Harris recurrence for strongly degenerate stochastic systems, with application to stochastic Hodgkin-Huxley models

Reinhard Höpfner, Universität Mainz
and
Eva Löcherbach, Université Cergy-Pontoise
and
Michele Thieullen, Université Paris VI

DYNSTOCH Warwick 10.09.2014

talk based on

Michele Thieullen, Eva Löcherbach, Reinhard Höpfner

- Transition densities for strongly degenerate time inhomogeneous random models. `arXiv:1310.7373`
- Ergodicity for a stochastic Hodgkin-Huxley model driven by Ornstein-Uhlenbeck type input. `arXiv:1311.3458`
- A general scheme for ergodicity in strongly degenerate stochastic systems. Ongoing work.

I: strongly degenerate stochastic systems – main result

for $m < d$, consider d-dim diffusion driven by m-dim Brownian motion

$$dX_t \;=\; b(t, X_t)\, dt \;+\; \sigma(X_t)\, dW_t \;\;,\;\; t \geq 0$$

with coefficients

$$b(t,x) = \begin{pmatrix} b^1(t,x) \\ \vdots \\ b^d(t,x) \end{pmatrix} \;,\; \sigma(x) = \begin{pmatrix} \sigma^{1,1}(x) & \dots & \sigma^{1,m}(x) \\ \vdots & & \vdots \\ \sigma^{d,1}(x) & \dots & \sigma^{d,m}(x) \end{pmatrix}$$

for $t \geq 0$, $x \in E$: state space $(E, \mathcal{E})$ Borel subset of $\mathbb{R}^d$ (with some properties)

coefficient smooth, but neither bounded nor globally Lipschitz

assume: unique strong solution exists, has infinite life time in E

aim: ask for Harris properties of $(X_t)_{t\geq 0}$ (non homogeneous in time)

when drift is time-periodic and when some Lyapunov function is at hand:

write $P_{s,t}(x, dy)$ $(0 \le s < t < \infty,\ x, y \in E)$ for the semigroup of $(X_t)_{t \ge 0}$

assumption A: i) the drift is T-periodic in the time argument

$$b(t,x) = b\left(i_T(t), x\right) \quad , \quad i_T(t) := t \text{ modulo } T$$

ii) we have a Lyapunov function:

$$\begin{cases} V : E \to [1, \infty) \ \ \mathcal{E}\text{-measurable, and for some compact } K: \\ P_{0,T}V \ \text{ bounded on } K \ , \ P_{0,T}V \le V - \varepsilon \ \text{ on } E \setminus K \end{cases}$$

T-periodicity of the drift implies that the semigroup is T-periodic

$$P_{s,t}(x, dy) \ = \ P_{s+kT, t+kT}(x, dy) \quad , \quad k \in \mathbb{N}_0 \ , \ x, y \in E$$

thus the grid chain $(X_{kT})_{k \in \mathbb{N}_0}$ is a time homogeneous Markov chain

Lyapunov condition grants that grid chain will visit K infinitely often alternative under assumption A: define torus $\mathbb{T} := [0, T]$, define $\overline{E} := \mathbb{T} \times E$,

add time as 0-component to the process X:

$$\overline{X}_t := (i_T(t)\,,\, X_t) \;\;,\;\; t \geq s \;\;,\;\; \overline{X}_0 = (s, x)$$

$\overline{X}$ is time homogeneous, $(1{+}d)$-dim, state space $(\overline{E}, \overline{\mathcal{E}})$

assumption B: i) for some $U \subset \mathbb{R}^d$ open and containing E, coefficients

$$(t,x) \to b^i(t,x)\;,\; x \to \sigma^{i,j}(x)\;,\; 1 \leq i \leq d\;,\; 1 \leq j \leq m$$

of SDE are real analytic functions on $\mathbb{T} \times U$

ii) there exists some $x^* \in \mathrm{int}(E)$ with the following two properties:

- x^* is of full weak Hoermander dimension (I explain on the blackboard)
- x^* is attainable in a sense of deterministic control (cf. next slide)

'attainable in a sense of deterministic control':

in view of control arguments, put the SDE in Stratonovich form

$$dX_t \;=\; \widetilde{b}(t, X_t)\, dt \;+\; \sigma(X_t) \circ dW_t$$

with Stratonovich drift: $\widetilde{b}(t, x)$ has components

$$\widetilde{b}^i(t, x) \;=\; b^i(t, x) \;-\; \frac{1}{2} \sum_{\ell=1}^{m} \sum_{j=1}^{d} \sigma^{j,\ell}(x)\, \frac{\partial \sigma^{i,\ell}}{\partial x^j}(x) \quad , \quad 1 \le i \le d$$

definition 1: call $x^* \in \mathrm{int}(E)$ attainable in a sense of deterministic control

if for every starting point $x \in E$ we can find some function $\dot{\mathrm{h}} : [0, \infty) \to \mathbb{R}^m$ depending on x and x^*, all components $\dot{\mathrm{h}}^\ell(\cdot)$ in L^2_{loc}, $1 \le \ell \le m$,

which drives a deterministic control system

$$\varphi = \varphi^{h,x,x^*} \quad \text{solution to} \quad d\varphi_t = \widetilde{b}(t, \varphi_t)dt + \sigma(\varphi_t)\, \dot{\mathrm{h}}(t) dt$$

from $x = \varphi_0$ towards $x^* = \lim_{t\to\infty} \varphi_t$

(control theorem: see Millet and Sanz-Sole 1994) theorem 1: assume A + B, then

i) (d-dim:) the grid chain $(X_{kT})_{k\in\mathbb{N}_0}$ is positive Harris recurrent with invariant probability μ on $(E,\mathcal{E})$

ii) (1+d-dim:) the process $\overline{X} := (\, i_T(t)\,,\, X_t\,)_{t\geq 0}$ is positive Harris recurrent with invariant probability $\overline{\mu}$ on $(\overline{E},\overline{\mathcal{E}})$

and both invariant measures are related by

$$\overline{\mu} \;=\; \frac{1}{T}\int_0^T ds\,(\epsilon_s \otimes \mu P_{0,s})$$

corollary 1: for functions $G : E \to \mathbb{R}$ in $L^1(\mu)$ and $F : \overline{E} \to \mathbb{R}$ in $L^1(\overline{\mu})$

$$\frac{1}{n}\sum_{k=1}^{n} G(X_{kT}) \;\longrightarrow\; \int \mu(dy)\, G(y)$$

$$\frac{1}{t}\int_0^t F(i_T(s), X_s)\,\Lambda(ds) \;\longrightarrow\; \frac{1}{T}\int_0^T \Lambda(ds)\int_E (\mu P_{0,s})(dy)\, F(s,y)$$

Q_x-almost surely as $n, t \to \infty$, for every choice of a starting point $x \in E$, for any T-periodic ms Λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, i.e. $\Lambda([0,T]) < \infty$ and $\Lambda(B) = \Lambda(B + kT)$

II: example, a stochastic Hodgkin-Huxley system

V membran potential in a neuron, n, m, h gating variables, ξ dendritic input

autonomous diffusion $(\xi_t)_{t\geq 0}$ modelling dendritic input, analytic coefficients, carrying T-periodic deterministic signal $t \to S(t)$ encoded in its semigroup

describe temporal dynamics of the neuron by a 5d stochastic system (ξHH):

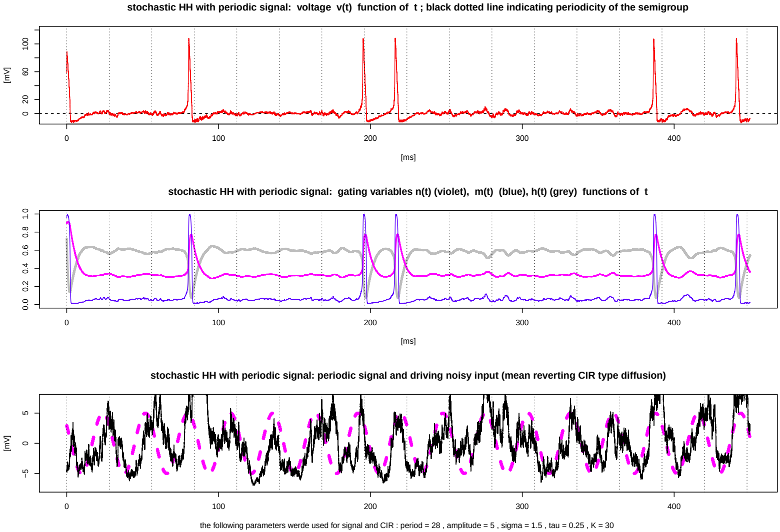
$$t \longrightarrow (V_t, n_t, m_t, h_t, \xi_t) =: X_t$$

5d SDE driven by 1d BM with state space $E = \mathbb{R}\times[0,1]^3\times\mathbb{R}$ defined by

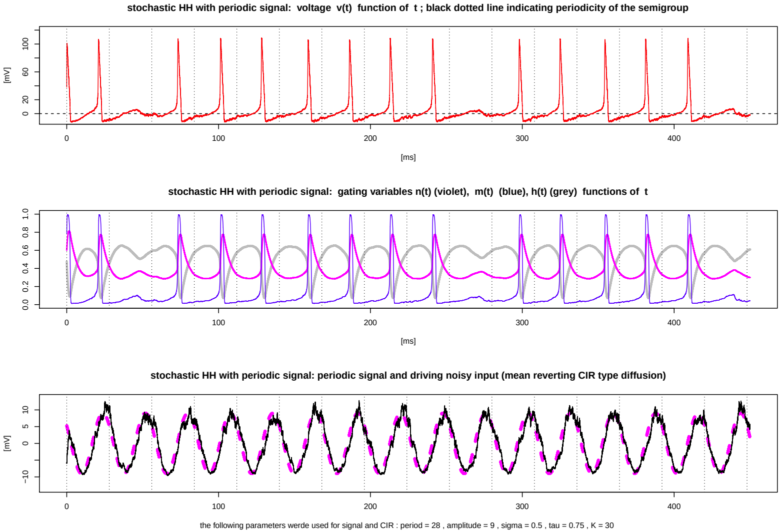
$$\begin{aligned}
dV_t &= d\xi_t - F(V_t, n_t, m_t, h_t)\,dt \\
dn_t &= [\alpha_n(V_t)(1-n_t) - \beta_n(V_t)n_t]\,dt \\
dm_t &= [\alpha_m(V_t)(1-m_t) - \beta_m(V_t)m_t]\,dt \\
dh_t &= [\alpha_h(V_t)(1-h_t) - \beta_h(V_t)h_t]\,dt \\
d\xi_t &= (S(t) - \xi_t)\,dt + dW_t
\end{aligned}$$

specific power series $F(V, n, m, h)$, strictly positive analytic fcts $\alpha_j(V)$, $\beta_j(V)$, $j = n, m, h$, see Izhikevich (2007), or Hodgkin and Huxley (1951)

trajectories may look like this (except that simulation here uses CIR type input)



or like this (depending on signal and choice of parameters for process $(\xi_t)_{t\geq 0}$)



classical deterministic HH systems with periodic deterministic signal $t \to \widetilde{S}(t)$:

$$
\begin{aligned}
dV_t &= \widetilde{S}(t)dt \; - \; F(V_t, n_t, m_t, h_t)\, dt \\
dn_t &= [\alpha_n(V_t)(1-n_t) - \beta_n(V_t)n_t]\, dt \\
dm_t &= [\alpha_m(V_t)(1-m_t) - \beta_m(V_t)m_t]\, dt \\
dh_t &= [\alpha_h(V_t)(1-h_t) - \beta_h(V_t)h_t]\, dt
\end{aligned}
$$

may show – depending on $\widetilde{S}(\cdot)$ – qualitatively quite different behaviour (spiking or non-spiking; single spikes or spike bursts, periodic or chaotic solutions; if periodic, periodicity of output may equal $\ell \geq 1$ periods of input; see interesting tableau based on numerical solutions in Endler 2012)

proposition 1: 'chamaeleon property' of (ξHH):

the stochastic HH system $(X_t)_{0 \leq t \leq T}$ carrying signal $t \to S(t)$ imitates

with positive probability over arbitrarily long (but fixed) time intervals

any deterministic HH with smooth and T-periodic signal $\widetilde{S}(\cdot) \neq S(\cdot)$

(a consequence of the control theorem)

the stochastic Hodgkin-Huxley neuron (ξHH)

$$X_t = (V_t, n_t, m_t, h_t, \xi_t) \quad , \quad t \geq 0$$

is a strongly degenerate diffusion with state space E, and we can show

- all assumptions A + B made above do hold, thus
- grid chain $(X_{kT})_k$ is positive Harris, invariant probability μ on $(E, \mathcal{E})$
- process $\overline{X} = (i_T(t), X_t)_t$ positive Harris, invariant probability $\overline{\mu}$ on $(\overline{E}, \overline{\mathcal{E}})$

Harris recurrence allows to analyze spiking patterns in the neuron via SLLN's:

$$F := \{x = (v, n, m, h, \zeta) : m > h\} \quad \text{(during a spike)}$$
$$Q := \{x = (v, n, m, h, \zeta) : m < h\} \quad \text{('quiet', or: between spikes)}$$

events in $\mathcal{E}$, count spikes as follows: $\sigma_0 \equiv 0$, then for $n = 1, 2, \ldots$

$$\tau_n := \inf\{t > \sigma_{n-1} : X_t \in F\} \quad (n\text{-th spike beginning})$$
$$\sigma_n := \inf\{t > \tau_n : X_t \in Q\} \quad (n\text{-th spike ending})$$

using decompositions into iid life periods and SLLN's (here we use Nummelin splitting in a sequence of 'accompanying' Harris processes with artificial atoms)

we can determine asymptotically a 'typical interspike time (ISI)' for the neuron

in the sense of a distribution function depending on the signal $t \to S(t)$ and the parameters of the SDE governing stochastic input $d\xi_t$

proposition 2: (Glivenko-Cantelli) define empirical distribution functions

$$\widehat{F}_n(t) = \frac{1}{n} \sum_{j=1}^{n} 1_{[0,t]}(\tau_{j+1} - \tau_j) \quad , \quad t \geq 0$$

then there is a honest distribution function F such that

$$\lim_{n\to\infty} \sup_{t\geq 0} \left| \widehat{F}_n(t) - F(t) \right| \quad = \quad 0$$

we may view F as the distribution function of 'the typical interspike time'

(although successive interspike times have no reason to be independent, there may be spike bursts, etc.)

III: proof of theorem 1, sketch of main arguments

back to setting of section I: d-dim SDE driven by m-dim BM, $m < d$,

$$dX_t \;=\; b(t, X_t)\, dt \;+\; \sigma(X_t)\, dW_t \quad , \quad t \geq 0$$

under assumptions A + B: drift T-periodic in time, existence of a Lyapunov function, analytic coefficients, existence of a point x^* which is of full weak Hoermander dimension and attainable in a sense of deterministic control

proof of theorem 1 consists of 3 main steps valid under assumptions A + B:

- control paths do transport weak Hoermander dimension
- all points in the state space are of full weak Hoermander dimension
- transition probabilities $P_{0,T}(\cdot, \cdot)$ locally admit continuous densities

then continue:

- rewrite this into a Nummelin minorization condition for the grid chain, with 'small set' some neighbourhood of x^*
- do Nummelin splitting in the grid chain $(X_{kT})_k$

Arnold, Kliemann (1987): ... degenerate diffusions. Stochastics **21**
DeMarco (2011): Smoothness and asymptotic estimates of densities for SDE with locally smooth coefficients ..., Ann. Appl. Probab. **21**
Endler (2012): Periodicities in the Hodgkin-Huxley model ..., Master th. Mainz
Hodgkin, Huxley (1952): A quantitative description of ion currents and its application to ... in nerve membranes. J. Physiol. **117**
Izhikevich (2007): Dynamical systems in neuroscience ... MIT Press
Kusuoka, Strook (1985): Applications of the Malliavin calculus, Part II. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **32**
Millet, Sanz-Sole (1994): A simple proof of the support theorem for diffusion processes. Sem. Proba. Strasb. **28**
Nualart (1995): The Malliavin calculus and related topics. Springer
Nummelin (1978): A splitting technique for Harris chains. ZW **43**
Nummelin (1985): General irreducible Markov chains ... Cambridge 1985
Meyn, Tweedie (1992/1993): Stability of Markovian processes I+II+III. AAP
Sussmann (1966): Orbits of families of vector fields ... TAMS **180**
Strook, Varadhan (1972): On the support of diffusion processes ... Proc. 6th Berkeley Symp. Math. Statist. Prob. III. Univ. California Press