Parametric estimation problem for a time-periodic signal with additive periodic noise

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Introduction

Model of periodic signal disturbed by noise whose variance is periodic

$$d\zeta_t = f(t,\theta)dt + \sigma(t)dW_t, \qquad t \geq 0, \tag{1}$$

where

1. $f(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is continuous, periodic in the first variable with period P;
2. $\sigma(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ is continuous periodic with the same period P;
3. $W = \{W_t, t \geq 0\}$ is a standard Brownian motion;
4. $\theta \in \Theta$ is an unknown parameter, Θ is a compact of $\mathbb{R}$.

Our target is

1. Estimation of the unknown parameter θ when we observe a continuous observation along the interval $[0, T]$
2. Estimation of the unknown parameter θ when we observe a discrete observation along the interval $[0, T]$.

We are going to use the maximum of likelihood method for the first estimation and the maximum of contrast for the second.

Then we show the consistency of these estimators.

Application, ξ_t such that $d\zeta_t = \frac{d\xi_t}{\xi_t}$, however

$$\zeta_t - \zeta_0 \neq \ln(\xi_t) - \ln(\xi_0).$$

So ξ_t is the solution of the geometric linear SDE

$$d\xi_t = f(t,\theta)\xi_t dt + \sigma(t)\xi_t dW_t, \tag{2}$$

and the estimation of the drift component in the model (1) is identical to this estimation in the model (2).

The equations of this type appear in several areas :

1. Finance (Karatzas and Shreve, 1991; Klebaner, 2006) (Black-Scholes-Merton model);
2. Mechanic (Has'minskii, 1980 ; Jankunas and Khas'minskii, 1997)

In the continuous case, for the parametric estimation several works are available (See for instance, Ibragimov and Has'minskii, 1981; Kutoyants, 1984...)

However it's difficult to obtain a complete observation of the sample path. So problem of discretization are considered.

On the drift estimation for a diffusion process, Le Breton (1976) has shown that maximum likelihood estimators based on the discrete schemas has asymptotically the same behaviour as the maximum likelihood estimators based on the continuous observation.

Kasonga (1988) has used the least square method to show the consistency of a estimators based on the discrete schemas.

The case of ergodic diffusion models is studied in Dacunha-Castelle and Florens-Zmirou (1986), Florens-Zmirou (1989).

Using maximum contrast and for small variance diffusion models, Genon-Catalot (1990) has shown, under some classical assumptions, asymptotic results. Harison (1996) has used this method to estimate the drift parameter for one-dimentional nonstationary Gaussian diffusion models.

In these works $\sigma(t)$ is assumed to be positive.

Maximum likelihood estimation

Likelihood and convergence in probability

Recall that ζ_t is given by

$$d\zeta_t = f(t,\theta)dt + \sigma(t)dW_t, \qquad t \geq 0.$$

To define the likelihood function we can use Theorem 7.18 of Liptser and Shiryaev (2001).
We apply this theorem to the next two processes

$$d\zeta_t^\theta = f(t,\theta)dt + \sigma(t)dW_t, \tag{3}$$

$$d\eta_t = \sigma(t)dW_t. \tag{4}$$

Under the condition

$$\sup_{t\in[0,P]} \left|\frac{f(t,\theta)}{\sigma(t)}\right| \mathbf{1}_{\{\sigma(t)\neq 0\}} < \infty,$$

these two processes fulfil the conditions of this Theorem 7.18. So $\mu_\theta^T \sim \nu^T$, where

$$\mu_\theta^T := \mathcal{L}(\zeta_t^\theta,\ 0 \le t \le T),\ \nu^T := \mathcal{L}(\eta_t,\ 0 \le t \le T).$$

In addition, the conditions of the Corollary which follows this Theorem 7.18 are satisfied and we have P-a.s.

$$\frac{d\mu_\theta}{d\nu}(\zeta^\theta) = \exp\left(\int_0^T \frac{f(s,\theta)}{\sigma^2(s)}\mathbf{1}_{\{\sigma(s)\neq 0\}} d\zeta_s^\theta - \frac{1}{2}\int_0^T \rho^2(s,\theta)ds\right)$$

where $\rho(s,\theta) := \frac{f(s,\theta)}{\sigma(s)}\mathbf{1}_{\{\sigma(s)\neq 0\}}$.

Denote the likelihood function

$$L_T(\theta) := \exp\left(\int_0^T \frac{f(s,\theta)}{\sigma^2(s)} \mathbf{1}_{\{\sigma(s)\neq 0\}} d\zeta_s^\theta - \frac{1}{2}\int_0^T \rho^2(s,\theta)ds\right) \quad (5)$$

Using the assumptions under $f(\cdot,\cdot)$ and $\sigma(\cdot)$ there exist $\hat{\theta}_T$ such that

$$L_T(\hat{\theta}_T) = \arg\sup_{\theta\in\Theta} L_T(\theta).$$

To show the convergence in P_θ of this estimator :
first we check that the log-likelihood is a contrast in the sense of Dacunha-Castelle and Duflo, 1983 (Definition 3.2.7) and then we apply a version of (Theorem 3.2.4, Dacunha-Castelle and Duflo, 1983, see also Theorem 5.7 of van der Vaart, 2005).

For $\alpha \in \Theta$ let

$$\Lambda_T(\alpha) := \log(L_T(\alpha))$$

$$\begin{aligned}\Lambda_T(\alpha) &= \frac{1}{T}\int_0^T \frac{\rho(s,\alpha)}{\sigma^2(s)}\mathbf{1}_{\{\sigma(s)\neq 0\}}d\zeta_s^\theta - \frac{1}{2T}\int_0^T \frac{f^2(s,\theta)}{\sigma^2(s)}\mathbf{1}_{\{\sigma(s)\neq 0\}}ds \\ &= \frac{1}{T}\int_0^T \left(\rho(s,\alpha)\rho(s,\theta) - \frac{1}{2}\rho^2(s,\alpha)\right)ds + \frac{1}{T}\int_0^T \rho(s,\alpha)dW_s\end{aligned}$$

Take $T = nP$, $\Lambda_n(\alpha)$ converges P_θ-p.s. to the contrast function

$$K(\theta,\alpha) := -\frac{1}{P}\int_0^P \left(\rho(s,\alpha)\rho(s,\theta) - \frac{1}{2}\rho^2(s,\theta)\right)ds$$

Recall now Theorem 3.2.4 of Dacunha-Castelle and Duflo.

Theorem 1

Let $(\Omega, \mathcal{F}, (\mathcal{F}_x)_{x>0}, (\mathrm{P}_\theta)_{\theta\in\Theta})$ *be a probability space, assume that the next two conditions are fulfilled*

1. Θ *is a compact of* $\mathbb{R}$, *the functions* $\alpha \mapsto \Lambda_n(\alpha)$, $\alpha \mapsto K(\theta, \alpha)$ *are continuous;*
2. *for all* $\epsilon > 0$, *there exists* $\eta > 0$ *such that*

$$\lim_{n\to\infty} \mathrm{P}_\theta\left(\sup_{|\alpha-\alpha'|<\eta} \left|\Lambda_n(\alpha) - \Lambda_n(\alpha')\right| > \epsilon\right) = 0.$$

Then the maximum contrast estimator $\hat{\theta}_n$ *is consistent in* θ.

$$\hat{\theta}_n \xrightarrow{\mathrm{P}_\theta} \theta.$$

$$\Lambda_n(\alpha) - \Lambda_n(\alpha') =$$
$$\frac{1}{2T}\int_0^T \left(\rho(s,\alpha) - \rho(s,\alpha')\right)\left(2\rho(s,\theta) - \rho(s,\alpha) - \rho(s,\alpha')\right) ds$$
$$+\frac{1}{T}\int_0^T (\rho(s,\alpha) - \rho(s,\alpha'))dW_s$$

The absolute value of the first term of this equality is bounded by a multiple of η where $|\alpha - \alpha'| \leq \eta$. We show that the second term converges in mean to 0 when $n \to \infty$.

So

$$\hat{\theta}_n \overset{\mathrm{P}_\theta}{\to} \theta.$$

Convergence in the case $f(t,\theta) = \theta f(t)$

Now consider the particular case $f(t,\theta) = \theta f(t)$ so ζ_t is given by this equation

$$d\zeta_t = \theta f(t)dt + \sigma(t)dW_t, \; t \in [0, T].$$

For the function $f(\cdot)$ non-parametric estimators are provided in (Ibragimov and Has'minskii, 1981; Dehay and El Waled, 2013).

For the parameter θ we are going to give the expression of its estimator and establish its convergence : convergence almost sure, mean square convergence, asymptotic normality and the asymptotic efficiency when $T \to \infty$.

Thanks to (5) the likelihood function in this case is

$$L_T(\theta) := \frac{d\mu_\theta}{d\nu}(\zeta^\theta) = \exp\left(\theta \int_0^T \frac{f(s)}{\sigma^2(s)} \mathbf{1}_{\{\sigma(s)\neq 0\}} d\zeta_s - \frac{\theta^2}{2}\int_0^T \rho^2(s)ds\right).$$

So the MLE is

$$\hat{\theta}_T := \frac{\int_0^T \frac{f(s)}{\sigma^2(s)} \mathbf{1}_{\{\sigma(s)\neq 0\}} d\zeta_s}{\int_0^T \rho^2(s)ds}.$$

Remark

When we observe a continuous trajectory of ξ_t defined in (2) on $[0,T]$

$$d\xi_t = \theta f(t)\xi_t dt + \sigma(t)\xi_t dW_t.$$

Then the conditions of the Theorem 7.18 and the Corollary which follows it are satisfied and we deduce that the MLE $\hat{\theta}_T$ is defined as

$$\hat{\theta}_T := \frac{\int_0^T \frac{f(s)}{\sigma^2(s)\xi_s} \mathbf{1}_{\{\sigma(s)\neq 0\}} d\xi_s}{\int_0^T \rho^2(s)ds}.$$

When $\zeta_s = \zeta_s^\theta$

$$d\zeta_t^\theta = \theta f(t)dt + \sigma(t)dW_t.$$

Hence we can write $\hat{\theta}_T$ as :

$$\hat{\theta}_T = \theta + \frac{\int_0^T \rho(s)dW_s}{\int_0^T \rho^2(s)ds} = \theta + \frac{V_T}{J_T}.$$

Here we show that $\hat{\theta}_T$ is unbiased, moreover we get the almost sure convergence, mean square convergence, the asymptotic normality and the asymptotic efficiency.

Almost sure convergence

Theorem 2

$\hat{\theta}_T$ converges almost surely to θ.

Proof.

$$J_T = \int_0^{nP} \rho^2(s)ds = n\int_0^{P} \rho^2(s)ds \Rightarrow \lim_{T\to\infty} \frac{J_T}{T} = \frac{1}{P}\int_0^{P} \rho^2(s)ds.$$

$$V_T = V_{nP} \quad = \quad \sum_{k=0}^{n-1}\int_{kP}^{(k+1)P} \rho(s)dW_s = \sum_{k=0}^{n-1}\int_0^{P} \rho(s)dW_s^{(kP)},$$

where $W_u^{(kP)} := W_{kP+u} - W_{kP}$. As

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\int_0^{P} \rho(s)dW_s^{(kP)} = \mathrm{E}\left[\int_0^{P} \rho(s)dW_s^{(kP)}\right] = 0 \;\; \mathrm{P}_\theta - p.s.$$

we deduce the convergence □

As

$$\mathcal{L}\left(\int_0^T \rho(s)dW_s\right)=\mathcal{N}\left(0,\int_0^T \rho^2(s)ds\right) \text{ and } \hat{\theta}_T-\theta=\frac{\int_0^T \rho(s)dW_s}{\int_0^T \rho^2(s)ds},$$

we deduce that

$$\mathcal{L}\left(\hat{\theta}_T-\theta\right)=\mathcal{N}\left(0,\frac{1}{\int_0^T \rho^2(s)ds}\right).$$

So we get the mean square convergence as well as the asymptotic normality.

Mean square convergence, asymptotic normality

Theorem 3

$\hat{\theta}_T$ converges in mean square to θ, and $\bar{\theta}_T=\sqrt{T}(\hat{\theta}_T-\theta)$ is asymptotically normal.

Asymptotic efficiency of $\hat{\theta}_T$

To justify the relevance of this estimator we see if it is asymptotically efficient. In order to show the asymptotic efficiency we use the Hájek-Le Cam inequality (see Kutoyants 1984, van der Vaart 1998 for further details).
We show firstly that the family $\mathrm{P}_\theta^{(T)}$ is locally asymptotically normal (see Definition 1.2.1 in Kutoyants 1984).

Proposition 1

$\mathrm{P}_\theta^{(T)}$ *is locally asymptotically normal.*

Proof.

After computation we get

$$\frac{d\mathrm{P}^{(T)}_{\theta+\Phi_T u}}{d\mathrm{P}^{(T)}_{\theta}}(\zeta_T) = \exp\left\{u\Delta_T(\zeta_T) - \frac{1}{2}u^2\right\}$$

where

$$\Phi_T := \left(\int_0^T \rho^2(s)\mathbf{1}_{\{\sigma(s)\neq 0\}}ds\right)^{-\frac{1}{2}},$$

$$\Delta_T(\zeta_T) := \left(\int_0^T \rho^2(s)ds\right)^{-\frac{1}{2}} \int_0^T \rho(s)dW_s.$$

□

Theorem 4

The estimator $\hat{\theta}_T$ is asymptotically efficient for the square error (see Definition 1.2.2 in Kutoyants 1984).

Maximum contrast estimation

Definition of the contrast

$$d\zeta_t = f(t,\theta)dt + \sigma(t)dW_t.$$

First, we discretize the interval $[0, T]$ in the following way.

Let $t_i := i\Delta_n$, $i \in 0 \cdots n-1$, where $\Delta_n = \frac{T}{n}$.
Following Genon-Catalot (1990) we can approximate the likelihood of this process by the next function

$$L_n(\theta,\zeta) := L_n(\theta) = \sum_{i=0}^{n-1} f(t_i,\theta)(\zeta_{t_{i+1}} - \zeta_{t_i}) - \frac{1}{2}\sum_{i=0}^{n-1} f^2(t_i,\theta)\Delta_n. \quad (6)$$

Assume that $T = n\Delta_n = N_n P$, $P = p_n\Delta_n$ fixed, $p_n \in \mathbb{N}$,
$T = n\Delta_n \to \infty$, $\Delta_n \to 0$ when $n \to \infty$.

To show the consistency of the maximum contrast estimator we firstly show that $U_n(\alpha) := \frac{L_n(\alpha)}{n\Delta_n}$ is a contrast where $\alpha \in \Theta$.

That is to show that $U_n(\alpha)$ converges in P_θ to some real contrast function $K(\theta, \alpha)$, where

$$K(\theta,\alpha) := -\frac{1}{2P}\int_0^P \left(f(s,\theta) - f(s,\alpha)\right)^2 ds + \frac{1}{2P}\int_0^P f^2(s,\theta) ds.$$

To prove this convergence we use the next two results

Lemma 1

For a continuous periodic function $f(\cdot,\cdot)$ *defined on* $[0, T] \times \Theta$ *where* Θ *is a compact of* $\mathbb{R}$ *we have*

$$\lim_{n\to\infty} \frac{1}{n\Delta_n} \sum_{i=0}^{n-1} f^2(t_i, \theta)\Delta_n = \frac{1}{P}\int_0^P f^2(t, \theta)dt. \tag{7}$$

$$\lim_{n\to\infty} \frac{1}{n\Delta_n} \sum_{i=0}^{n-1} f(t_i, \alpha) \int_{t_i}^{t_{i+1}} f(t, \theta)dt = \frac{1}{P}\int_0^P f(t, \alpha) f(t, \theta)dt. \tag{8}$$

Theorem 5

Under the above conditions and for $\sigma(s) \neq 0$ if there exists an s such that $f(s,\theta) \neq f(s,\alpha)$ then $U_n(\alpha)$ is a contrast.

To prove that $U_n(\alpha)$ converges in P_θ to $K(\theta,\alpha)$ we prove the convergence in mean square

Proof.

$K(\theta,\alpha)$ is a contrast function which has a strict maximum for $\alpha=\theta$

$$K(\theta,\theta)=\frac{1}{2P}\int_0^P f^2(s,\theta)ds.$$

$$\mathrm{E}_\theta\left[\left|U_n(\alpha)-K(\theta,\alpha)\right|^2\right] = \left|\mathrm{E}_\theta\left[U_n(\alpha)\right]-K(\theta,\alpha)\right|^2+\mathrm{var}_\theta\left(U_n(\alpha)\right)$$

Using (7) and (8) one can show that

$$\lim_{n\to\infty}\mathrm{E}_\theta\left[U_n(\alpha)\right]=K(\theta,\alpha),$$

$$\lim_{n\to\infty}\mathrm{var}_\theta\left(U_n(\alpha)\right)=0.$$

□

Now we apply again Theorem 3.2.4 of Dacunha-Castelle and Duflo

Corollary 1

In our case the two conditions of this theorem are fulfilled.

Proof.

1. The functions $U_n(\alpha)$, $K(\theta, \alpha)$ are continuous .
2. for all $\epsilon > 0$, there exists $\eta > 0$ such that

$$\lim_{n\to\infty} \mathrm{P}_\theta \left(\sup_{|\alpha-\alpha'|<\eta} \left| \frac{L_n(\alpha) - L_n(\alpha')}{n\Delta_n} \right| > \epsilon \right) = 0.$$

□

Study of the case $f(t,\theta) = \theta f(t), \sigma(t) = 1$

Consider $f(t,\theta) = \theta f(t)$, $\sigma(t) = 1$. So we have the next model

$$d\zeta_t = \theta f(t)dt + dW_t. \tag{9}$$

In the discrete case, let's make again the next discretization of the interval $[0, T]$. $\{\zeta_{t_i}\}$ $i = 0, \cdots, n-1$, where $t_i = i\Delta_n$.

Then we have the next contrast

$$L_n(\theta) = \sum_{i=0}^{n-1} \theta f(t_i)(\zeta_{t_{i+1}} - \zeta_{t_i}) - \sum_{i=0}^{n-1} \theta^2 f^2(t_i)\Delta_n.$$

The estimator of θ can be explicitly written as

$$\hat{\theta}_n = \frac{\sum_{i=0}^{n-1} f(t_i)(\zeta_{t_{i+1}} - \zeta_{t_i})}{\sum_{i=0}^{n-1} f^2(t_i)\Delta_n}. \tag{10}$$

Therefore

$$\hat{\theta}_n = \theta + \theta R_n + \frac{1}{\Delta_n} \frac{\sum_{i=0}^{n-1} f(t_i)}{\sum_{i=0}^{n-1} f^2(t_i)} (W_{t_{i+1}} - W_{t_i})$$

where

$$R_n := \frac{\sum_{i=0}^{n-1} f(t_i) \int_{t_i}^{t_{i+1}} (f(t) - f(t_i))dt}{\sum_{i=0}^{n-1} f^2(t_i)\Delta_n}.$$

Proposition 2

The estimator $\hat{\theta}_n$ is asymptotically unbiased.

Mean square convergence

Theorem 6

Assume that $n\Delta_n$ goes to ∞ when n goes to ∞, then the estimator $\hat{\theta}_n$ converges in mean square to θ. Moreover if $f(\cdot)$ is continuously derivable then we have

$$\lim_{n\to\infty} n\Delta_n \mathrm{E}\left[|\hat{\theta}_n - \theta|^2\right] = \left(\frac{1}{P}\int_0^P f^2(t)dt\right)^{-1}.$$

Proof.

$$\begin{aligned} \mathrm{E}\left[|\hat{\theta}_n - \theta|^2\right] &= \left(\mathrm{E}[\hat{\theta}_n - \theta]\right)^2 + \mathrm{var}(\hat{\theta}_n) \\ &= \theta^2 R_n^2 + \frac{1}{\sum_{i=0}^{n-1} f^2(t_i)\Delta_n}. \end{aligned}$$

To finish the proof we use the next lemma

Lemma 2

Under the above conditions on $f(\cdot)$ and T we have

$$\lim_{n\Delta_n \to \infty} \frac{1}{n\Delta_n} \sum_{i=0}^{n-1} f^2(t_i)\Delta_n = \frac{1}{P}\int_0^P f^2(t)dt.$$

□

Asymptotic normality

Theorem 7

Assume that $f(\cdot)$ is continuously derivable and that $n\Delta_n^3$ goes to 0 when n goes to ∞ then $\sqrt{n\Delta_n}(\hat{\theta}_n - \theta)$ converges in law to $\mathcal{N}\left(0, \sigma^2\right)$, where

$$\sigma^2 = \left(\frac{1}{P}\int_O^P f^2(t)dt\right)^{-1}.$$

Therefore

$$\lim_{n\to\infty} \frac{\sqrt{n\Delta_n}}{\sigma}(\hat{\theta}_n - \theta) \overset{\mathcal{L}}{\sim} \mathcal{N}(0,1).$$

Simulation

$T = nP = 1000$ sample size , $P = 1$, $f(t) = cos(2\pi t)$, $\sigma(t) = 1$, $\delta = 10^{-2}$ discretization step, $\theta = 0$.

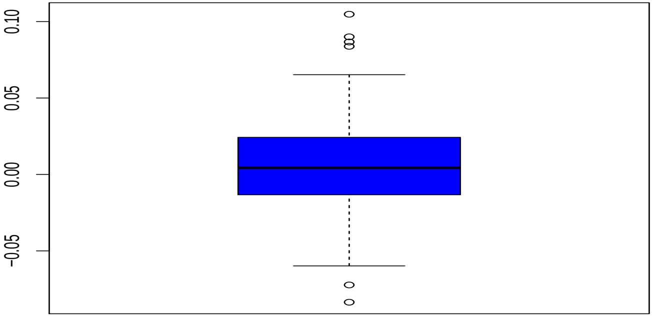

Figure: Boxplot of the values of the estimator $\hat{\theta}_n$ from 100 repetitions

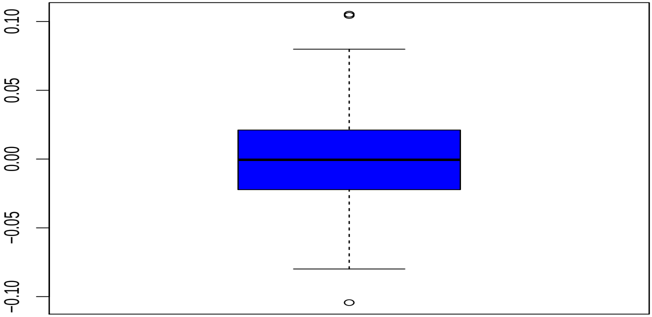

Figure: Boxplot of the values of the estimator $\hat{\theta}_n$ from 1000 repetitions

For $\theta = 1$, $\delta = 10^{-3}$

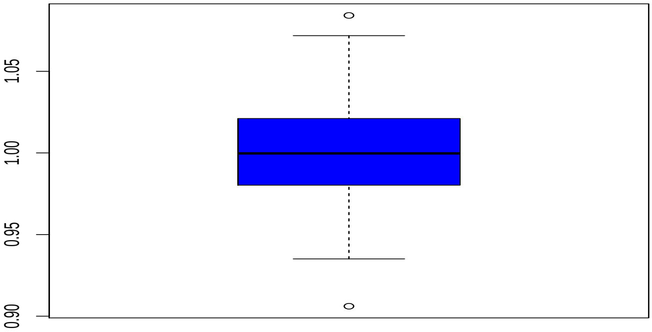

Figure: Boxplot of the values of the estimator $\hat{\theta}_n$ from 1000 repetitions

$$\bar{\theta}_n := \sqrt{n\Delta_n}(\hat{\theta}_n - \theta), \lim_{n\to\infty} \sqrt{n\Delta_n}(\hat{\theta}_n - \theta) \sim \mathcal{N}\left(0, \frac{P}{\int_0^P \rho^2(s)ds}\right) \textit{ in law}.$$

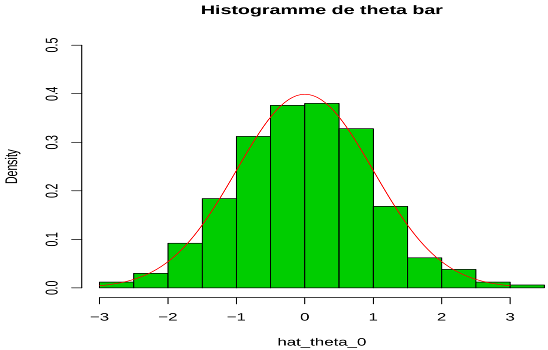

Figure: Histogram $\bar{\theta}_n$, $\theta = 0$ from 1000 repetitions

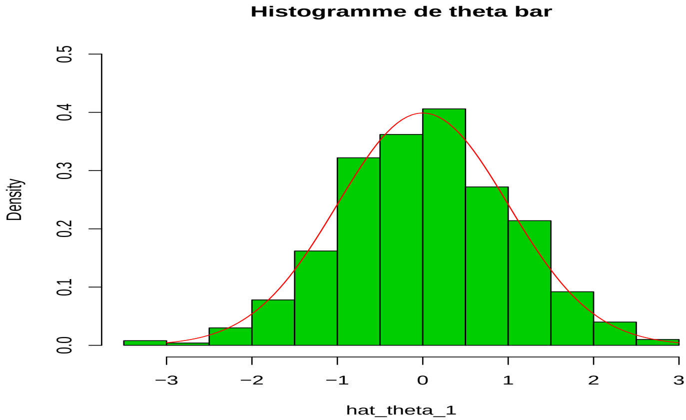

Figure: Histogram of $\bar{\theta}_n$, $\theta = 1$ from 1000 repetitions

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