Quantifying the uncertainty of activation periods in fMRI data via changepoint analysis

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Joint work with John A. D. Aston and Adam M. Johansen

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fMRI data from an Anxiety Induced Experiment

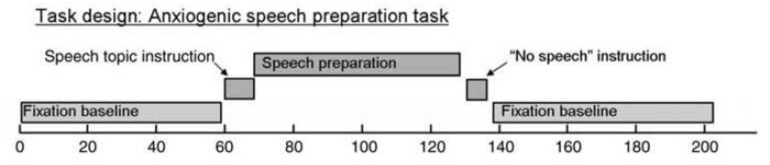


Figure: Design of Anxiety Induced Experiment

- 215 images/time points
- 24 subjects

Image from Lindquist et al. [2007]

fMRI Data

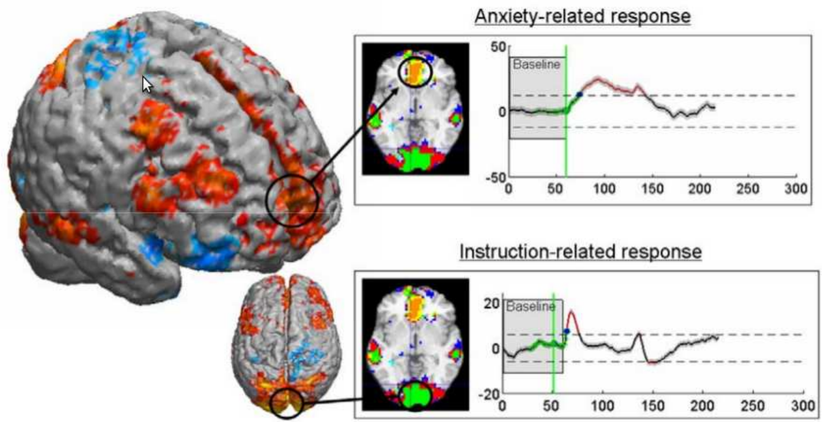
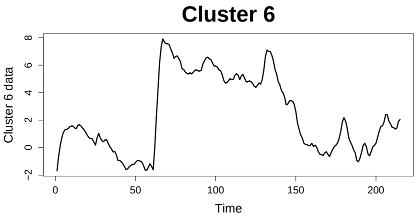
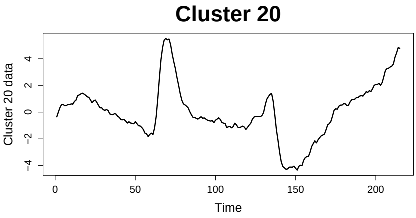


Image from Lindquist et al. [2007]

Brain Regions of interest



(a) RMPFC time series



(b) VC time series

Questions of interest

- The exact design of the experiment.
- When might activations have occurred?
- How many activations may have occurred?
- How long are these activation periods?

Aim

Propose a methodology to fully capture the uncertainty of changepoint characteristics given a sequence of data $y = y_{1:n} = (y_1, \ldots, y_n)$.

- Changepoint Probability (CPP) to a regime occurring at time t.
- Probability of m changepoints occurring in the data.

Focus our attention to these quantities although others are easily accessible.

Hidden Markov Models

- Assume y can be generated by a finite number of states.
- **Hidden Markov Models (HMMs)** to model time series y with $\{X_t\}$ as our underlying Markov Chain, with finite state space Ω_X.

General Finite State Hidden Markov Model

$$\text{Emission:} \quad y_t | y_{1:t-1}, x_{1:t}, \theta \sim f(y_t | x_{t-r:t}, y_{1:t-1}, \theta)$$
$$\text{Transition:} \quad p(x_t | y_{1:t-1}, x_{-r+1:t-1}, \theta) = p(x_t | x_{t-1}, \theta)$$

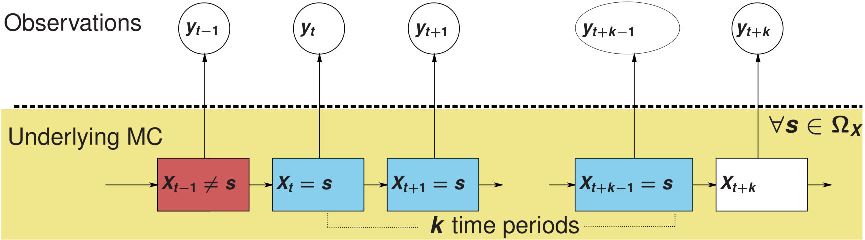
- Changepoints in this HMM framework.

Changepoint Definition

Changepoint Definition

Changepoint to a regime is said to have occurred at time t when a state persists for at least k time periods in $\{X_t\}$.

$$x_{t-1} \neq x_t = x_{t+1} = \ldots = x_{t+j} \quad \text{where } j \geq k - 1$$



Define $\tau^{(k)}$ and $M^{(k)}$ to be the time and number of changepoints.

Model Parameters

- θ indicates our model parameter vector which needs to be estimated.
- Components will vary; dependent on the particular type of HMM used.
  - **P**, the $|\Omega_X| \times |\Omega_X|$ probability transition matrix for $\{X_t\}$
  - Parameters for the emission distribution which can be dependent on the underlying state
    Means μ_{X_t}, variances $\sigma^2_{X_t}$, Poisson intensity rates λ_{X_t}, ...

Suppose we are interested in the changepoint probability $P(\tau^{(k)} = t|y)$.

$$P(\tau^{(k)} = t|y) = \int_{\theta} P(\tau^{(k)} = t, \theta|y)d\theta = \int_{\theta} P(\tau^{(k)} = t|\theta, y)p(\theta|y)d\theta$$

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$$\approx \widehat{P^N}(\tau^{(k)} = t|y) = \sum_{i=1}^{N} W^i P(\tau^{(k)} = t|\theta^i, y)$$

by standard Monte Carlo results.

- $\{W^i, \theta^i\}_{i=1}^N$ approximates $p(\theta|y)$
- $P(\tau^{(k)} = t|\theta^i, y)$

Proposed Methodology

1. Approximate $p(\theta|y)$ via SMC samplers to obtain a normalised weighted cloud of particles $\{W^i, \theta^i\}_{i=1}^N$.

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Combines recent work of Del Moral et al. [2006] and Aston et al. [2009].
Specific details can be found in Nam et al. [in press].

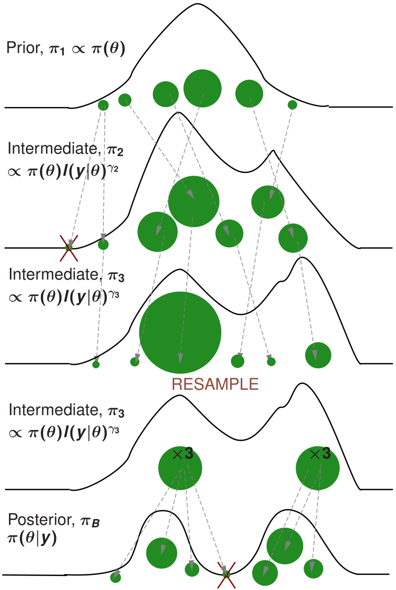
Sequential Monte Carlo (SMC) Samplers

SMC Samplers

$$\pi_b \propto p(\theta) l(y|\theta)^{\gamma_b}$$

where

- $p(\theta)$ = prior of the model parameters
- $l(y|\theta)$ = likelihood
- $0 = \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_B = 1$, a tempering schedule



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- $P(\tau^{(k)} = t|\theta^i, y) = \sum_{u=1,2,\dots} P(\tau_u^{(k)} = t|\theta^i, y)$.
- $P(\tau_u^{(k)} = t|\theta^i, y) \equiv P(W(k, u) = t + k - 1|\theta^i, y)$ re-express as a waiting time for runs.

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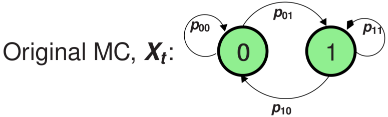
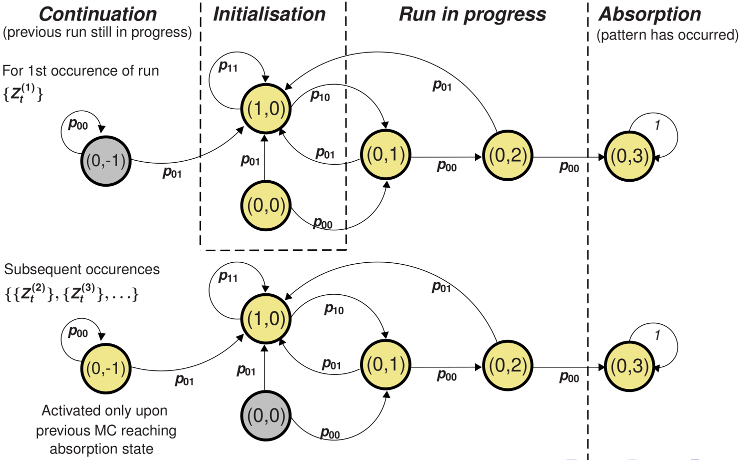
- $P(\tau_u^{(k)} = t|\theta^i, y) \equiv P(W(k,u) = t + k - 1|\theta^i, y)$ re-express as a waiting time for runs.
- Waiting time distribution for runs can be computed exactly via Finite Markov Chain Imbedding (FMCI).

Finite Markov Chain Imbedding (FMCI)



Run of interest: **000** $(s = 0, k = 3)$

Via FMCI: Auxiliary MCs $\{\{Z_t^{(1)}\}, \{Z_t^{(2)}\}, \ldots\}$



Waiting Time Distributions via FMCI

- $P(W(k,u) \leq t+k-1|\theta^i) = P(Z^{(u)}_{t+k-1} \in A|\theta^i)$ where A denotes the set of absorption states in Ω_Z.
- Computed by standard Markov Chain results.

FMCI in a HMM context

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- Results in a sequence of time dependent posterior transition probability matrices $\{\tilde{\mathbf{M}}_1, \tilde{\mathbf{M}}_2, \ldots, \tilde{\mathbf{M}}_n\}$ for the auxiliary MCs $\{Z_t\}$.

Exact Distributions for Changepoint characteristics

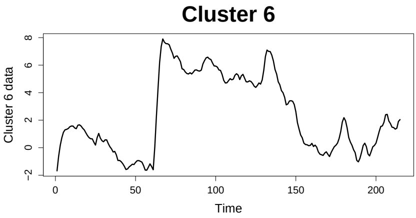
Probability of uth changepoint at specific time

$$P(\tau_u^{(k)} = t|\theta^i, y) = P(W(k,u) = t+k-1|\theta^i, y)$$
$$= P(W(k,u) \leq t+k-1|\theta^i, y) - P(W(k,u) \leq t+k-2|\theta^i, y)$$

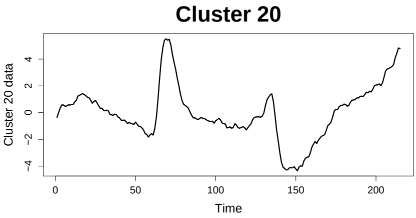
Distribution of the number of changepoints,

$$P(M^{(k)} = u|\theta^i) = P(W(k,u) \leq n|\theta^i, y) - P(W(k,u+1) \leq n|\theta^i, y)$$

fMRI data



(c) RMPFC time series



(d) VC time series

Preliminary I

- AR error process leads to a HMS-AR model.
- Detrending within model.

HMS-AR(r) with detrending

$$\begin{aligned} y_t - \mu_{x_t} - \mathbf{m}_t^{'}\beta &= a_t \\ a_t &= \phi_1 a_{t-1} + \ldots + \phi_r a_{t-r} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2) \end{aligned}$$

- Detrending parameters within model $\rightarrow$ Estimated within SMC samplers.

Preliminary II

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  No detrending, Polynomial of order 3, Discrete Cosine Basis

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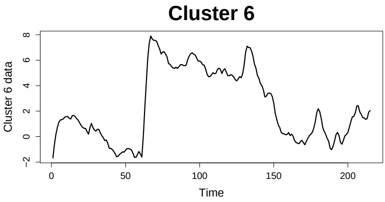
- SMC samplers: $500 = N$ particles, $100 = B$ iterations

Cluster 6, CPP HMS-AR(0)



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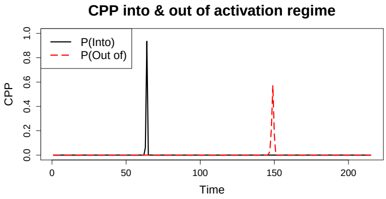
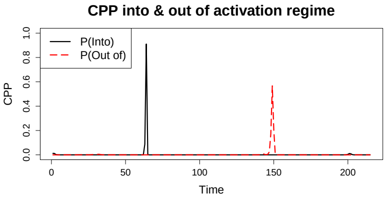
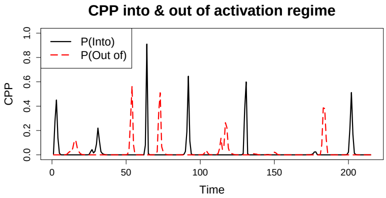


(f) AR(0), No detrending



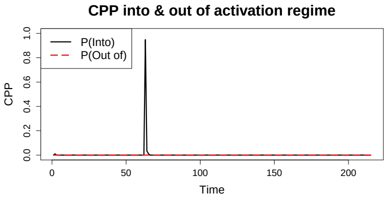
(g) AR(0), Poly detrending



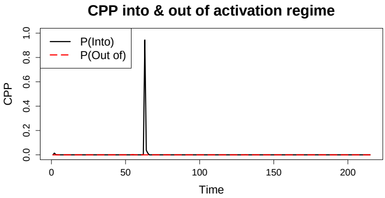
(h) AR(0), DCB detrending

Cluster 6, CPP, HMS-AR(1)

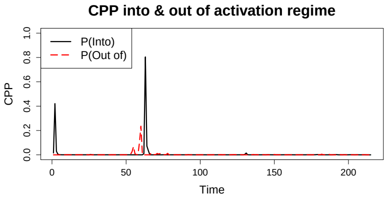
(i) RMPFC time series



(j) AR(1), No detrending

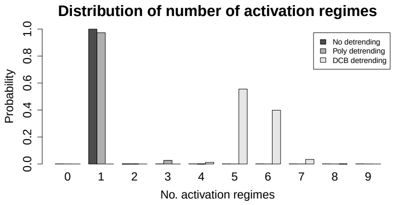


(k) AR(1), Poly detrending

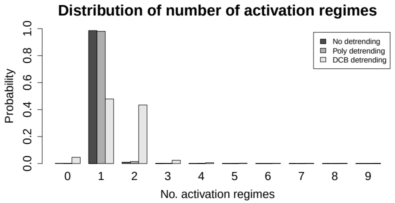


(l) AR(1), DCB detrending

Cluster 6, Distribution of Number Activation Regimes

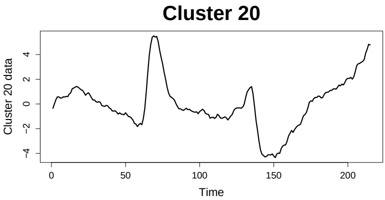


(m) AR(0)

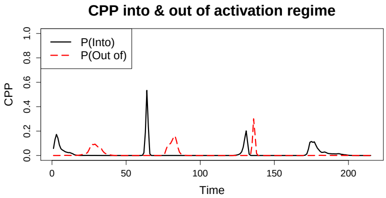


(n) AR(1)

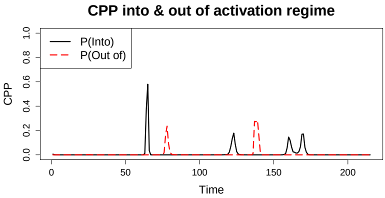
Cluster 20, CPP HMS-AR(0)



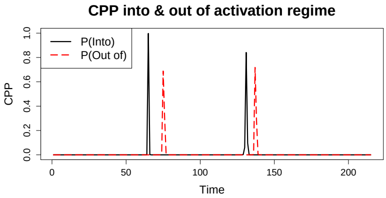
(o) VC time series



(p) AR(0), No detrending

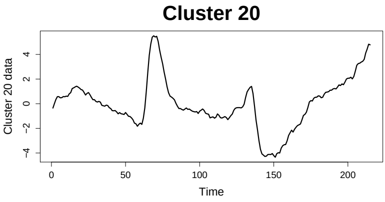


(q) AR(0), Poly detrending

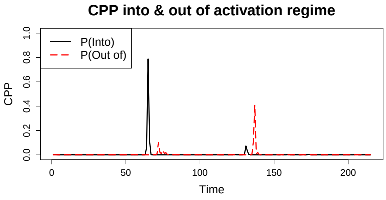


(r) AR(0), DCB detrending

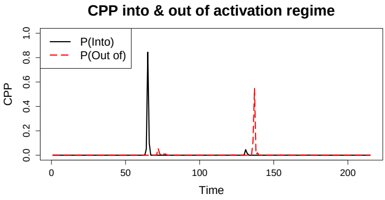
Cluster 20, CPP, HMS-AR(1)



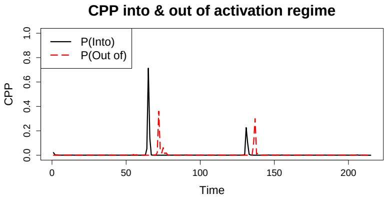
(s) VC time series



(t) AR(1), No detrending

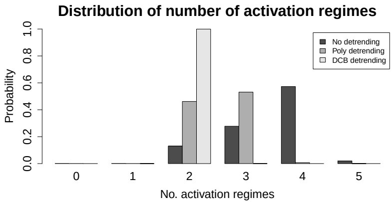


(u) AR(1), Poly detrending

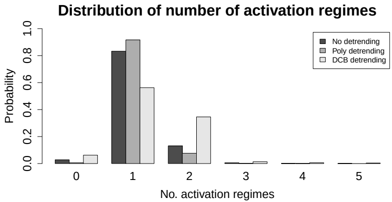


(v) AR(1), DCB detrending

Cluster 20, Distribution of Number Activation Regimes



(w) AR(0)



(x) AR(1)

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- Application to fMRI data.
- Effects of error process assumptions and types of detrending.

References

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Implementation of SMC samplers I

- Consider the model parameters for a 2-state Hamilton's MS-AR(r) model. $\theta = (\mathbf{P}, \mu_1, \mu_2, \sigma^2, \phi_1, \ldots, \phi_r)$
- Simple linear tempering schedule,
$\gamma_b = \frac{b-1}{B-1}, \qquad b = 1, \ldots, B.$

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- Simple linear tempering schedule, $\gamma_b = \frac{b-1}{B-1}, \qquad b = 1, \ldots, B.$
- Reparameterisation of variances to precisions, $\lambda = 1/\sigma^2$ -AR parameters to Partial Autocorrelation Coefficients (PAC).

Implementation of SMC samplers II

- **Initialisation:** Assume independence between the components of θ, and sample from Bayesian priors.

Approximation of posterior

$$p(\theta|y) \approx \{W_B^{(i)}, \theta_B^{(i)}\}_{i=1}^N = \{W^i, \theta^i\}_{i=1}^N$$

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- **Intermediary Output:** Weighted cloud of N particles, $\{W_b^{(i)}, \theta_b^{(i)}\}_{i=1}^{N}$ approximates the distribution π_b.

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Waiting Time Distributions for Runs in HMM context

For $t = 1, \ldots, n$

$$\Psi_t = \Psi_{t-1}\tilde{\mathbf{M}}_t$$
$$\psi_t^{(u)} \leftarrow \psi_t^{(u)} + \psi_{t-1}^{(u-1)}(\tilde{\mathbf{M}}_t - \mathbf{I})\Upsilon, \quad u = 2, \ldots, \lfloor n/k \rfloor$$
$$P(W(k, u|\theta^i, y) \leq t) = P(Z_t^{(u)} \in A) = \psi_t^{(u)} U(A)$$

where :$\psi_t^{(u)} = 1 \times |\Omega_Z|$ vector storing probabilities of the uth chain being in the corresponding state.

$\Psi_t = \lfloor n/k \rfloor \times |\Omega_Z|$ with $(\psi_t^{(1)}, \psi_t^{(2)}, \ldots)$ as row vectors.

$\mathbf{I} = |\Omega_Z| \times |\Omega_Z|$ identity matrix

$\Upsilon = |\Omega_Z| \times |\Omega_Z|$ matrix connecting absorption state to the corresponding continuation state

$U(A) = |\Omega_Z| \times 1$ vector with 1s in location of absorption states 0s elsewhere.