(x^2 + y^2) - xy - y^2 + xy(x^2 + y^2) \\
&= -r^2
\end{align*}
\]

ie: \( \frac{dr}{dt} = r(1 - r^2) \), \( \frac{d\theta}{dt} = -1 \)

Integrate directly –

\[
\theta = \theta_0 - t \quad \left[ r^2 = \frac{Ae^{2t}}{1 + Ae^{2t}} \right]
\]

don't need to integrate \( r \) equation to see the limit cycle.

\[
\frac{dr}{dt} = 0 \quad r = 1 \quad \text{for any } \theta \quad \text{(as well as } r = 0 \text{ the fixed point)}
\]

trajectory sits on circle \( r = 1 \).

For \( r > 1 \quad r(1 - r^2) < 0 \) \( r < 1 \quad r(1 - r^2) < 0 \) by inspection.

Therefore, solution is attracted to \( r = 1 \) circle.

No single cast iron method to find limit cycles – see course texts for some advanced methods.
Example of limit cycle – Van der Pol oscillator

Van der Pol, 1926 – Electric circuit with valve (model of heartbeat)

Identical to Rayleigh, 1883 – Nonlinear Vibrations

1st experimentally shown limit cycle

\[ \frac{d^2x}{dt^2} + \varepsilon(x^2 - 1) \frac{dx}{dt} + x = 0 \]

cause of trouble

Write as

\[ \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x - \varepsilon(x^2 - 1)y \]

If \( \varepsilon = 0 \) → linear pendulum \( \omega = 1 \).

Symmetries – invariant for \( \varepsilon \to -t; \quad \varepsilon \to -\varepsilon \)

Therefore, solve for \( \varepsilon > 0 \)
- reverse time for \( \varepsilon < 0 \)

ie: \( \varepsilon > 0 \) growth is \( \varepsilon < 0 \) damping, etc.

Fixed points

\( \bar{x} = 0, \quad \bar{y} = 0 \)

Stability

Find \( F = \frac{dx}{dt} = y \)

\( G = \frac{dy}{dt} = -x - \varepsilon(x^2 - 1)y \)

or work out

\[ \frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 1, \quad \frac{\partial G}{\partial x} = -1 - 2\varepsilon x, \quad \frac{\partial G}{\partial y} = -\varepsilon(x^2 - 1) \]
Evaluate at $\bar{x}, \bar{y} = 0$ \[ \frac{\partial F}{\partial x} = 0 \quad \frac{\partial F}{\partial y} = 1 \quad \frac{\partial G}{\partial x} = -1 \quad \frac{\partial G}{\partial y} = \varepsilon \]

then
\[ p = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \varepsilon \]
\[ q = \frac{\partial F}{\partial x} \cdot \frac{\partial G}{\partial y} - \frac{\partial F}{\partial y} \cdot \frac{\partial G}{\partial x} = 1 \]

$\varepsilon > 0 \quad p > 0 \quad q > 0$ unstable and spiral if $p^2 < 4q$.

Guess there is more ….

Since damping term is $\varepsilon(x^2 - 1)$

this is $+ve$ for large $x$ (damping)

changes sign as $x \to 1$ (growth)

is zero at $x = 1$! (neither!)

Solve – multiple timescale analysis (Rowlands, appendix)
- method of averaging (Drazin, p 193) - handout for result

Pendulum by formula

We have
\[ \frac{d\theta}{dt} = 0 \delta\theta + 1.\delta y \equiv F \]
\[ \frac{dy}{dt} = -\omega^2 (-1)^n \delta\theta + 0\delta y \equiv G \]

or
\[ \frac{dy}{dt} = 0 \delta y - \omega^2 (-1)^n \delta\theta \]
\[ \frac{d\theta}{dt} = 1.\delta y + 0 \delta\theta \]

- same thing since

\[ J = \begin{pmatrix} 0 & 1 \\ -\omega^2 (-1)^n & 0 \end{pmatrix} \]

\[ F = a\delta x + b\delta y \]
\[ G = c\delta x + d\delta y \]

\[ p = a + d = 0 \]
\[ q = ad - bc = \omega^2 (-1)^n \]

So, for $n$ even $q > 0$ centre, $n$ odd $q < 0$ saddle (see handout)