autoSPEC
Adaptive Spectral Estimation for Nonstationary Time Series

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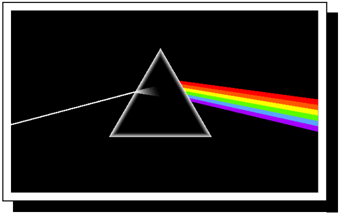

Ori Rosen
UTEP

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FOURIER TRANSFORM AND PERIODOGRAM

Collect stationary time series $\{X_t;\ t = 1, \ldots, n\}$ with interest in cycles.
Rather than work with the data $\{X_t\}$, we transform it into the frequency domain:

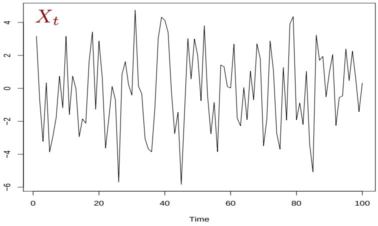
Discrete Fourier Transformation (DFT)

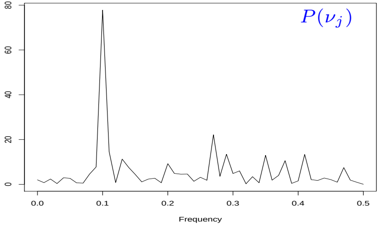
$$X_t \mapsto d_j = n^{-1/2} \sum_{t=1}^{n} X_t\, e^{-2\pi i t\, j/n}$$

Periodogram $(j = 0, 1, \ldots, n-1)$

$$P(\nu_j) = |d_j|^2 = \left[\frac{1}{n}\sum_{t=1}^{n} X_t \cos(2\pi t \tfrac{j}{n})\right]^2 + \left[\frac{1}{n}\sum_{t=1}^{n} X_t \sin(2\pi t \tfrac{j}{n})\right]^2$$

That is, match (correlate) data with [co]sines oscillating at freqs $\nu_j = \frac{j}{n}\frac{cycles}{points}$.





Spectral Density

The periodogram $P(\nu_{j:n}) = \left|n^{-1/2} \sum_{t=1}^{n} X_t \exp(-2\pi i t \nu_{j:n})\right|^2$ is a sample concept. Its population counterpart is the ($\nu_{j:n} = \frac{j_n}{n} \to \nu$)

Spectral Density

$$f(\nu) = \lim_{n\to\infty} \mathbb{E}\{P(\nu_{j:n})\} = \sum_{h=-\infty}^{\infty} \gamma(h) \exp(2\pi i \nu h)$$

provided the limit exits (i.e. $\sum |\gamma(h)| < \infty$ where $\gamma(h) = \text{cov}\{X_{t+h}, X_t\}$). It follows that $f(\nu) \geq 0$, $f(1+\nu) = f(\nu)$, $f(\nu) = f(-\nu)$, and because

$$\gamma(h) = \int_{-1/2}^{1/2} f(\nu) \exp(-2\pi i \nu h)\, d\nu$$

The sample equivalent of the integral equation is:

$$\sum_{j=1}^{n-1} P(j/n)\, n^{-1} = S^2.$$

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$$\int_{-1/2}^{1/2} f(\nu)\, d\nu = \text{var}(X_t) \qquad [= \gamma(0)].$$

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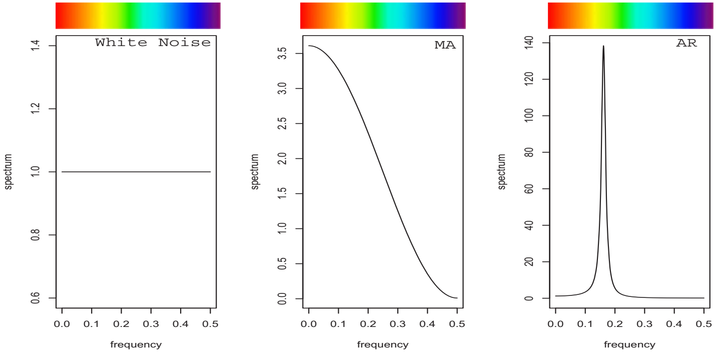
Some Examples

- WN: W_t is white noise if $EW_t = 0$ and $\gamma(h) = \sigma_w^2 \delta_0^h$. The spectral density
$$f(\nu) = \sum \gamma(h) \exp(-2\pi i \nu h) = \sigma_w^2 \qquad -1/2 \le \nu \le 1/2,$$
is uniform (think of white light).
- MA: $X_t = W_t + .9W_{t-1}$
- AR: $X_t = X_{t-1} - .9X_{t-2} + W_t$



SOME ASYMPTOTIC RESULTS

$$d(\nu_{j:n}) = n^{-1/2} \sum_{t=1}^{n} X_t \overbrace{\exp(-2\pi i t \nu_{j:n})}^{\cos(2\pi t \nu_{j:n}) - i \sin(2\pi t \nu_{j:n})}$$
$$= d_c(\nu_{j:n}) - i\, d_s(\nu_{j:n})$$

Under general conditions on $\{X_t\}$ $(n \to \infty,\ \ \nu_{j:n} \to \nu)$:

- $d_c(\nu_{j:n}) \sim \text{AN}\big(0, \frac{1}{2} f(\nu)\big)$
- $d_s(\nu_{j:n}) \sim \text{AN}\big(0, \frac{1}{2} f(\nu)\big)$
- $d_{\substack{c\\s}}(\nu_{j:n}) \perp d_{\substack{c\\s}}(\nu_{k:n}) \quad \forall\, j, k$ ($\nu_{k:n} \to \nu' \neq \nu$ and terms not the same)

$$P_n(\nu_{j:n}) = d_c^2(\nu_{j:n}) + d_s^2(\nu_{j:n}), \quad \text{thus} \quad 2P_n(\nu_{j:n})/f(\nu) \Rightarrow \chi_2^2,$$

so

$$\mathbb{E}[P_n(\nu_{j:n})] \to f(\nu), \ \text{but var}[P_n(\nu_{j:n})] \to f^2(\nu) \longleftrightarrow \text{BAD}$$

One remedy? Kernel smooth for consistency:

$$\widehat{f}(\nu) = \int_{-1/2}^{1/2} P_n(\lambda) K_n(\nu - \lambda) d\lambda$$

Some Asymptotic Results

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Whittle Likelihood

Given time series data $\boldsymbol{x} = (X_1, \dots, X_n)$, for large n,

$$\mathcal{L}(f \mid \boldsymbol{x}) \approx (2\pi)^{-n/2} \prod_{k=0}^{n-1} \exp\left\{ -\frac{1}{2} \left[\log f(\nu_k) + \frac{P_n(\nu_k)}{f(\nu_k)} \right] \right\} ,$$

$\nu_k = k/n$, and $k = 0, \dots, [n/2]$.

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Stationary Case

Estimation of Spectra via Smoothing Splines

In the stationary case, let $P_n(\nu_k)$ denote the periodogram. For large n, approximately [recall $2P_n(\nu_{k:n})/f(\nu) \Rightarrow \chi^2_2$]

$$P_n(\nu_k) = f(\nu_k)U_k$$

where $f(\nu_k)$ is the spectrum and $U_k \overset{iid}{\sim}$ Gamma$(1,1)$.

Taking logs, we have a GLM

$$y(\nu_k) = g(\nu_k) + \epsilon_k$$

where $y(\nu_k) = \log P_n(\nu_k)$, $g(\nu_k) = \log f(\nu_k)$ and ϵ_k are iid $\log(\chi^2_2/2)$s.

Want to fit the model with the constraint that $g()$ is smooth.
Wahba (1980) suggested smoothing splines. This can be done in a Bayesian framework.

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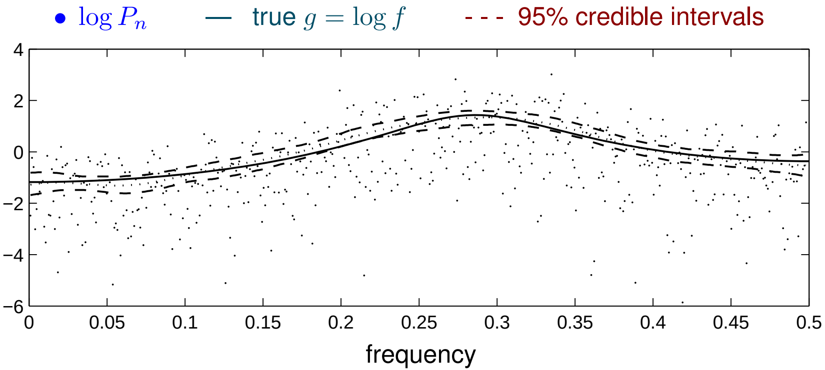
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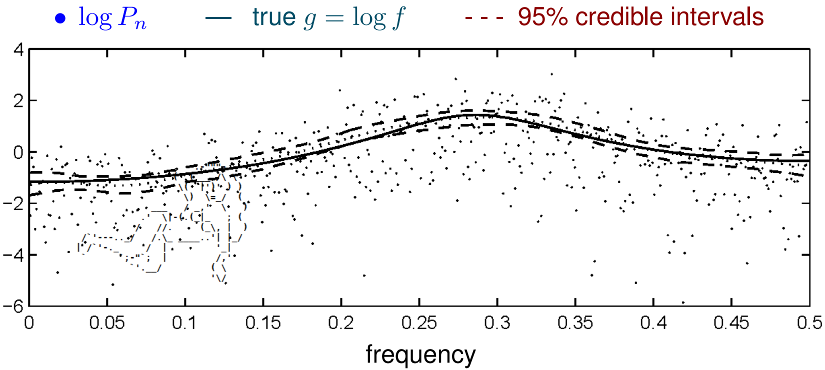
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How It's Done: Prior Distributions

Place a linear smoothing spline prior on $\log f(\nu)$. Let $y_n(\nu_k) = \log P_n(\nu_k)$ and $g(\nu_k) = \log f(\nu_k)$.

- $g(\nu_k) = \alpha_0 + \alpha_1 \nu_k + h(\nu_k)$ linear [α] + nonlinear [$h()$]
- $\alpha_0 \sim N(0, \sigma_\alpha^2)$, $\alpha_1 \equiv 0$, since $(\partial g(\nu)/\partial \nu)|_{\nu=0} = 0$.
- $\boldsymbol{h} = D\boldsymbol{\beta}$, is a linear combination of basis functions where $\boldsymbol{h} = (h(\nu_0), \dots, h(\nu_{n/2}))'$, and the jth column of D is $\sqrt{2}\cos(j\pi\boldsymbol{\nu})$, $\boldsymbol{\nu} = (\nu_0, \dots, \nu_{n/2})'$.
- $\boldsymbol{\beta} \sim N(0, \tau^2 I)$
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Wood, Jiang, Tanner (2002). Bayesian mixture of splines for ... nonparametric regression. *Biometrika*

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SAMPLING SCHEME ~ METROPOLIS-HASTINGS (M-H)

The parameters α_0, $\boldsymbol{\beta}$ and τ^2 are drawn from the posterior distribution $p(\alpha_0, \boldsymbol{\beta}, \tau^2 \mid \boldsymbol{y})$, where $\boldsymbol{y} = (y_n(\nu_0), \ldots, y_n(\nu_{n/2}))'$, using MCMC:

- α_0 and $\boldsymbol{\beta}$ are sampled jointly via an M-H step from

$$p(\alpha_0, \boldsymbol{\beta} \mid \tau^2, \boldsymbol{y}) \propto \exp\Big\{-\frac{1}{2}\sum_{k=0}^{n-1}\Big[\alpha_0 + \boldsymbol{d}_k'\boldsymbol{\beta} + \exp\big(y_n(\nu_k) - \alpha_0 - \boldsymbol{d}_k'\boldsymbol{\beta}\big)\Big] - \frac{\alpha_0^2}{2\sigma_\alpha^2} - \frac{1}{2\tau^2}\boldsymbol{\beta}'\boldsymbol{\beta}\Big\},$$

  where $\boldsymbol{d}_k'$ is the kth row of D.

- τ^2 is sampled from the truncated inverse gamma distribution,

$$p(\tau^2 \mid \boldsymbol{\beta}) \propto (\tau^2)^{-J/2} \exp\Big(-\frac{1}{2\tau^2}\boldsymbol{\beta}'\boldsymbol{\beta}\Big), \ \tau^2 \in (0, c_{\tau^2}],$$

  where J = number of frequencies.

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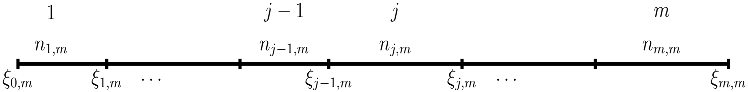
  where $\boldsymbol{d}_k'$ is the kth row of D.

- τ^2 is sampled from the truncated inverse gamma distribution,

$$p(\tau^2 \mid \boldsymbol{\beta}) \propto (\tau^2)^{-J/2} \exp\Big(-\frac{1}{2\tau^2}\boldsymbol{\beta}'\boldsymbol{\beta}\Big), \ \tau^2 \in (0, c_{\tau^2}],$$

  where $J =$ number of frequencies.

PIECEWISE STATIONARY



Suppose $\boldsymbol{x} = \{X_1, \ldots, X_n\}$ is a time series with an unknown number of stationary segments.

m: unknown number of segments ($m = 1$ means stationary)

$n_{j,m}$: number of observations in the jth segment, $n_{j,m} \geq t_{\min}$.

$\xi_{j,m}$: location of the end of the jth segment, $j = 0, \ldots, m$, $\xi_{0,m} \equiv 0$ and $\xi_{m,m} \equiv n$.

$f_{j,m}$: spectral densities

$P_{n_{j,m}}$: periodograms at $\nu_{k_j} = k_j / n_{j,m}$, $0 \leq k_j \leq n_{j,m} - 1$.

Whittle Likelihood

$$\mathcal{L}(f_{1,m},\ldots,f_{m,m} \mid \boldsymbol{x} \text{ [data]}, \boldsymbol{\xi}_m \text{ [partition]}) \approx \prod_{j=1}^{m} (2\pi)^{-n_{j,m}/2} \prod_{k_j=0}^{n_{j,m}-1} \exp\left\{-\frac{1}{2}\left[\log f_{j,m}(\nu_{k_j}) + \frac{P_{n_{j,m}}(\nu_{k_j})}{f_{j,m}(\nu_{k_j})}\right]\right\}$$

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Prior Distributions

- Priors on $g_{j,m}(\nu) = \log f_{j,m}(\nu)$, $j = 1, \ldots, m$, as before.
- $\Pr(\xi_{j,m} = t \mid m) = 1/p_{jm}$, for $j = 1, \ldots, m-1$, where $p_{jm} = n - \xi_{j-1,m} - (m - j + 1)t_{\min} + 1$ is the number of available locations for split point $\xi_{j,m}$.
- The prior on the number of segments

$$\Pr(m = k) = 1/M, \quad \text{for} \quad k = 1, \ldots, M.$$

SAMPLING SCHEME

LIFE GOES ON WITHIN MOVES AND WITHOUT MOVES

Within-model moves: (location of end points)

- Given m, $\xi_{k^*,m}$ is proposed to be relocated.
- The corresponding and βs are updated (absorb α_0s into βs).
- These two steps are jointly accepted or rejected in a M-H step.
- The τ^2s are then updated in a Gibbs step.

Between-model moves: (number of segments)

- $m^p = m^c + 1^\dagger$
  - Select a segment to split
  - Select a new split point in this segment.
  - Two new τ^2s are formed from the current τ^2
  - Two new βs are drawn.
- $m^p = m^c - 1$
  - Select a split point to be removed.
  - A single τ^2 is then formed from the current τ^2s
  - A new β is proposed.

Accept or Reject in a M-H step.

† c=current, p=proposed

Between-Model Moves: $m^c \to m^p$

Let $\boldsymbol{\theta}_m = \{\boldsymbol{\xi}_m, \boldsymbol{\tau}^2_m, \boldsymbol{\beta}_m\}$ and suppose the chain is currently at $(m^c, \boldsymbol{\theta}^c_{m^c})$. We propose to move to $(m^p, \boldsymbol{\theta}^p_{m^p})$ by drawing $(m^p, \boldsymbol{\theta}^p_{m^p})$ from a proposal density $q(m^p, \boldsymbol{\theta}^p_{m^p} \mid m^c, \boldsymbol{\theta}^c_{m^c})$ and accepting this draw with probability

$$\alpha = \min\left\{1, \frac{p(m^p, \boldsymbol{\theta}^p_{m^p}|\boldsymbol{x}) \times q(m^c, \boldsymbol{\theta}^c_{m^c} \mid m^p, \boldsymbol{\theta}^p_{m^p})}{p(m^c, \boldsymbol{\theta}^c_{m^c}|\boldsymbol{x}) \times q(m^p, \boldsymbol{\theta}^p_{m^p} \mid m^c, \boldsymbol{\theta}^c_{m^c})}\right\},$$

where $p(\cdot)$ is the approximate likelihood. The M-H transition kernel is composed of the $q(m^p|m^c) \times \alpha$. These are essentially a likelihood ratios. Thus the decision of whether or not to change m via the posterior is essentially based on the likelihood ratio.

Within-model moves, relocation of end points, is similar.

... and model averaging!
... and all data used for estimation, not just segmented data!

Between-Model Moves: $m^c \rightarrow m^p$

Let $\boldsymbol{\theta}_m = \{\boldsymbol{\xi}_m, \boldsymbol{\tau}_m^2, \boldsymbol{\beta}_m\}$ and suppose the chain is currently at $(m^c, \boldsymbol{\theta}_{m^c}^c)$. We propose to move to $(m^p, \boldsymbol{\theta}_{m^p}^p)$ by drawing $(m^p, \boldsymbol{\theta}_{m^p}^p)$ from a proposal density $q(m^p, \boldsymbol{\theta}_{m^p}^p \mid m^c, \boldsymbol{\theta}_{m^c}^c)$ and accepting this draw with probability

$$\alpha = \min\left\{1, \frac{p(m^p, \boldsymbol{\theta}_{m^p}^p | \boldsymbol{x}) \times q(m^c, \boldsymbol{\theta}_{m^c}^c \mid m^p, \boldsymbol{\theta}_{m^p}^p)}{p(m^c, \boldsymbol{\theta}_{m^c}^c | \boldsymbol{x}) \times q(m^p, \boldsymbol{\theta}_{m^p}^p \mid m^c, \boldsymbol{\theta}_{m^c}^c)}\right\},$$

where $p(\cdot)$ is the approximate likelihood. The M-H transition kernel is composed of the $q(m^p | m^c) \times \alpha$. These are essentially a likelihood ratios. Thus the decision of whether or not to change m via the posterior is essentially based on the likelihood ratio.

Within-model moves, relocation of end points, is similar.

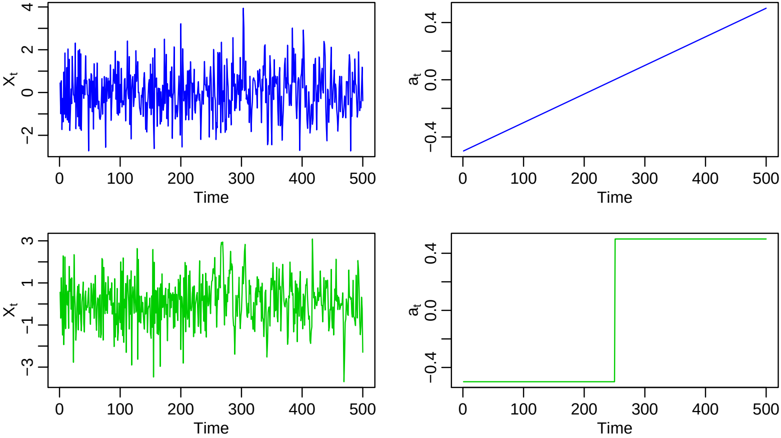
... and model averaging!
... and all data used for estimation, not just segmented data!

Example

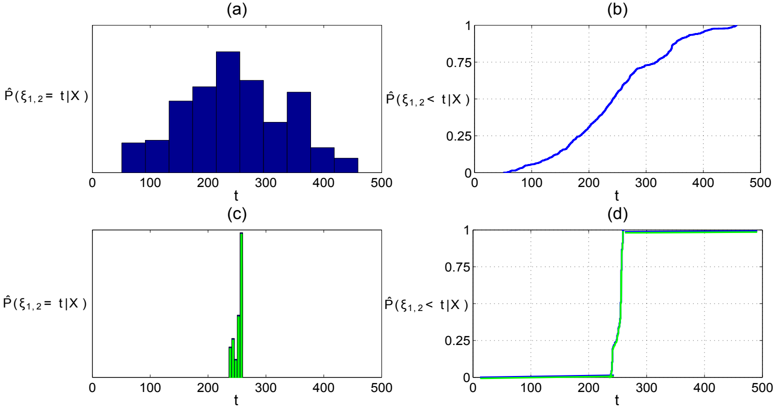
Consider two tvAR(1) models $X_t = a_t X_{t-1} + W_t$ for $t = 1, \ldots, 500$

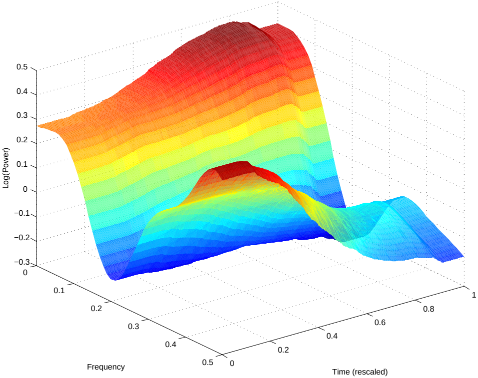
(blue) $a_t = t/500 - .5$ There is no optimal segmentation in this case.

(green) $a_t = .5\,\text{sign}(t - 250)$



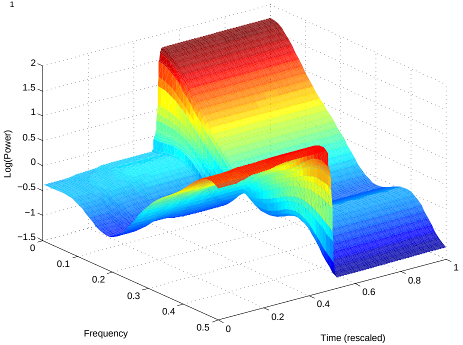
In each case, $m = 2$ is the modal value [posteriors in paper] on the number of partitions. Plotted below are $\Pr(\xi_{1,2} = t \mid data)$ and $\Pr(\xi_{1,2} < t \mid data)$, where $\xi_{1,2}$ is the change point when $m = 2$.



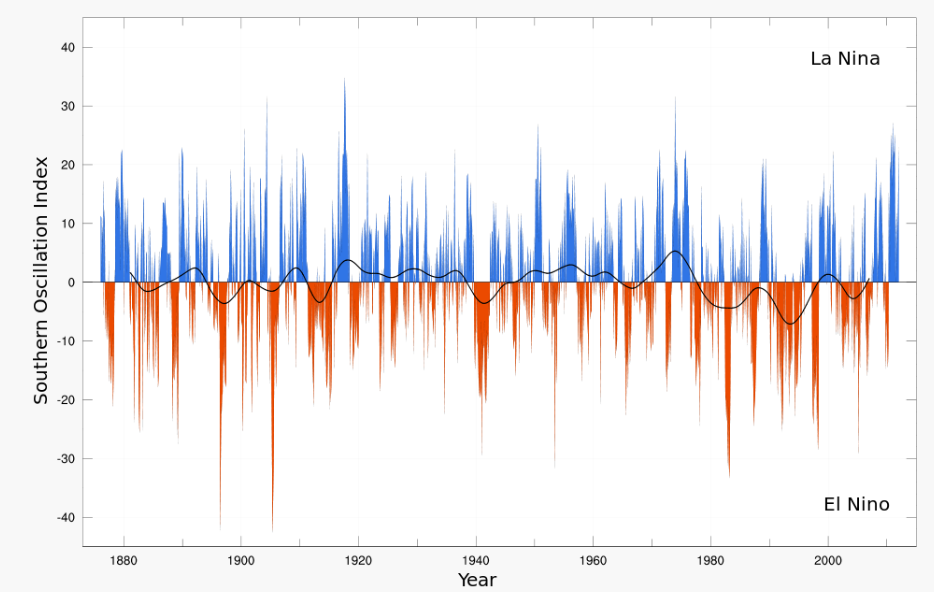


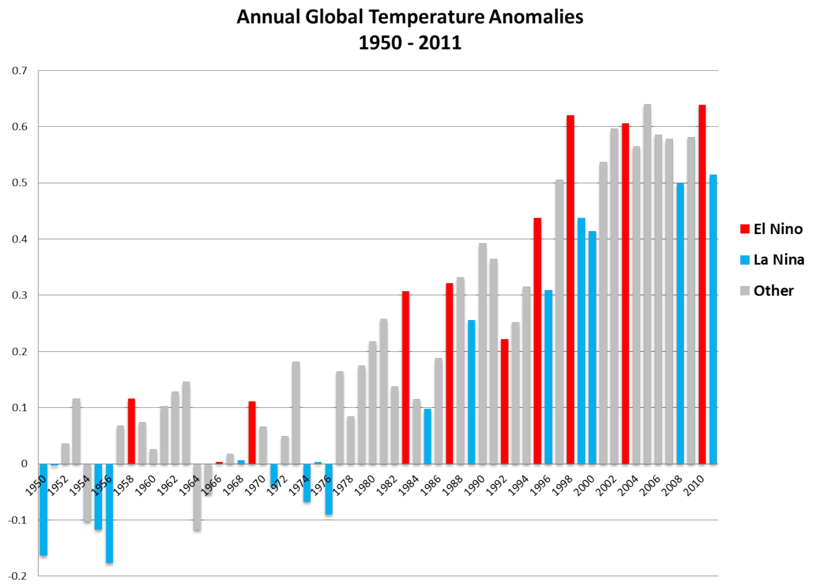
← time-varying

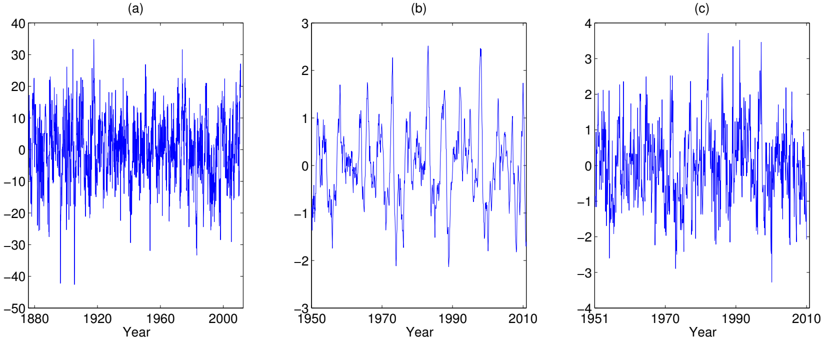
change-point →



El Niño – Southern Oscillation




Annual Global Temperature Anomalies
1950 - 2011
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
1950
1952
1954
1956
1958
1960
1962
1964
1966
1968
1970
1972
1974
1976
1978
1980
1982
1984
1986
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
2010
El Nino
La Nina
Other



Plots of (a) SOI from 1876–2011; (b) Niño3.4 index from 1950-2011; (c) DSLPA from 1951–2010.

The Posteriors – $\Pr(m = k \mid \boldsymbol{x})$

k	SOI	Niño3.4	DSLPA
1	0.95	0.93	0.99
2	0.05	0.07	0.01
3	0.00	0.00	0.00

The Posteriors – $\Pr(m = k \mid \boldsymbol{x})$

k	SOI	Niño3.4	DSLPA
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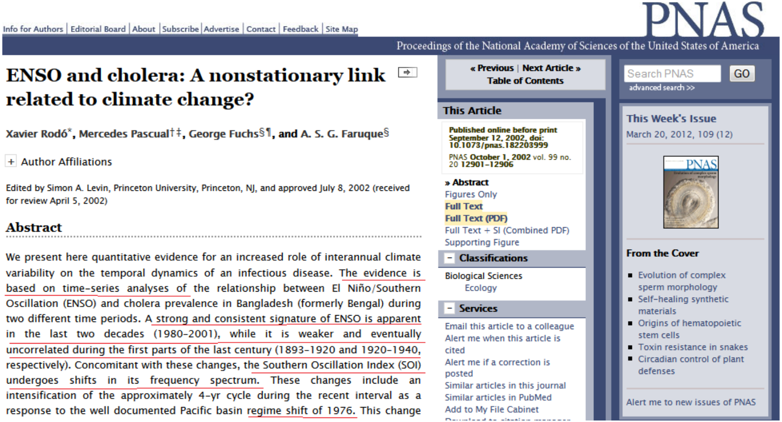

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ENSO and cholera: A nonstationary link related to climate change?

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Edited by Simon A. Levin, Princeton University, Princeton, NJ, and approved July 8, 2002 (received for review April 5, 2002)

Abstract

We present here quantitative evidence for an increased role of interannual climate variability on the temporal dynamics of an infectious disease. The evidence is based on time-series analyses of the relationship between El Niño/Southern Oscillation (ENSO) and cholera prevalence in Bangladesh (formerly Bengal) during two different time periods. A strong and consistent signature of ENSO is apparent in the last two decades (1980–2001), while it is weaker and eventually uncorrelated during the first parts of the last century (1893–1920 and 1920–1940, respectively). Concomitant with these changes, the Southern Oscillation Index (SOI) undergoes shifts in its frequency spectrum. These changes include an intensification of the approximately 4-yr cycle during the recent interval as a response to the well documented Pacific basin regime shift of 1976. This change

« Previous | Next Article »
Table of Contents

This Article

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» Abstract
Figures Only
Full Text
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