Sheet 1 Question 1

(i) Particle motion in \( B \) field

\[
m \frac{dv}{dt} = qv \wedge B \quad \frac{dr}{dt} = v
\]

Normalise \( v^* = \frac{v}{v_0} \), \( t = t' T \) \( r = r' L \) \( B = B' B_0 \)

sub in

\[
m \frac{d(v^* v_0^*)}{dt T} = q v_0^* v^* \wedge B^* B_0
\]

\[
\frac{dv^*}{dt} = T \frac{qB_0}{m} v^* \wedge B^* \quad \text{which is normalised if } T = \left( \frac{qB_0}{m} \right)^{-1} = \frac{1}{\Omega}
\]

also

\[
\frac{dr^* L}{dt T} = v^* v_0 \quad \text{ie: } v_0 = \frac{L}{T}
\]

so \( L = v_0 T = \frac{v_0}{\Omega} \)

solving the equations yields circular motion about \( B \) with frequency \( \Omega \), radius \( L \).

Frequency is independent of velocity (particle energy), whereas gyroradius \( (L) \) depends on velocity.

(ii) Wave equation (ID here)

\[
\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2}
\]

Normalise:

\[
\frac{1}{c^2} \frac{\partial^2 \psi^*}{\partial t^*} \frac{\psi_0}{T^2} = \frac{\partial^2 \psi^* \psi_0}{\partial x^* L^2}
\]

which is normalised (dimensionless) if

\[
\frac{\partial^2 \psi^*}{\partial t^*} = \frac{\partial^2 \psi^*}{\partial x^*} \quad \frac{L}{T} = c.
\]

Therefore, \( c \) is characteristic velocity of all structures regardless of length scale and is independent of amplitude \( \psi \). Solutions are of the form \( \psi = f(x + ct) + g(x - ct) \).
(iii) Conservation of quantity $Q$ with number density $n$

\[
\frac{\partial (nQ)}{\partial t} = \nabla \cdot (nQ \mathbf{v}),
\]

where $Q$ is carried by "particles" of density $n$.

Normalise

\[
\frac{\partial (n^*Q^*)}{\partial t^*} - \frac{1}{T} \frac{1}{L} Q_0 = \nabla \cdot (n^*Q^* \mathbf{v}^*) \frac{1}{L} \frac{1}{L^*} L^* Q_0
\]

ie \[ \mathbf{v}^* = \frac{\mathbf{v}}{v_0} = \frac{\mathbf{v}}{\left(\frac{T}{L}\right)} \]

then:

\[
\frac{\partial}{\partial t^*}(n^*Q^*) = \nabla \cdot (n^*Q^* \mathbf{v}^*).
\]

There is no characteristic scale if $v_0 = \frac{L}{T}$ equation just specifies that structures on all length and timescales are conserved.
Sheet 1 Question 2

\[ F = F_0 + F_1 M + F_2 M^2 + F_3 M^3 + F_4 M^4 \]

can always be written as

\[ F = F_0' + F_2' (M - M_0)^2 + F_3' (M - M_0)^3 + F_4' (M - M_0)^4 \]

since both are general polynomials up to degree 4 then \( M \rightarrow M - M_0 \) is the required transformation.

(i) For symmetry \( F_3 = 0 \).

We then have (dropping 's)

\[ F(M) = F_0 + \alpha (T - T_c) M^2 + \beta M^4 \]

extrema

\[ \frac{\partial F}{\partial M} = 2\alpha (T - T_c) M + 4\beta M^3 = 2M (\alpha (T - T_c) + 2\beta M)^2 \]

ie: at \( M = 0 \) or \( M^2 = \frac{\alpha (T_c - T)}{2\beta} \).

But \( M \) is real so:

\[ M = \pm \sqrt{\frac{\alpha (T_c - T)}{2\beta}} \]

is an extreme for \( T < T_c \)

look for minima

\[ \frac{\partial^2 F}{\partial M^2} = 2\alpha (T - T_c) + 12\beta M^2. \]

\[ M = 0: \quad \text{min for } T > T_c \quad \text{max for } T < T_c. \]

\[ M = \pm \sqrt{\frac{\alpha (T_c - T)}{2\beta}} \]

\[ \frac{\partial^2 F}{\partial M^2} = 2\alpha (T - T_c) + 12\beta \frac{\alpha (T_c - T)}{2\beta} = -4\alpha (T - T_c) \]

\( \text{min for } T < T_c \quad \text{max for } T > T_c \)

pitchfork bifurcation at \( T = T_c \)
As we go from $T > T_c$ to $T < T_c$ system "falls" into one of the potential walls – which one is determined by fluctuations at $T = T_c$.
(ii) Asymmetric, now $F_3 = \gamma \neq 0$

$$\frac{\partial F}{\partial M} = 2\alpha(T - T_c)M + 3\gamma M^2 + 4\beta M^4$$

extrema now $\frac{\partial F}{\partial M} = 0 = M\left\{2\alpha(T - T_c) + 3\gamma M + 4\beta M^2\right\}$

$$M = 0, \quad M = \frac{-3\gamma \pm \sqrt{(9\gamma^2 - 4.2\alpha(T - T_c)4\beta)}}{2.4\beta}$$

Two real values of $M$ when

$$9\gamma^2 > 32\alpha\beta(T - T_c)$$

write $M$ as

$$M = \frac{-3\gamma \pm \sqrt{\gamma^2 - \gamma_c^2}}{8\beta}.$$  

Consider

$$\frac{\partial^2 F}{\partial M^2} = 2\alpha(T - T_c) + 6\gamma M + 12\beta M^2$$

$$M = 0 \quad \text{is min for} \quad T > T_c.$$  

For $M \neq 0$ extrema given by $2\alpha(T - T_c) + 3\gamma M + 4\beta M^2 = 0$ which gives

$$\frac{\partial^2 F}{\partial M^2} = 3\gamma M + 8\beta M^2,$$

or

$$\frac{\partial^2 F}{\partial M^2} = M\left(\pm3\sqrt{\gamma^2 - \gamma_c^2}\right)$$

Then in addition to $M = 0$ solution

$$\gamma^2 > \gamma_c^2 \quad \text{2 real } M \neq 0 \text{ roots, one max, one min}$$

$$\gamma^2 = \gamma_c^2 \quad M = \frac{-3\gamma}{8\beta} \quad \gamma_c^2 = \frac{32\alpha\beta}{9}(T - T_c) \quad \Rightarrow T > T_c.$$  

$$\gamma^2 < \gamma_c^2 - M \text{ imaginary } \quad \text{no max/min.}$$

Also at $\gamma_c^2 = 0 \quad T = T_c \quad M = \frac{-3\gamma \pm 3\gamma}{8\beta} \quad \text{ie:} \quad M = \frac{-6\gamma}{8\beta}$$

(a)
(b) is \((-\) ve root hence \(\frac{\partial^2 F}{\partial M^2} > 0\) is a min

(a) is inflexion. Finally, for \(\gamma_c^2 < 0\) 2 real roots, both min and \(M = 0\) is max

graphically
Now fluctuations are unimportant.
iii) Van der Vaal

Expand for \( bm << 1 \)

using

\[
\ln(1 - bm) = - \left[ bM + \left( \frac{bM}{2} \right)^2 + \left( \frac{bM}{3} \right)^3 + \ldots \right]
\]

Substitute into \( F \)

\[
F = \frac{T}{b} \left[ -bM - \left( \frac{bM}{2} \right)^2 - \left( \frac{bM}{3} \right)^3 + (bM)^2 + \left( \frac{bM}{2} \right)^3 + \left( \frac{bM}{3} \right)^4 \right] + MT - \frac{aM^2}{2}
\]

\[
= M^2 \left( \frac{bT}{2} - \frac{a}{2} \right) + M^3 \frac{b^2}{6} T + b^3 \frac{T M^4}{12}
\]

then \( \alpha (T - T_c) \equiv \frac{bT - a}{2} = b \left( T - \frac{a}{b} \right) \)

\( T_c = \frac{a}{b} \).
Sheet 1 Question 3

(i) \( \frac{dq}{dt} = \sin q \)

fixed points \( \sin \bar{q} = 0 \quad \bar{q} = n\pi \quad n \text{ integer} \)

linearize about fixed points

\[ q(t) = \bar{q} + \delta q \]

\[ \frac{d\delta q}{dt} = \sin(\bar{q} + \delta q) = \sin \bar{q} \cos \delta q + \cos \bar{q} \sin \delta q = 0 \]

\( \sin \delta q = \delta q, \cos \delta q = 0 \) as \( \delta q \) is small

then \( \frac{d\delta q}{dt} = (-1)^n \delta q \)

solution is of form \( \delta q = \delta q_0 e^{nt} \)

\( s + ve \) for \( n \) even – unstable

\( s - ve \) for \( n \) odd – stable

Phase plane analysis

\[ \begin{array}{c}
\text{flow arrows } +ve \quad \text{for } \frac{dq}{dt} -ve \quad q \text{ increases with time} \\
\text{flow arrows } -ve \quad \text{for } \frac{dq}{dt} -ve \quad q \text{ decreases with time}
\end{array} \]

\[ \text{\square \ stable} \]

\[ \bullet \text{ unstable} \]
ii) \( \frac{dq}{dt} = \alpha q - \beta q^2 \)

fixed points \( \alpha \bar{q} - \beta \bar{q}^2 = 0 \)
\[ \bar{q}(\alpha - \beta \bar{q}) = 0 \]

ie: \( \bar{q} = 0 \) or \( \bar{q} = \frac{\alpha}{\beta} \).

Stability \( q(t) = \bar{q} + \delta q(t) \)

Sub in
\[ \frac{d}{dt}(\delta q) = \alpha(\bar{q} + \delta q) - \beta(\bar{q} + \delta q)^2 \]
\[ = \alpha \bar{q} - \beta \bar{q}^2 + \delta q(\alpha - 2\beta \bar{q}) + 0(\delta q^2) \]

but \( \alpha \bar{q} - \beta \bar{q}^2 = 0 \)

So, \( \frac{d(\delta q)}{dt} = \delta q(\alpha - 2\beta \bar{q}) \),

then, assuming that \( \delta q = \delta q_0 e^{\mu t} \)

we will have \( s + ve \) for \( \alpha - 2\beta \bar{q} > 0 \)
\( s - ve \) for \( \alpha - 2\beta \bar{q} < 0 \).

Take \( \alpha, \beta > 0 \)
then \( \bar{q} = 0 \) is \( s + ve \), ie: unstable (repellor)
\( \bar{q} = \frac{\alpha}{\beta} \) is \( s - ve \), ie: stable (attractor)

Phase plane – sketch \( \frac{dq}{dt}, vz q \)
Problem Sheet 2 – Non Linearity, Chaos and Complexity Solutions

Sheet 2 Question 1.

i) Undamped oscillator

\[ \frac{d^2 x}{dt^2} = -\omega^2 \sin x. \]

Can integrate this once

\[ \frac{d}{dt} \left( \frac{dx}{dt} \right) = -\omega^2 \sin x \frac{dx}{dt} \]

\[ \Rightarrow \frac{1}{2} \left( \frac{dx}{dt} \right)^2 - \omega^2 \cos x = E = \text{constant}. \]

To obtain the dynamics – obtain fixed points, phase plane, etc.

first write as two coupled first order DE

\[ \frac{dx}{dt} = y \quad \frac{dy}{dt} = -\omega^2 \sin x \]

fixed points \( \bar{y} = 0, \sin \bar{x} = 0 \) or \( \bar{x} = n\pi \).

Stability

Linearize

\[ y = \bar{y} + \delta y \quad x = \bar{x} + \delta x \]

then

\[ \frac{d \delta x}{dt} = \delta y \quad \frac{d \delta y}{dt} = -\omega^2 \sin (\bar{x} + \delta x) \]

\[ = -\omega^2 \sin (n\pi + \delta x) \]

use

\[ \sin (A + B) = \sin A \cos B + \cos A \sin B \]

\[ \sin (n\pi + \delta x) = \sin n\pi \cos \delta x + \cos n\pi \sin \delta x \]

\[ = 0 \]

\[ \cos (n\pi) = (-1)^n \quad \text{and} \quad \sin \delta x = \delta x \quad \text{since} \ \delta x \ \text{small} \]

so

\[ \frac{d \delta x}{dt} = \delta y \quad \frac{d \delta y}{dt} = -\omega^2 (-1)^n \delta x. \]

Sufficiently simple to go direct to second order DE
\[
\frac{d^2\delta x}{dt^2} = -\omega^2 (1)^N \delta x \quad \text{for which we know solutions of form } \delta x = Ae^{i\lambda t} + Be^{-i\lambda t}.
\]

Then \( n \) even
\[
\frac{d^2\delta x}{dt^2} = -\omega^2 \delta x \quad \delta x = Ae^{i\omega t} + Be^{-i\omega t},
\]

\( n \) odd
\[
\frac{d^2\delta x}{dt^2} = +\omega^2 \delta x \quad \delta x = Ae^{\omega t} + Be^{-\omega t}.
\]

So, \( n \) even are centre fixed points
\[
\delta x \quad \text{is oscillatory and } \quad \delta y = \frac{d\delta x}{dt} = i\omega Ae^{i\omega t} - i\omega Be^{-i\omega t}
\]

recall \( i = e^{\frac{\pi}{2}} \) and \( -i = e^{-\frac{\pi}{2}} \) (complex numbers \( x + iy = re^{i\theta} \))

So,
\[
\delta y = \omega Ae^{(i\omega t + \frac{\pi}{2})} + \omega Be^{-(i\omega t + \frac{\pi}{2})}
\]

- out of phase \( \frac{\pi}{2} \) with \( \delta x \)

\( n \) odd
\[
\delta x = Ae^{\omega t} + Be^{-\omega t} \quad \delta y = \omega Ae^{\omega t} - \omega Be^{-\omega t}
\]

Saddle point

Separatrix has lines given by
\[
t \to \infty \quad \frac{\delta y}{\delta x} = \frac{\omega Ae^{\omega t}}{Ae^{\omega t}} = \omega
\]
\[
t \to -\infty \quad \frac{\delta y}{\delta x} = \frac{-\omega Be^{-\omega t}}{Be^{-\omega t}} = -\omega.
\]

Topology: constant of the motion defines the phase plane orbits: and
\[
E = \frac{y^2}{2} - \omega^2 \cos x \text{ has symmetry in } y \text{ and } x
\]

Phase plane: see lecture notes and handouts for sketch.

Separatrix has \( x = \pm \pi \to \cos x = -1 \) when \( y = 0 \), \( E_c = \omega^2 \) on the separatrix.
ii) **Damped oscillator**

\[
\frac{d^2 x}{dt^2} + \lambda \frac{dx}{dt} + \omega^2 \sin x = 0
\]

Now we will have first order DE:

\[
\frac{dx}{dt} = y
\]

\[
\frac{dy}{dt} = -\omega^2 \sin x - \lambda y.
\]

Fixed point \( y = 0, \omega^2 \sin x = 0 \),

ie: as undamped case \( y = 0, x = n\pi \).

**Stability analysis**

\( y = \delta y \quad x = \bar{x} + \delta x \)

So \( \frac{d\delta x}{dt} = \delta y \quad \frac{d\delta y}{dt} = -\delta y - \omega^2 (-1)^n \delta x \) (as before – same identities).

Now more complicated – solve using general formula as in lectures (given in detail here).

We write \( \delta x = \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \)

then pair of equations are just

\[
\frac{d\delta x}{dt} = \mathbf{J} \cdot \delta x \quad \mathbf{J} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

where we use notation

\[
\frac{d\delta x}{dt} = a \delta x + b \delta y \
\frac{d\delta y}{dt} = c \delta x + d \delta y
\]

We then have solutions of the form

\[
\delta x = C_1 e^{\delta t}\mathbf{u}_+ + C_2 e^{\delta t}\mathbf{u}_-\]
where the eigenvalues \( s_{\pm} \) are solutions of

\[
\begin{vmatrix}
  a-s & b \\
  c & d-s
\end{vmatrix} = 0
\]

ie:

\[
0 - (a-s)(d-s) - bc = s^2 - s(a+d) + ad - bc
\]

thus

\[
s = \frac{1}{2} \left( (a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)} \right)
\]

here, this is

\[
s_{\pm} = \frac{1}{2} \left( -\lambda \pm \sqrt{\lambda^2 - 4(\omega^2 (-1)^n)} \right)
\]

Two cases:

\( n \) odd

\[
s_{\pm} = \frac{1}{2} \left( -\lambda \pm \sqrt{\lambda^2 + 4\omega^2} \right)
\]

\( n \) even

\[
s_{\pm} = \frac{1}{2} \left( -\lambda \pm \sqrt{\lambda^2 - 4\omega^2} \right)
\]

**n odd:**

\( s_{\pm} \) are real, distinct.

\[
s_{\pm} = \frac{1}{2} \left( -\lambda \pm \sqrt{\frac{4\omega^2}{\lambda^2}} \right)
\]

for \( \lambda \) +ve or -ve

\( s_{\pm} \) are real and of opposite sign – saddle points (as before).

**n even:**

\( s_{\pm} \) may be complex

\[
s_{\pm} = \frac{1}{2} \left( -\lambda \pm \sqrt{\frac{4\omega^2}{\lambda^2}} \right)
\]

complex if \( 4\omega^2 > \lambda^2 \) otherwise real.

For \( \lambda > 0 \) – decay to stable fixed point
\( \lambda < 0 \) – growth – unstable fixed point

If \( 4\omega^2 > \lambda^2 \) these are spiral.

Note that if \( \lambda = 0 \) we have

\[
s_{\pm} = \pm \omega \quad n \text{ odd} - \text{saddle and}
\]

\[
s_{\pm} = \pm i\omega \quad n \text{ even} - \text{circle fixed points}
\]

So, essentially here, circle points \( \rightarrow \) spiral fixed points for \( 4\omega^2 > \lambda^2 \).
**Topology**

Look for symmetries in original DE.

\[
\frac{d^2 x}{dt^2} + \lambda \frac{dx}{dt} + \omega^2 \sin x = 0
\]

\[
x \to -x \quad \frac{d^2 x}{dt^2} + (-1) \lambda \frac{dx}{dt} + \omega^2 \sin x(-1) = 0
\]

Same equation \( x > -x \) is this symmetry by reflection? Check what happens to \( y \) (below).

\[
t \to -t \quad (-1)^2 \frac{d^2 x}{dt^2} + (-1) \lambda \frac{dx}{dt} + \omega^2 \sin x = 0
\]

\( t \to -t \) is \( \lambda \to -\lambda \),

ie: damping and increasing \( t \equiv \) growth and decreasing \( t \)

Sufficient to sketch one of these and note that

\[
y = \frac{dx}{dt} \quad \text{so } x \to -x \text{ gives } y \to -y \text{ rotational symmetry.}
\]

See course handout for sketch
Lotka-Volterra

In our original notation

\[
\frac{dx}{dt} = (\lambda - \alpha y) x \\
\frac{dy}{dt} = -(\eta - \beta x) y
\]

Fixed points

\[
(\lambda - \alpha \bar{y}) \bar{x} = 0 \quad \bar{x} = 0 \text{ or } \bar{y} = \frac{\lambda}{\alpha}
\]

\[-(\eta - \beta \bar{x}) \bar{y} = 0 \quad \bar{y} = 0, \text{ or } \bar{x} = \frac{\eta}{\beta}
\]

ie: \( \bar{x} = 0, \bar{y} = 0 \quad \bar{x} = \frac{\eta}{\beta}, \quad \bar{y} = \frac{\lambda}{\alpha} \).

Stability – linearise

\[
x = \bar{x} + \delta x \quad y = \bar{y} + \delta y
\]

\[
\frac{d\delta x}{dt} = \lambda (\bar{x} + \delta x) - \alpha (\bar{y} + \delta y)(\bar{x} + \delta x)
\]

\[
= \lambda \bar{x} - \alpha \bar{y} \bar{x} + (\lambda - \alpha \bar{y}) \delta x - \alpha \bar{x} \delta y - \alpha \delta x \delta y
\]

\[
= 0
\]

\[
\frac{d\delta x}{dt} = (\lambda - \alpha \bar{y}) \delta x - \alpha \bar{x} \delta y
\]

\[
\frac{d\delta y}{dt} = -\eta (\bar{y} + \delta y) + \beta (\bar{x} + \delta x)(\bar{y} + \delta y)
\]

\[
= -\eta \bar{y} + \beta \bar{x} \bar{y} + \delta y(-\eta + \beta \bar{x}) + \delta x(\beta \bar{y}) + \beta \delta x \delta y
\]

\[
= 0
\]

\[
\frac{d\delta y}{dt} = (-\eta + \beta \bar{x}) \delta y + \beta \bar{y} \delta x
\]

again – can use formula but shown in full here: write in the form \( \frac{d}{dt} \delta x = J \cdot \delta x \)

then in notation of notes

\[
J = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (\lambda - \alpha \bar{y}) & -\alpha \bar{x} \\ \beta \bar{y} & (\beta \bar{x} - \eta) \end{bmatrix}
\]
with eigenvalues

\[ s_{\pm} = \frac{1}{2} \left\{ (a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)} \right\} \]

Consider two fixed points

\[ \bar{x} = 0, \quad \bar{y} = 0 \quad \text{J} = \begin{bmatrix} \lambda & 0 \\ 0 & -\eta \end{bmatrix} \]

\[ s_{\pm} = \frac{1}{2} \left\{ (\lambda - \eta) \pm \sqrt{(\lambda - \eta)^2 + 4(\lambda \eta)} \right\} \]

\[ \lambda^2 - 2\lambda \eta + \eta^2 + 4\lambda \eta = (\lambda + \eta)^2 \]

\[ s_{\pm} = \frac{1}{2} \left\{ (\lambda - \eta) \pm (\lambda + \eta) \right\} \]

ie: \[ s_+ = \lambda \quad s_- = -\eta \quad \text{saddle point.} \]

Consider fixed point

\[ \bar{x} = \frac{\eta}{\beta}, \quad \bar{y} = \frac{\lambda}{\alpha} \]

\[ \text{J} = \begin{bmatrix} 0 & -\alpha \eta \\ \beta & 0 \\ \frac{\beta \lambda}{\alpha} \end{bmatrix} \]

\[ S_{\pm} = \frac{1}{2} \left\{ \pm \sqrt{0 - 4 \left( \frac{\beta \lambda}{\alpha} \right) \left( \frac{\alpha \eta}{\beta} \right)} \right\} \]

\[ = \pm \sqrt{-\lambda \eta} \]

ie: wholly imaginary – centre fixed point.

Topology: no \( t \) symmetry since

\[ t \rightarrow -t \quad -\frac{dx}{dt} = (\lambda - \alpha y) x \]

\[ -\frac{dy}{dt} = - (\eta - \beta x) y \]
Similarly, no symmetries in $x - y$ except change of sign in $\lambda, \eta, \beta, \alpha$ – unrealistic.

Phase plane:

\[
C = (\eta \ln R - \beta R) - (\alpha F - \lambda \ln F)
\]

\[
\frac{dC}{dt} = \frac{\eta}{R} \frac{dR}{dt} - \beta \frac{dR}{dt} - \alpha \frac{dF}{dt} + \lambda \frac{1}{F} \frac{dF}{dt}
\]

\[
= (\lambda - \alpha F)(\eta - \beta R) - (\lambda - \alpha F)(\eta - \beta R)
\]

\[
= 0.
\]

Hence $C$ is a constant and different values of $C$ specify trajectories (closed) about the centre fixed point.
Sheet 2 Question 3

Proof of existence of a limit cycle:

given \( \frac{dx}{dt} = x - y - x(x^2 + 2y^2), \frac{dy}{dt} = x + y(x^2 + y^2) \)

convert to plane polar coordinates \( r, \theta \) use

\[
x = r \cos \theta \quad y = r \sin \theta
\]

and

\[
x \frac{dx}{dt} + y \frac{dy}{dt} = r \frac{dr}{dt} \quad x \frac{dy}{dt} - y \frac{dx}{dt} = r^2 \frac{d\theta}{dt}
\]

then

\[
r^2 \frac{d\theta}{dt} = x \left[ x + y' - y(x^2 + y^2) \right] - y \left[ x' - y(x^2 + 2y^2) \right] = x^2 + y^2 + xy^3 = r^2 + r^4 \cos \theta \sin^3 \theta
\]

\[
r \frac{dr}{dt} = x \left[ x - y' - x(x^2 + 2y^2) \right] + y \left[ -y' + y(x^2 + y^2) \right]
\]

\[
= x^2 + y^2 - x^4 + 3y^2x^2 - y^4
\]

\[
= x^2 + y^2 - (x^2 + y^2)^2 - x^2y^2
\]

\[
= r^3 - r^4 - r^4 \cos^2 \theta \sin^2 \theta.
\]

Identity:

\[
\sin(A + B) = \sin A \cos B + \cos A \sin B
\]

\[
\sin 2A = 2 \sin A \cos A
\]

\[
r^2 \frac{d\theta}{dt} = r^2 + r^4 \frac{1}{2} \sin^2 \theta \sin 2\theta
\]

Giving

\[
r \frac{dr}{dt} = r^3 - r^4 \left( 1 + \frac{1}{4} \sin^2 2\theta \right)
\]

now

\[
r \frac{dr}{dt} = r^2 - r^4 \left( 1 + \frac{1}{4} \sin^2 2\theta \right) = r^2 \left( 1 - r^2 B \right)
\]

Bracket \( B \) is bounded \([1, \frac{5}{4}]\)
Minimum value of $B = 1$ has $\frac{dr}{dt} = 0$ for $r = 1$

Maximum $B = \frac{5}{4}$ has $\frac{dr}{dt} = 0$ for $r = \frac{\sqrt{4}}{\sqrt{5}}$

If $r > 1, \frac{dr}{dt} < 0$

If $r < \frac{\sqrt{4}}{\sqrt{5}}, \frac{dr}{dt} > 0$

orbits are attracted into the annulus for any $\theta$

and $\frac{d\theta}{dt} \neq 0$ in annulus

therefore, limit cycle.
Problem Sheet 3 – Non Linearity, Chaos and Complexity Solutions

Sheet 3 Question 1

Lyapunov exponent.

For a general map \( x_{n+1} = f(x_n) \)

This has iterates \( x_1, x_2, \ldots, x_n \) initial condition \( x_0 \) so \( x_1 = f(x_0), \ x_2 = f(x_1) \), etc.

For initially neighbouring points \( x_0 = x_0 + \epsilon_0, \ x_0 \) with \( \epsilon_0 << 1 \).

After one iterate \( \bar{x}_1 = f(x_0) = f(x_0 + \epsilon_0) = f(x_0) + \epsilon_0 \frac{df}{dx}(x_0) + \ldots \) by Taylor expansion.

Now, two points separated by \( \epsilon_1 \) after one iterate, i.e.

\[ \bar{x}_1 = x_1 + \epsilon_1 = f(x_0 + \epsilon_0) = f(x_0) + \epsilon_0 \frac{df}{dx}(x_0) + \ldots \] so \( \epsilon_1 = \epsilon_0 \ f'(x_0) \) to first order in \( \epsilon_0 \).

Generally, for \( j^{th} \) iterate we have \( \bar{x}_j = x_j + \epsilon_j \) thus \( \epsilon_j = \epsilon_{j-1} \ f'(x_{j-1}) \) provided \( \epsilon_j << 1 \ 0 < j < n \).

Then,

\[ \bar{x}_n = x_n + \epsilon_n = x_n + \epsilon_{n-1} f'(x_{n-1}) \]
\[ = x_n + \epsilon_{n-2} f'(x_{n-2}) f'(x_{n-1}) \]
\[ = x_n + \epsilon_0 f'(x_0) f'(x_1) \ldots f'(x_{n-1}) \]

or

\[ \bar{x}_{n+1} = x_{n+1} \epsilon_0 f'(x_0) \ldots f'(x_n) \]
\[ \bar{x}_n = x_n + \epsilon_0 \prod_{j=0}^{n-1} f'(x_j) \]

Now write

\[ f'(x_j) = e^{\ln[f'(x_j)]} \]

and neglecting signs of \( f' \) we can write

\[ \bar{x}_n = x_n + \epsilon_0 \exp \left[ \sum_{j=0}^{n-1} \ln|f'(x_j)| \right] \]

and hence Lyapunov exponent defined as:

\[ \lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln|f'(x_j)| \]

which is a measure of exponential divergence

\[ \bar{x}_n - x_n = \epsilon_0 e^{\lambda n} \]

If \( \lambda < 0 \) then \( \bar{x}_n \to x_n \) for large \( n \), converging – this is attractor (attractive fixed point).

If \( \lambda > 0 \) – exponential divergence for large \( n \). repellor (repulsive fixed point).
Sheet 3 Question 2

The map

\[ x_{n+1} = \frac{x_n}{a}, \quad 0 < x < a \]

\[ x_{n+1} = \frac{(1-x_n)}{(1-a)}, \quad a < x < 1 \]

where \(0 < a < 1\).

Consider fixed points

\[ \bar{x} = 0 \] and

\[ \bar{x} \] in the range \([a,1]\)

ie: \( \bar{x} = \frac{1-\bar{x}}{(1-a)} \)

\[ \bar{x} - a\bar{x} = 1 - \bar{x} \]

or \((2-a)\bar{x} = 1\)

thus fixed points \(\bar{x} = 0 \quad \bar{x} = \frac{1}{(2-a)}\).

Stability

Linearize

\[ x_n = \bar{x} + \delta x_n \quad x_{n+1} = \bar{x} + \delta x_{n+1}. \]

sub into

\[ x_{n+1} = \frac{(1-x_n)}{(1-a)} \]

\[ \bar{x} + \delta x_{n+1} = \frac{(1-\bar{x} - \delta x_n)}{1-a} \]

\[ \bar{x} + \delta x_{n+1} = \frac{(1-\bar{x})}{(1-a)} - \frac{\delta x_n}{(1-a)} \]

ie: \( \delta x_{n+1} = \frac{-\delta x_n}{1-a} = \frac{\delta x_n}{(a-1)} \)

hence unstable for all \(0 < a < 1\): \( \delta x_{n+1} = \frac{1}{(a-1)} \delta x_n \)
Find "folding points" such that $M^2(x) = 0$ or $M^2(x) = 1$.

$M^2(x) = 0$

Clearly, $M^2(x) = 0$ for $M(x) = 0$ or $1$, i.e.: $M(x) = 0$, $x_R = 0$ or $x_R = a$

$M^2(x) = 1$

Since $M(a) = 1$ we seek $x_R$ such that $M(x_R) = a$.

Two possibilities

$0 < x < a$ \hspace{1cm} $M(x) = \frac{x}{a}, \quad a = \frac{x_R}{a}, \quad x_R = a^2$

$a < x < 1$ \hspace{1cm} $M(x) = \frac{1-x}{1-a}, \quad a = \frac{1-x_R}{1-a}, \quad x_R = 1 - a(1 - a)$

Sketch:

here \( a > \frac{1}{2} \) thus

\( a^2 > \frac{a}{2} \) (try it!).

\( 1 - a(1 - a) < \frac{1 - a}{2} \)

Same topology as symmetric case (stretching and folding) just asymmetric.
Lyapunov exponent for $M(x)$

Fixed point is in the range $[a, 1]$

so $M(x) = \frac{(1-x)}{(1-a)}$

\[
\frac{dM}{dx} = \frac{1}{1-a} \quad \text{and} \quad 0 < a < 1
\]

so $\frac{dM}{dx} > 1$ hence $\lambda = \ln\left(\frac{1}{1-a}\right)$

$\lambda > 0$ exponential divergence

Special cases $a = 0$ and $a = 1$

\[a = 0\]

Now $M(x) = 1 - x$

fixed point $\bar{x} = 1 - \bar{x}$

$\bar{x} = \frac{1}{2}$

gradient $\frac{dM}{dx} = -1$ everywhere.

Lyapunov exponent $\lambda = \ln|-1| = 0$

$\lambda = 0$ is marginally stable –

now $M(\bar{x}) = \bar{x} = \frac{1}{2}$

for any $0 < x < 1, \ x \neq \frac{1}{2}$ write $\bar{x}_0 = \bar{x} + \varepsilon$

\[
M(x_0) = 1 - \bar{x} - \varepsilon = x_1
\]

\[
M^2(x_0) = M(x_1) = 1 - (1 - \bar{x} - \varepsilon) = \bar{x} + \varepsilon = x_0
\]
hence \( M^2(x_0) = x_0 \) these are period two orbits

\[
M(x) = 1
\]

\[
\text{graphically}
\]

or by simply calculating \( M^2(x) = 1 - (1 - x) = x \)

\[
M^2(x) = x
\]

This is a return map \( M^2(x) = x \)

\[
1
\]

\[
x_0
\]

\[
x_0
\]

\[
1
\]

\[
a=1
\]

\[
M(x) = x \quad \text{again, a return map}
\]

\[
\frac{dM}{dx} = \frac{d(x)}{dx} = 1 \quad \text{so Lyapunov exponent} \quad \gamma = \ln|\| = 0 \quad \text{marginally stable}
\]

true for both orbits of \( M(x, a = 1) \) and of \( M^2(x, a = 0) \) [period 2 orbits of \( M \)]
Sheet 3 Question 3

We have

\[ \frac{dg}{dt} = \lambda_s g - eR \quad \frac{dR}{dt} = \lambda_g g - \alpha FR \]

and from Lotka-Volterra equations \( \frac{dF}{dt} = (\eta - \beta R) F \)

fast growing grass \( \lambda_s \gg \lambda_g \)

then we assume the grass is enslaved to the rabbits –

\[ \frac{dg}{dt} = 0 \quad \lambda_s g - eR = 0 \quad g = \frac{eR}{\lambda_s} \]

giving \( \frac{dR}{dt} = \frac{e\lambda_b}{\lambda_s} R - \alpha FR = (\lambda - \alpha F) R \)

where \( \lambda = \frac{e\lambda_b}{\lambda_s} \)

which are the original Lotka-Volterra equations so dynamics of foxes and rabbits are the same and the grass is enslaved to rabbits.
Problem Sheet 4 – Non Linearity, Chaos and Complexity Solutions

Sheet 4 Question 1

(a) $B = 0$ case

$$F(M) = \alpha (T - T_c)M^2 + \beta M^4$$

minima $M = 0, \ M = \pm \sqrt{\frac{\alpha (T_c - T)}{2\beta}}$

Thus, if we normalise $M$ to some $\tilde{M}$  $M^* = \frac{M}{\tilde{M}}$

$$M^* = \pm \sqrt{\frac{\alpha T_c}{2\beta M^2}} \left(1 - \frac{T}{T_c}\right)$$

Two dimensionless groups $\pi_1 = \frac{\alpha T_c}{2\beta M^2}$, $\pi_2 = \frac{T}{T_c}$.

$B = B_0$ case

$$F(M) = \alpha (T - T_c)M^2 + \gamma M^3 + \beta M^4$$

extrema at $M = 0$ and $M = -3\gamma \pm \sqrt{9\gamma^2 - 32\alpha \beta (T - T_c)}$

Normalise $M$ to $\tilde{M}$  $M^* = \frac{M}{\tilde{M}}$

$$M^* = \frac{-3\gamma}{8\beta M} \pm \left[ \frac{9\gamma^2}{(8\beta \tilde{M})^2} - \frac{32\alpha \beta T_c}{(8\beta \tilde{M})^2} \left(1 - \frac{T}{T_c}\right) \right]$$

3 dimensionless groups

$$\pi_1 = \frac{3\gamma}{8\beta M}, \quad \pi_2 = \frac{32\alpha \beta T_c}{(8\beta \tilde{M})^2}, \quad \pi_3 = \frac{T}{T_c}.$$
(b) Microscopic model

<table>
<thead>
<tr>
<th>Quantity</th>
<th>dimension</th>
<th>what it is</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>$[M^c] = [M]^{1/2}/[L]^{1/2}[T]$</td>
<td>Magnetization/spin</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$[L]$</td>
<td>Spin separation</td>
</tr>
<tr>
<td>$L_0$</td>
<td>$[L]$</td>
<td>box size</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>$[T]$</td>
<td>time step</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>$[M^c][T]^{-1}$</td>
<td>average charge in magnetization due to random fluctuations per spin</td>
</tr>
<tr>
<td>$B_0$</td>
<td>$[M^c]$</td>
<td>externally applied field</td>
</tr>
</tbody>
</table>

Since $\text{Tesla} = [M]^{1/2}/[L]^{1/2}[T]$.

In absence of $B_0$  

| $N = 5$ | $R = 3$ | 2 groups |

With applied $B_0$  

| $N = 6$ | $R = 3$ | 3 groups |

These are:

$$\pi_1 = \frac{\varepsilon}{m} \Delta t \quad \pi_2 = \frac{L_0}{\eta} \quad \pi_3 = \frac{B_0}{m},$$

so in absence of applied $B_0$ we have $\pi_1$ and $\pi_2$ only. With applied $B_0$ we have $\pi_3$ as well.

Then we can identify

$$\frac{\varepsilon}{m} \Delta t \equiv \frac{T}{T_c} \quad \frac{\alpha T_c}{2\beta M^2} \equiv \frac{L_0}{\eta} \quad \frac{3\gamma}{8\beta M} \equiv \frac{B_0}{m}.$$
Sheet 4 Question 2

Fireflies

Fly around at random, and each has a "clock" to tell it when to flash

![Clock Diagram]

cycle length $\tau_c$

firefly flashes as $t=12$ say……

all start at random time $\tau_s$

flash duration $\tau_d$

<table>
<thead>
<tr>
<th>Quantity</th>
<th>diversion</th>
<th>what it is</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_c$</td>
<td>$[T]$</td>
<td>cycle length</td>
</tr>
<tr>
<td>$\langle \tau_s \rangle$</td>
<td>$[T]$</td>
<td>average start time</td>
</tr>
<tr>
<td>$\tau_d$</td>
<td>$[T]$</td>
<td>duration</td>
</tr>
<tr>
<td>$R$</td>
<td>$[L]$</td>
<td>interaction radius</td>
</tr>
<tr>
<td>$N_f$</td>
<td>–</td>
<td>No of flashes to reset</td>
</tr>
<tr>
<td>$L_0$</td>
<td>$[L]$</td>
<td>Size of box</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>$[T]$</td>
<td>timestep</td>
</tr>
<tr>
<td>$v$</td>
<td>$[L] [T]^{-1}$</td>
<td>speed</td>
</tr>
<tr>
<td>$N$</td>
<td>–</td>
<td>number of fireflies</td>
</tr>
</tbody>
</table>

$N = 9 \quad R = 2 \quad 7$ parameters
(most are trivial)
There are some 'trivial' and 'non-trivial' parameters here.

### Trivial

1. \( \pi_1 = \frac{R}{L_0} \) if \( \pi_1 > 1 \) fireflies all see each other

2. \( \pi_2 = \frac{v \Delta t}{L_0} \) \( \pi_2 > 1 \) fireflies cross box in one timestep

3. \( \pi_3 = \frac{R}{v \Delta t} \) \( \pi_3 < 1 \) fireflies rush past each other

4. \( \pi_4 = \frac{\tau_d}{\tau_c} \) \( \pi_4 > 1 \) fireflies 'always switched on'

5. \( \pi_5 = \frac{\tau_s}{\tau_c} \) – only relevant if no synchronization – otherwise system 'forgets' initial phase

6. \( \pi_6 = \frac{\tau_c}{\Delta t} \) need \( \pi_6, \pi_7 \gg 1 \) to resolve the dynamics

7. \( \pi_7 = \frac{\tau_d}{\Delta t} \)

Thus, to realise the 'interesting' dynamics on computer we need

\[ \pi_1 \ll 1, \quad \pi_2 \ll 1, \quad \pi_3 \ll 1, \quad \pi_4 \ll 1, \quad \pi_6, \pi_7 \gg 1. \]

In this case these are 'trivial'.

### Non-trivial parameters

For synchronization a firefly must see \( N_f \) flashes within \( R \) – at least 'some of the time'.

Let number of flashes seen with \( R \) be \( \alpha \)

\[ \alpha = \frac{R^2 N \tau_d}{L_0^2 \tau_c} \]

\( \uparrow \quad \uparrow \)

number within \( R \) fraction of these ‘on’

want \( \alpha \geq N_f \) for synchronization.

Thus, non-trivial parameters are \( \pi_1 = \alpha, \pi_2 = N_f \) and for synchronization \( \alpha \geq N_f \).