Change-Point Detection in Parametric Stochastic Models

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Online change detection

Our goal is to test if there is a change in the dynamics of a stochastic process $X_1, X_2, \ldots$, by using a general online framework. We assume that the so-called noncontamination assumption holds, meaning that there is no change during the first m observations. The online method is based on suitable statistics of the form $\tau_{m,k} = \tau_{m,k}(X_1, \ldots, X_{m+k})$. The null hypothesis, that there is no change in the dynamics during the time points $m+1, \ldots, m+Tm$, is rejected if and only if the variable $\sup_{1\le k\le Tm} \tau_{m,k}$ exceeds a corresponding critical value. This method can be applied if

$$\sup_{1\le k\le Tm} \tau_{m,k} \xrightarrow{\mathcal{D}} \tau_T, \qquad m \to \infty,$$

with a nondegenerate variable τ_T, resulting that the critical values can be determined based on the distribution of the limit.

Our model

In our model the observations are $\mathbb{R}^p \times \mathbb{R}^q$-valued pairs $(\mathbb{X}_n, \mathbb{Y}_n)$, $n \in \mathbb{N}_+$, where the process $\mathbb{X}_n, n \in \mathbb{N}_+$, is stationary and ergodic, or it is an aperiodic positive Harris recurrent Markov chain. We assume that there exists a measurable function $f : \mathbb{R}^p \times \Theta \to \mathbb{R}^q$ defined on some parameter space Θ such that

$$E(\mathbb{Y}_n \mid \mathbb{X}_k, \mathbb{Y}_{k-1}, k \le n) = f(\mathbb{X}_n, \theta_n), \qquad n = 1, 2, \ldots,$$

with some $\theta_n \in \Theta$. In this model the noncontamination assumption means that $\theta_m = \cdots = \theta_1 = \theta_0$ for some known m and unknown $\theta_0 \in \Theta$. The goal is to test the null hypothesis

$$\mathcal{H}_0 : \quad \theta_{m+1} = \cdots = \theta_{\lfloor m+Tm \rfloor} = \theta_0,$$

where $T \in (0, \infty]$ is a fixed parameter.

To perform the test we introduce $\mathbb{U}_n = \mathbb{Y}_n - f(\mathbb{X}_n, \theta_0)$, $n \in \mathbb{N}_+$, that is a martingale difference sequence under the null hypothesis. We suppose that there is a nonsingular deterministic matrix $\mathbf{I}_0$ such that $\sum_{n=1}^m \mathbb{U}_n \mathbb{U}_n^\top / m \xrightarrow{P} \mathbf{I}_0$ as $m \to \infty$. Also, we consider suitable estimators $\widehat{\theta}_m$ and $\widehat{\mathbf{I}}_m$ of θ_0 and $\mathbf{I}_0$ based on the training sample $(\mathbb{X}_1, \mathbb{Y}_1), \ldots, (\mathbb{X}_m, \mathbb{Y}_m)$. Then, we can estimate $\mathbb{U}_n$ by $\widehat{\mathbb{U}}_{m,n} = \mathbb{Y}_n - f(\mathbb{X}_n, \widehat{\theta}_m)$. Our testing method is based on the $\mathbb{R}^q$-valued random vectors

$$\mathbb{S}_{m,k} = \widehat{\mathbf{I}}_m^{-1/2} \frac{\sum_{n=m+1}^{m+k} \widehat{\mathbb{U}}_{m,n} - \frac{k}{m} \sum_{n=1}^{m} \widehat{\mathbb{U}}_{m,n}}{\sqrt{m}(1+k/m)(k/(m+k))^\gamma}, \qquad m, k = 1, 2, \ldots,$$

where $\gamma \in [0, 1/2)$ is the tuning parameter of the weight function.

Main result

Consider any continuous function $\psi : \mathbb{R}^q \to \mathbb{R}$. If $\mathcal{H}_0$ holds and the model satisfies the above assumptions along with certain additional regularity conditions, then

$$\sup_{1\le k\le Tm} \psi(\mathbb{S}_{m,k}) \xrightarrow{\mathcal{D}} \sup_{0\le t\le T/(1+T)} \psi\big(\mathcal{W}(t)/t^\gamma\big), \qquad m \to \infty,$$

where $\mathcal{W}$ is the q-dimensional standard Wiener process. It is a consequence that we can test the null hypothesis $\mathcal{H}_0$ with the statistics $\tau_{m,k} = \psi(\mathbb{S}_{m,k})$ as described above.

Using the norm-like functions $\psi(\mathbf{x}) = \|\mathbf{x}\|$, $\psi(\mathbf{x}) = \max_{1\le i\le q} |x_i|$, and $\psi(\mathbf{x}) = |\mathbf{c}^\top \mathbf{x}|$, the limit variable can be represented in a nicer form. (Here $\mathbf{x} = (x_1, \ldots, x_q) \in \mathbb{R}^q$ and $\mathbf{c} \in \mathbb{R}^q$.)

Alternative hypothesis

We investigate the power of the test under the alternative hypothesis that there is only a single change in the dynamics of the system, which occurs at the time point $m + k_m^*$. We assume that the dynamics after the change does not depend on m, and the last observation $(\mathbb{X}_{m+k_m^*}, \mathbb{Y}_{m+k_m^*})$ of the pre-change dynamics is the initial value of the post-change dynamics. The first time of valid rejection is defined as the smallest time point $m+k > m+k_m^*$ when the statistics $\tau_{m,k}$ exceeds the corresponding critical value.

Power of the test

If $T = \infty$ and the system satisfies certain regularity conditions, then the test is consistent under the single-change alternative. Additional results on the rate of the first time of rejection, and the first time of valid rejection are also available.

Examples

1. Regression models. Consider the standard regression model $\zeta_n = \phi(\xi_n, \theta) + \eta_n$, $n \in \mathbb{N}_+$, with zero mean error terms. To test the change of the parameter θ define $\mathbb{X}_n = \xi_n$ and $\mathbb{Y}_n = \zeta_n$, resulting $f(\mathbf{x}, \theta) = \phi(\mathbf{x}, \theta)$ and $\mathbb{U}_n = \zeta_n - \phi(\xi_n, \theta)$. With a variant of this setup a change in the variances of the error terms can also be tested.
2. Time series. Consider the time series defined by the recursion $\xi_n = \phi(\xi_{n-1}, \ldots, \xi_{n-p}, \theta) + \eta_n$, $n \in \mathbb{N}_+$, with zero mean error terms. Define $\mathbb{X}_n = (\xi_{n-1}, \ldots, \xi_{n-p})^\top$ and $\mathbb{Y}_n = \xi_n$ in order to test the change of θ. These result that $f(\mathbf{x}, \theta) = \phi(\mathbf{x}, \theta)$ and $\mathbb{U}_n = \xi_n - \phi(\xi_{n-1}, \ldots, \xi_{n-p}, \theta)$.
3. Galton–Watson processes. Let $\xi_0, \xi_1, \ldots$ be a single-type Galton–Watson process with immigration, and let m_1 and m_2 stand for the expectations of the offspring and the immigration distributions. To test the change of m_1 and m_2, let $\mathbb{X}_n = \xi_{n-1}$ and $\mathbb{Y}_n = \xi_n$, resulting $f(\mathbf{x}, m_1, m_2) = m_1 \mathbf{x} + m_2$. With a variant of this setup the change of the variances can also be tested, largely increasing the power of the test.
4. Independent observations. To test the change of the first r moments of the independent random variables $\xi_0, \xi_1, \ldots$ define $\mathbb{X}_n = \xi_{n-1}$ and $\mathbb{Y}_n = (\xi_n, \ldots, \xi_n^r)^\top$, resulting that $f(\mathbf{x}, \theta) = \theta := (E\xi_1, \ldots, E\xi_1^r)^\top$ and $\mathbb{U}_n = (\xi_n, \ldots, \xi_n^r)^\top - \theta$.

Simulation study

To illustrate the method we generated single type Galton–Watson processes with immigration distribution Poisson(10). We tested the model under the single-change alternative, where the offspring distribution was switched from Bernoulli(p) to Poisson(λ). The significance level was 5% and we made 300 repetitions. The additional parameters and the percentages of rejections:

- $m = 500$, $T = 2$,
- $k_m^* = 500$,
- $\gamma = 0.25$, $\psi(x) = |x|$.

	$\lambda = 0.2$	$\lambda = 0.5$	$\lambda = 0.8$
$p = 0.2$	4.3	97.6	100.0
$p = 0.5$	67.6	5.0	100.0
$p = 0.8$	100.0	97.3	13.6