

A Survey of Mathematical Finance

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Abstract

Finance is one of the fastest growing areas in mathematics. In some senses it is not a discipline in its own right, but rather an application area in which mathematicians with backgrounds in probability theory, statistics, optimal control, convex and functional analysis and partial differential equations can bring to bear experiences and results from their own fields to problems of real world interest.

In this survey we begin with the simplest possible financial model, and then give an account of the Black-Scholes option pricing formula, in which the key ideas are the replication of option payoffs and pricing under the risk-neutral measure. Then we move on to discuss other important problems in finance, including the general theory for semi-martingale price processes, pricing in incomplete markets, interest rate models and credit risk. The emphasis is on techniques and methodologies from stochastic processes.

Keywords and Phrases: Derivative pricing, Black-Scholes, incomplete markets, stochastic calculus, martingale measures.

1 Preamble

Despite the comparatively recent origins of the subject, mathematical finance is one of the most important application areas of mathematics today. Three decades ago the subject barely registered as a research area, but when in the early 1970s Fisher Black, Myron Scholes and Robert Merton linked the well developed notions of Brownian motion and Itô calculus to the problems of derivative pricing and hedging, a new and vibrant discipline was created. The celebrated Black-Scholes option pricing formula (the discovery and development of which earned Nobel prizes in 1997 for Scholes and Merton, Black having died a couple of years previously) revolutionised the finance industry, facilitating the subsequent rapid expansion in the trading of financial derivatives. The growth in volume of trading of these instruments has been matched by the growth of mathematical finance as a research endeavour. This has helped create new topics for mathematical inquiry, reinvigorating many existing areas, and developing bridges between previously unconnected subjects. Now many mathematics departments in the United Kingdom and throughout the world are developing research and teaching programmes in finance, and the output of these programmes, both in terms of the research and the graduates, provides an important resource for the City of London and elsewhere.

Mathematician's Brownian motion was first introduced by Bachelier (1900) who was motivated by an attempt to model the fluctuations of asset prices and to price derivatives. Although he was the first researcher to characterise Brownian motion and his work was well known to Kolmogorov and Doob, the impact of his work was not recognised by the Finance community for many years. (His name is, however, honoured by the main international Mathematical Finance society.) Indeed it was much later that Samuelson (1965) suggested using exponential Brownian motion to model stock prices. In the exponential Brownian model the proportional price changes are generated by a Brownian motion. Over a small time interval the proportional price changes are Gaussian random variables with a variance proportional to the length of the interval, and price changes over disjoint intervals are uncorrelated. The exponential Brownian model reflects the limited liability (non-negativity) property of share prices and whilst it is not appropriate for all financial assets in all market conditions, it remains the reference model against which any alternative dynamics are judged.

It was in a model with exponential Brownian assets that Black and Scholes (1973) constructed a replicating portfolio and with it proposed a 'fair' price for a financial derivative. (A derivative security or contingent claim is a financial instrument whose payoff is derived from, or contingent upon, the behaviour of some other underlying asset. For example a call option on a stock or share gives the option holder the right, but not the obligation, to purchase one unit of the stock at a prespecified price called the strike.) Their ideas were quickly advanced by Merton (1973). The key insight was that if it was possible to replicate the payoff of the derivative as the gains from trade from a dynamic, self-financing hedging strategy, then the initial fortune required to finance that strategy was exactly the arbitrage-free price for the option. Furthermore, since all the risks associated with the option were removed by hedging, the price is independent of the risk preferences of the agent.

This argument was developed into a mathematical theory by Harrison and Kreps (1979) and Harrison and Pliska (1981). These authors emphasised the central role of probability theory and martingales (a martingale is a random process which is as likely to go up as down, on average) and it is their stochastic theory that we explain here, and which

provides the foundation for much of the subsequent development of the subject. Their key conclusion is that option prices are given by expectations — but not expectations with respect to the real world or physical measure. Instead prices are expectations with respect to the risk neutral measure under which the discounted price of the underlying asset is a martingale.

In this survey we concentrate on the problems of derivative pricing. We begin with an analysis of option pricing in the simplest possible one-period binomial model, the conclusions from which — including the fact that there is a unique, preference independent, fair option price — are subsequently mirrored in the Black-Scholes world. We then investigate the extent to which the Black-Scholes model can be generalised without destroying these key features.

When all options can be priced via replication the model is complete. Otherwise the model is incomplete. In this situation there is no universal scheme for pricing options. Instead we compare and contrast some of the possible alternatives, and this topic is the main theme of the article. In particular we discuss in some simple but canonical settings how options can be priced and hedged under various investment criteria.

No survey of mathematical finance can cover all areas of the subject in equal depth, and any summary inevitably reflects the background and interests of the author. The fact that this article stresses stochastic methods for derivative pricing in complete and incomplete markets is a case in point. In the final few sections we cover, briefly, some of the other important topics in finance, including interest rate models and credit risk.

2 Derivative Pricing: A First Pass

Consider the following model of a financial market. There is a single risky asset whose price is given by $(X_t)_{0 \leq t \leq T}$ and a risk-less bank account. The market in these assets is perfect, by which we mean that there are no transaction costs or taxes, the risky and risk-less assets can be bought in arbitrary quantities and agents are price takers.

A derivative security, or contingent claim, is a financial security whose value is contingent upon the value of the risky asset. For example a call option (with strike K , and maturity T) gives the holder the right, but not the obligation, to buy one unit of the risky asset at time T for price K . If $X_T > K$ then the option holder can exercise this right, and (perhaps by selling the asset) make a profit of $(X_T - K)$, whereas if $X_T \leq K$ the option matures worthless. At maturity the call option is worth $(X_T - K)^+$.

The fundamental problem in mathematical finance is to give a fair price for the random payoff of a derivative security given a stochastic model for the behaviour of the underlying.

2.1 The simplest case: the binomial model.

Suppose $X_0 = x$ and that at time T , X_T takes one of the values xu and xd where $u > d$. (More formally we let $\Omega = \{\omega_u, \omega_d\}$ and define $X_T(\omega_u) = xu$, $X_T(\omega_d) = xd$ and we suppose $0 < \mathbb{P}(\{\omega_u\}) < 1$.) There is also a bank account which pays a fixed and constant rate of interest r over the period $[0, T]$ so that one unit invested in the account at time 0 is worth $R = (1 + r)$ at time T . We assume $R \in (d, u)$ to prevent simple arbitrages.

The problem is to price a derivative security which pays off $h_u = h(xu)$ in a year when the price has moved ‘up’, and $h_d = h(xd)$ otherwise.

Suppose we can find θ, ϕ which solve

$$h_u = \theta xu + \phi R, \tag{1}$$

$$h_d = \theta x d + \phi R. \quad (2)$$

Then the agent is indifferent between receiving the derivative and holding an initial portfolio of θ units of risky asset and investing ϕ units in the bank. Hence the time-zero fair value for the option is $C = \theta x + \phi$, the cost of financing the strategy implicit in the right-hand-side of (1) and (2). This is our first example of pricing by arbitrage; if the derivative trades at any price other than C then there are risk-free profits to be made, either by selling the derivative and purchasing the portfolio (θ, ϕ) or by following the reverse strategy. Since this cannot happen in any sensible market — there would be infinite demand for the derivative if it traded for a price below C , and infinite supply if it traded above C — the derivative must trade for the arbitrage free price C .

In this simple binary model the values of θ and ϕ can be calculated from (1) and (2). We find $\theta = (h_u - h_d)/(x(u - d))$ and $\phi = (uh_d - dh_u)/(R(u - d))$, so that an expression for the derivative price is

$$\frac{1}{R} \left\{ \frac{R-d}{u-d} h_u + \frac{u-R}{u-d} h_d \right\} \quad (3)$$

There are two key observations to be made in this simple model which will inspire our future analysis.

The first is that the key to option pricing is the concept of *replication*; the fact that the fair price is determined by a trading strategy which creates the same payoff as the option. In the binomial model it is always possible to find θ and ϕ to solve (1) and (2) so that replication is possible for all contingent claim payoffs h .

The second key observation relates to the concept of martingale pricing. If we write $q = (R - d)/(u - d)$ then $q \in (0, 1)$ and the derivative price (3) can be written as

$$\frac{1}{R} \{qh_u + (1 - q)h_d\} = \frac{1}{R} \mathbb{E}^q [h(X_T)]$$

so that the option price is the discounted expected payoff of the option, where *the expectation is taken with respect to the risk-neutral probabilities* $(q, 1 - q)$. The probability q has the special property that the expected value of the discounted asset price under the probabilities $(q, 1 - q)$ is the initial value; ie q satisfies

$$x = \frac{1}{R} (qxu + (1 - q)xd).$$

The discounted asset price is a martingale if we take expectations using the q -probabilities. Note that we have completed a full analysis of the problem without reference to the probabilities of the various events under the real-world measure \mathbb{P} .

Rather than focusing on the measure or probabilities, we can consider instead the state price density. Let $p = \mathbb{P}(\{\omega_u\})$ and define ζ via $\zeta_0 = 1$ and

$$\begin{aligned} \zeta_T(\omega_u) &= \frac{q}{pR} = \frac{1}{pR} \frac{(R-d)}{(u-d)}; \\ \zeta_T(\omega_d) &= \frac{(1-q)}{(1-p)R} = \frac{1}{(1-p)R} \frac{(u-R)}{(u-d)}. \end{aligned}$$

Then $(\zeta_t X_t)_{t=0,T}$ is a martingale, and the fair price of the option is $\mathbb{E}[\zeta_T h(X_T)]$.

The above model, which is essentially due to Cox *et al* (1979) can be made more realistic by extending it to cover several time-steps. (Indeed, since a random walk converges to

Brownian motion, the suitably scaled limit will be the continuous-time model of the next section.) The contingent claim pricing problem can be solved by backward induction and the derivative price is precisely the discounted expected payoff where the probabilities have been modified to make the discounted prices of traded assets into martingales.

Note that if it is possible for the risky asset to take on more than two price values at the end of the time-step then the replication argument fails. For example in a trinomial model in which X_T may take the values xu, xR, xd say, then the analogue to (1) and (2) is a triple of simultaneous equations in two unknowns for which there is no solution in general. Conversely there are many choices of probabilities which make the price process into a martingale.

2.2 The Black-Scholes model: Pricing and hedging

We now consider the derivative pricing problem in continuous time. Following Samuelson (1965) the model is based on a Brownian motion or a Wiener process W_t . The stochastic process W_t is not finite variation and so the standard rules of calculus do not apply. Instead we use stochastic calculus. For a very brief introduction to the key concepts see the appendix, or one of the many introductory (Mikosch (1998), Steele (2001)) or more specialist texts (Revuz and Yor (1998), Rogers and Williams (2000)).

We suppose that we have a perfect frictionless model (as before, zero transaction costs, zero taxes and dividends, the same interest rate for both borrowing and lending, agents as price takers) in which trading takes place in continuous time. The economy consists of a single risky asset with price process $(X_t)_{0 \leq t \leq T}$ which follows an exponential Brownian motion, and a bank account which pays a constant rate of interest r . The dynamics for the risky asset are specified under the physical measure \mathbb{P} and are exogenous to the model. This reflects the fact that agents are taken to be small investors, and their actions do not affect the market price. The risky asset price and the value of R_0 units of currency invested in the bank account are given by

$$X_t = X_0 \exp \left\{ \sigma W_t + \left(\nu - \frac{1}{2} \sigma^2 \right) t \right\}, \quad R_t = R_0 \exp \{ rt \},$$

or, in differential notation (using Itô's formula (36))

$$dX_t = X_t(\sigma dW_t + \nu dt), \quad dR_t = rR_t dt. \quad (4)$$

Here the parameters $\sigma > 0, \nu$ and r (respectively the volatility and drift of the risky asset and the interest rate) are taken to be constants. The value of monies invested in the bank account R_t obeys standard Newtonian calculus and the ordinary differential equation for R_t in (4) might more usually be written $dR(t)/dt = rR(t)$. We use the form $dR_t = rR_t dt$ as an analogy to a stochastic differential equation, and to remind us that in a more complicated model the interest rate may itself be stochastic. We call the asset with price R_t a bond.

Our goal, as in the binomial model, is to consider the wealth process which results from holding a portfolio consisting of θ_t units of the risky asset and ϕ_t units of the bond. The elements of the portfolio θ_t and ϕ_t must be chosen based on information available at time t . We assume this information set or filtration is generated by the price process X_t , which means in our current context that it is the Brownian filtration generated by W_t . The value of the portfolio is then given by

$$V_t = \theta_t X_t + \phi_t R_t. \quad (5)$$

We further assume that the dynamics of the portfolio value satisfy

$$V_t = V_0 + \int_0^t \theta_s dX_s + \int_0^t \phi_s dR_s \quad (6)$$

or, in differential notation,

$$dV_t = \theta_t dX_t + \phi_t dR_t. \quad (7)$$

It should be emphasised that (7) is not obtained by taking the Itô derivative of the products in (5). Instead it is postulated as a modelling assumption, motivated by the situation in discrete time. See the remarks in Section 3 for a further discussion of this issue.

A value process V_t which satisfies (7) is said to be self-financing. The term captures the idea that no inputs or outputs of cash are needed to create V_t ; instead all fluctuations in value come from the investment in the risky asset and bond. Further, if V_t solves (5) then once θ_t has been chosen, ϕ_t is determined via the relationship $\phi_t R_t = V_t - \theta_t X_t$. In particular, we do not need to model ϕ explicitly; ϕ_t merely represents the number of bonds we can buy with the cash surplus after we purchase θ_t units of X_t . Sometimes we write V^θ to stress the dependence of the self-financing value process on the strategy θ , or $V^{v,\theta}$ if we also wish to stress the starting wealth. It follows that we can rewrite (7) as

$$dV_t^\theta = \theta_t (dX_t - rX_t dt) + rV_t^\theta dt, \quad (8)$$

which, given the stochastic dynamics of X_t is equivalent to

$$dV_t^\theta = \theta_t X_t \sigma (dW_t + \lambda dt) + rV_t^\theta dt, \quad (9)$$

where $\lambda = (\nu - r)/\sigma$ is the Sharpe ratio of the risky asset. It turns out to be much more convenient to work with the Sharpe ratio λ rather than the drift ν , so that ν will not be mentioned again.

Consider now the problem of pricing a contingent claim with non-negative payoff $h(X_T)$ at time T .

Define a super-replicating strategy to be a pair (v, θ) such that the wealth process $V^{v,\theta}$, defined via $V_0^{v,\theta} = v$ and $V^{v,\theta}$ solves (8), satisfies $V_t^{v,\theta} \geq 0$ and $V_T^{v,\theta} \geq h(X_T)$, \mathbb{P} -almost surely. A replicating strategy has $V_T^{v,\theta} = h(X_T)$. The key idea is that if there exists a super-replicating strategy for initial wealth v , then an agent would be at least as happy to receive initial fortune v and to follow trading strategy θ , as to receive the option. Hence the no-arbitrage principle gives us that v is an upper bound on the fair price of the claim.

Consider $\tilde{X}_t = R_0 X_t / R_t$. We will use the notation $\tilde{\cdot}$ to denote a discounted quantity. We have

$$d\tilde{X}_t = d(R_0 X_t / R_t) = \frac{R_0 X_t}{R_t} \left(\frac{dX_t}{X_t} - r dt \right)$$

which in our case can be simplified to

$$d\tilde{X}_t = \tilde{X}_t \sigma (dW_t + \lambda dt). \quad (10)$$

Now consider the discounted process $\tilde{V}_t^\theta = R_0 V_t^\theta / R_t$. If V^θ is self-financing then V^θ solves (8) and, in terms of discounted quantities,

$$d\tilde{V}_t^\theta = \theta_t d\tilde{X}_t \quad (11)$$

or equivalently, $d\tilde{V}_t^\theta = \theta_t \tilde{X}_t \sigma (dW_t + \lambda dt)$. The simplicity of this equation shows the advantage we gain from switching to discounted variables. Now suppose $V^{v,\theta}$ is the value process associated with a replicating strategy (v, θ) . Then

$$\frac{R_0}{R_T} h(X_T) = \frac{R_0}{R_T} V_T^{v,\theta} = \tilde{V}_T^{v,\theta} = v + \int_0^T \theta_t d\tilde{X}_t \quad (12)$$

\mathbb{P} -almost surely.

Suppose for a brief moment, that $\lambda = 0$ and \tilde{X}_t is a martingale. Then we can take expectations in (12) and provided that $\int_0^T \theta_t d\tilde{X}_t$ is a true martingale and not just a local martingale, we can deduce a value for v . This value represents the replication price for the contingent claim.

Now remove the assumption that $\lambda = 0$, so that the discounted price is not a martingale. Suppose, however, that we can find a new probability measure \mathbb{Q} , equivalent to \mathbb{P} , such that the stochastic integral in (12) is a martingale under \mathbb{Q} . Then the identities in (12) hold \mathbb{Q} -almost surely and taking expectations under \mathbb{Q} we have the formula

$$v = \mathbb{E}^{\mathbb{Q}} \left[\frac{R_0}{R_T} h(X_T) \right]. \quad (13)$$

This gives us the fair price of the option. The measure \mathbb{Q} is a computational device, but it is extremely powerful in that it leads us to the option price.

Motivated by the above analysis, our goal is to find a measure \mathbb{Q} under which the price process is a martingale, or to use a language more familiar to economists, to find a state-price density process ζ_t such that $\zeta_t X_t$ is a martingale.

Define the change of measure density Z_t via

$$Z_t = \exp \left(-\lambda W_t - \frac{1}{2} \lambda^2 t \right)$$

and let \mathbb{Q} and ζ_t be given by

$$\mathbb{Q}(A) = \mathbb{E}[Z_T I_A] \quad \text{and} \quad \zeta_t = \frac{R_0}{R_t} Z_t. \quad (14)$$

Then the probability measure \mathbb{Q} is equivalent to \mathbb{P} and by the Cameron-Martin-Girsanov formula, see the appendix, $W^{\mathbb{Q}}$ defined via $W_t^{\mathbb{Q}} = W_t + \lambda t$ is a \mathbb{Q} -Brownian motion. Hence, see (10), $d\tilde{X}_t = \sigma \tilde{X}_t dW_t^{\mathbb{Q}}$ and \tilde{X} is a \mathbb{Q} -martingale. Alternatively

$$d(\zeta_t X_t) = d(Z_t \tilde{X}_t) = (\sigma - \lambda)(\zeta_t X_t) dW_t$$

so that $\zeta_t X_t$ is a \mathbb{P} -martingale. The above result is an example of the simple proposition that for any process Y_t , we have that \tilde{Y}_t is a (local) martingale under \mathbb{Q} if and only if $\zeta_t Y_t$ is a (local) martingale under \mathbb{P} .

Now suppose that $V^{v,\theta}$ is the value process of a super-replicating strategy for $h(X_T)$. Then, from (11), $\tilde{V}^{v,\theta}$ is a local \mathbb{Q} -martingale. Further $V^{v,\theta}$, and hence $\tilde{V}^{v,\theta}$, is non-negative and we conclude that \tilde{V} is a \mathbb{Q} -supermartingale. Thus

$$v \geq \mathbb{E}^{\mathbb{Q}}[\tilde{V}_T^{v,\theta}] \geq \mathbb{E}^{\mathbb{Q}}[R_0 h(X_T)/R_T] = \mathbb{E}[\zeta_T h(X_T)]$$

In particular $\mathbb{E}[\zeta_T h(X_T)]$ is a lower bound on the fair price of the derivative.

If $\mathbb{E}[\zeta_T h(X_T)] = \infty$ then there is no super-replicating strategy corresponding to a finite initial price. Henceforth we exclude this case.

Now we want to show that there is a super-replicating strategy with initial fortune $v = \mathbb{E}[\zeta_T h(X_T)]$. Define the martingale

$$\tilde{\Pi}_t = \mathbb{E}_t^{\mathbb{Q}} \left[\frac{R_0}{R_T} h(X_T) \right]$$

where \mathbb{E}_t denotes expectation given information available at time t . Observe that $\tilde{\Pi}_t \geq 0$, and $\tilde{\Pi}_T = R_0 h(X_T)/R_T$ \mathbb{Q} -almost surely (and hence \mathbb{P} -almost surely since \mathbb{P} and \mathbb{Q} are equivalent). By the Brownian martingale representation theorem (recall that the filtration is generated by W_t) we can write any \mathbb{Q} -martingale $\tilde{\Pi}_t$ as a stochastic integral with respect to the \mathbb{Q} -Brownian motion $W^{\mathbb{Q}}$. We have

$$\tilde{\Pi}_t = v + \int_0^t \psi_s dW_s^{\mathbb{Q}} = v + \int_0^t \theta_s^{\Pi} d\tilde{X}_s \quad (15)$$

where $\theta_t^{\Pi} = \psi_t/\sigma\tilde{X}_t$ and $d\tilde{X}_s = \tilde{X}_s\sigma dW_s^{\mathbb{Q}}$. Then Π_t defined via $\Pi_t = R_t\tilde{\Pi}_t/R_0$ satisfies $\Pi_0 = v$, $\Pi_t \geq 0$ and $\Pi_T = h(X_T)$, \mathbb{P} -almost surely, with dynamics

$$d\Pi_t = \theta_t^{\Pi}(dX_t - rX_t dt) + r\Pi_t dt.$$

Hence Π_t defines the value process of a self-financing, super-replicating (and indeed replicating) strategy with initial value $v = \mathbb{E}^{\mathbb{Q}}[R_0 h(X_T)/R_T]$ and it follows that v is the fair price for the derivative. The associated hedging strategy is given by θ_t^{Π} .

Note that, in exact parallel with the binomial model, the key ideas are the replication of the option payoff and the idea of finding a change of measure under which the discounted price process is a martingale. That measure is then used for pricing. The Sharpe ratio λ in the original model is irrelevant for pricing (as is the drift), and instead volatility σ is the crucial parameter. The fact that we price the option by replication means that an agent who sells the option for its fair price can remove all the risk via a hedging strategy. This explains why the risk preferences of the agent do not enter into the pricing formula.

To date we have identified the fair price of the option, but not the replicating strategy θ_t^{Π} . To do this in general we need to know how to represent a martingale as a stochastic integral in a Brownian filtration. This can be done by Clark's Theorem which is a special case of Malliavan calculus. Alternatively, for payoffs which are a function of X_T alone (or perhaps a function of X_T and a small number of other path-dependent state variables — see the examples below) we can exploit the Markov property to give an explicit form for the hedging strategy θ .

Suppose the option payoff depends only on the value of the underlying asset at time T . By the Markov property we can represent the time- t value V_t of the contingent claim via

$$V_t = V(X_t, t) = \mathbb{E}_t^{\mathbb{Q}} \left[\frac{R_t}{R_T} h(X_T) \right]. \quad (16)$$

Recall that $dX_t = \sigma X_t dW_t^{\mathbb{Q}} + rX_t dt$. Then, by Itô's formula, assuming that V is sufficiently smooth,

$$\begin{aligned} dV_t &= V'(X_t, t)dX_t + \frac{1}{2}V''(X_t, t)(dX_t)^2 + \dot{V}(X_t, t)dt \\ &= V'(X_t, t)X_t\sigma dW_t^{\mathbb{Q}} + \left[V'(X_t, t)rX_t + \frac{1}{2}V''(X_t, t)\sigma^2 X_t^2 + \dot{V}(X_t, t) \right] dt \end{aligned}$$

Conversely, if V is self-financing then from (9)

$$dV_t = \theta_t X_t \sigma dW_t^{\mathbb{Q}} + V_t r dt.$$

If V is the value function of a self-financing replicating portfolio then these representations must be almost surely identical, and for (almost every) path realisation we must have $\theta_t = V'(X_t, t)$ (for Lebesgue almost surely all $t \in [0, T]$). Further, when we equate finite variation terms we find that the value function must solve

$$\mathcal{L}V = 0 \quad \text{subject to } V(x, T) = h(x), \quad (17)$$

where

$$\mathcal{L}f(x, t) = rx f'(x, t) + \frac{1}{2} \sigma^2 x^2 f''(x, t) + \dot{f}(x, t) - r f(x, t). \quad (18)$$

The partial differential equation (17) for V can be shown to be equivalent to the stochastic pricing formula (16) using the Feynman-Kac formula and is sometimes called the Black-Scholes pricing pde. The hedging strategy $\theta_t = V'(X_t, t)$ is known as the delta-hedge.

2.3 Vanilla and Exotic Options

In the setting of the Samuelson-Black-Scholes exponential Brownian motion model for option pricing we have shown that it is possible to derive a unique fair price for contingent claims. The key mathematical tools that we used were Itô's formula, the Cameron-Martin-Girsanov change of measure and the Brownian martingale representation theorem. In later sections we discuss in more detail the class of admissible trading strategies and the extent to which the conclusions of the above analysis are robust to changes in the underlying model. We also consider the impact that the failure of the model assumptions has on hedging and pricing. However in the rest of this section we assume that the model holds and investigate the implications for the pricing of some common traded options.

The advantage of working with a simple model, albeit an overly simplistic one, is that it gives insights into the behaviour of derivative prices which might be hidden in a more realistic situation. For example, it allows us to investigate the comparative statics of the option price and to understand how prices depend on the key parameters such as volatility (Bergman *al* (1996), Renault and Touzi (1997), Hobson (1998a)). The true test of a model is partly how well does it explain option prices in the market (but as Figlewski (2002) argues one does not need the full power of the Black-Scholes call pricing function for that), and partly how well do the theoretical hedges perform.

2.3.1 Call options

Traditionally the first, simplest and most widely traded options are put and call options. A call option with maturity T and strike K has payoff $(X_T - K)^+$. The time- t price of the call option is

$$V(X_t, t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}} [(X_T - K)^+] = X_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-)$$

where Φ is the cumulative Normal distribution function and

$$d_{\pm} = \frac{\ln(X_t/K e^{-r(T-t)}) \pm \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}.$$

The delta-hedging strategy is given by $\theta_t = \Phi(d_+)$.

A put option gives the holder the right to sell the risky asset for price K . Since $(X_T - K) = (X_T - K)^+ - (K - X_T)^+$, there is a put-call parity result; namely that the price of a call option minus that of a put option equals $X_t - e^{-r(T-t)}K$.

2.4 American Options

If a claim is European in style then it is exercised at a fixed predetermined time T . American style options can be exercised at any (stopping) time τ up to the, possibly infinite, maturity T . The price becomes (see Myneni (1992))

$$\text{ess sup}_{\tau \leq T} \mathbb{E}^{\mathbb{Q}}[e^{-r\tau} h(X_\tau)],$$

where the ess sup is taken over all stopping times τ with $t \leq \tau \leq T$. If $h(x) = (x - K)^+$ (an American call) then provided there are no dividends it is never optimal to exercise the option early and the American call has the same price as a European call. However for an American put option with $h(x) = (K - x)^+$ the benefits of the convexity of the payoff can sometimes be outweighed by the losses associated with the fact that the undiscounted prices increase on average over time and the payoff function is decreasing. The pricing problem becomes an optimal stopping problem in which the optimal exercise strategy has to be determined.

One fruitful approach to this problem is to consider it as a dynamic programming problem. The martingale optimality principle allows us to write down a Hamilton-Jacobi-Bellman equation. The pricing function solves $V(x, t) \geq h(x)$ and $\mathcal{L}V = 0$ on $I_t = \{x : V(x, t) > h(x)\}$ where, as before,

$$\mathcal{L}f = \frac{1}{2}\sigma^2 x^2 f'' + rx f' - rf + \dot{f},$$

together with a smooth fit condition on ∂I_t . This is a free boundary problem for which there is no closed form solution. It is related to the Stefan problem from fluid dynamics (Friedman 2000).

The natural explanation for the European/American nomenclature would be that options of appropriate style were traded in the relevant geographical markets. However there is no strong evidence for this proposition. (Instead there is an anecdote which claims that the adjectives were coined by an American researcher who wanted to appropriate the more sophisticated and challenging option for his own continent.) Whatever the origins of the terminology, it began a trend for naming options after regions or countries — Asia, Bermuda, Paris, Russia and Israel each have an option named after them.

Puts and calls have simple payoffs and are sometimes called vanilla options in honour of the most basic flavour of ice cream. Options with more complicated payoffs are said to be exotic.

2.5 Exotic Options

2.5.1 Barrier Options

An example of an exotic option is an option whose payoff is contingent upon both the value of the underlying at maturity and the value of the maximum price attained by the underlying over some period. For example a knock-out call option has payoff $(X_T - K)^+ I_{\{\overline{X}_T \leq B\}}$ where \overline{X}_T is the maximum price attained by the underlying and B is the

barrier level. The option becomes worthless if ever the underlying exceeds the barrier. In the Black-Scholes model there are closed form expressions for the prices and associated hedging strategies for barrier options which involve the cumulative Normal distribution function.

In practice barrier options can be difficult instruments to hedge. The classical delta-hedge can involve very large positions, especially when the underlying asset is near the barrier and the time to maturity is small. In these cases practical issues tend to dominate (for example it can be useful to hedge using the call as well as the underlying, see Andersen and Andreasen (2000), Brown *et al* (2001)) and an alternative pricing rule and hedging strategy is needed, perhaps aiming to super-replicate the payoff rather than aiming to replicate exactly.

Barrier options are closely related to digital and lookback options. A digital option pays one if ever the underlying crosses the barrier, whilst the payoff of a lookback is contingent upon the maximum price attained by the underlying over the lifetime of the option. In the Black-Scholes model there are formulae for all of these, see for example Goldman *et al* (1979).

2.5.2 Asian Options

An Asian fixed-strike call has payoff $(A_T - K)^+$ where $A_T = (1/T) \int_0^T X_u du$. (Of course this is an idealised mathematical version of the real contract which is based upon discrete averaging.) Asian options are options on the average rate and were introduced partly to meet the need for commodity producers who sold their output at a constant rate over time, and partly to negate the effects of price manipulation.

The Asian pricing problem is to calculate the distribution of $\hat{A}_T = \int_0^T e^{\sigma W_s + (r - \sigma^2/2)s} ds$ in such a way that it is possible to give a simple representation formula for the price of the Asian call. In general there are no closed form solutions but the pricing problem motivated several attempts to give a stochastic characterisation of the distribution, see Geman and Yor (1993), as well as various ideas for the pricing of Asian options via Monte-Carlo methods (with carefully chosen control variates, see Rogers and Shi (1995)) or pdes (Vecer (2001)).

2.5.3 Passport Options

The Passport option (introduced by Hyer *et al* (1997)) is an example of an exotic option which was not widely traded, but which generated some novel research problems in mathematics. In the symmetric passport option problem the aim is to evaluate

$$\sup_{|\theta_t| \leq 1} \mathbb{E}^{\mathbb{Q}}[(\tilde{G}_t^\theta)^+]$$

where \tilde{G}^θ is the discounted gains from trade using a self-financing strategy θ . In particular

$$\tilde{G}_t^\theta = g + \int_0^T \theta_s d\tilde{X}_s.$$

It turns out (see for example Andersen *et al* (1998)) that the optimal strategy is to take $\theta_s = -\text{sgn}(\tilde{G}_s)$. Moreover, the price is related via the Skorokhod Problem and local times to that of a lookback option (Henderson and Hobson (2000), Delbaen and Yor (2002)).

2.6 Numéraires

We saw in the analysis of the Black-Scholes model that it is convenient to work with discounted prices. This switch can be described as a change of numéraire from cash to bond, and the fundamental and very sound economic principle upon which it is based is that the prices of contingent claims should not depend on the units in which they are denominated.

As well as cash and bond it is sometimes useful to use a risky asset, or the gains from trade of a portfolio of risky assets as numéraire, see Geman *et al* (1995) and Gouriéroux *et al* (1998). For example, consider pricing an exchange option (Margrabe (1978)) with payoff $(X_T - Y_T)^+$, where the price processes X_t and Y_t are given by correlated Brownian motions. Then a change of numéraire from cash to Y_t reduces the pricing problem to that of pricing a standard call in the Black-Scholes model on the single underlying X_t/Y_t .

In general the form of a martingale measure \mathbb{Q} depends on the choice of numéraire N (see Branger (2004)), and for clarity one should consider the pair (N_T, \mathbb{Q}^N) . Alternatively we can fix attention on the state-price-density

$$\zeta_T = \frac{N_0}{N_T} \frac{d\mathbb{Q}^N}{d\mathbb{P}}$$

which is numéraire independent.

2.7 Optimal consumption and investment problems

Consider an agent who can trade in a market as in Section 2.2. Suppose that, rather than trying to price a derivative, the aim of this agent is to maximise his utility of terminal wealth, or alternatively to maximise his utility of consumption over time.

Let $U : \mathbb{R}^+$ (or \mathbb{R}) $\mapsto \mathbb{R}$ be an increasing (to reflect the fact that agents prefer more to less) and concave (to reflect the law of diminishing marginal returns) utility function. Examples include power-law utilities $U(x) = x^{1-R}/(1-R)$, for $R > 0$, logarithmic utility $U(x) = \ln x$ and exponential utility $U(x) = -e^{-x}$, together with various other less tractable families such as

$$U(x) = \kappa^{-1} \left(1 + \kappa x - \sqrt{1 + \kappa^2 x^2} \right) \quad \kappa > 0.$$

The classical Merton problem (Merton, 1969) is to find the optimal trading strategy which maximises the expected utility of terminal wealth $\mathbb{E}U(V_T)$ where V_T is given by (6). In the Black-Scholes model there is a full solution to this optimal control problem. In the primal approach it is possible to write down a Hamilton-Jacobi-Bellman (HJB) equation for the value function of an agent, and then, at least for the case of power law, logarithmic and exponential utilities, to conjecture the form of the solution. In simple cases a standard verification theorem gives that indeed we have a solution of the HJB equation, and the optimal strategy. (In less simple cases the solution of the HJB equation may only exist in the sense of a viscosity solution see Duffie *et al* (1997).)

There is an alternative approach, called the dual method, which gives very powerful insights, see Karatzas (1989) for a survey. The problem is to maximise the expected utility of terminal wealth V_T subject to the wealth satisfying a budget constraint. If we write this in Lagrangian form

$$\mathbb{E}[U(V_T) - \mu(\zeta_T V_T - v)]$$

and introduce the Legendre transform $\hat{U}(y) = \sup_v \{U(v) - vy\}$ of U then

$$\mathbb{E}[U(V_T) - \mu(\zeta_T V_T - x)] \leq \mu v + \mathbb{E}\hat{U}(\mu\zeta_T), \quad (19)$$

with equality when $U'(V_T) = \mu\zeta_T$ almost surely. This inequality holds for all admissible strategies, and all (positive) Lagrange multipliers so we have

$$\sup_{V_T} \mathbb{E}[U(V_T)] \leq \inf_{\mu} \left\{ \mu v + \mathbb{E}\hat{U}(\mu\zeta_T) \right\}. \quad (20)$$

Further, in standard cases (when the asymptotic elasticity of utility is less than one, see Kramkov and Schachermayer (1999)), there is no duality gap and there is equality between the expressions in (20). The optimal solution given by a target wealth V_T^* and a Lagrange multiplier μ^* is such that $V_T^* = I(\mu^*\zeta_T)$ where I is the inverse to the derivative of U . (In fact μ^* is the value of the Lagrange multiplier such that $\mathbb{E}[\zeta_T V_T^*] = \mathbb{E}[\zeta_T I(\mu^*\zeta_T)] = v$.)

In the analysis of the Merton problem for the Black-Scholes model presented here, the dual problem is simpler than the primal problem since the minimisation takes place over a single real-valued Lagrange multiplier rather than a random-variable valued space of terminal wealths. If we think of the dual problem then it is natural to look for utilities whose Legendre transform \hat{U} takes a simple form. For example, consider the class of dual functions given by $\hat{U}''(y) = Ay^{q-2}$ for $q \in \mathbb{R}$ and A a positive constant. The class of associated utility functions is exactly the class of HARA utilities, which includes the power, logarithmic and exponential utilities as special cases, see Merton (1990, p137).

Instead of aiming to maximise expected utility of terminal wealth it is also natural to consider agents who wish to maximise expected discounted utility of consumption over time. Let the wealth process be described by the equation

$$dV_t = \theta_t dX_t + (r - \theta_t X_t)dt - c_t dt$$

where c_t is the consumption rate. Then the problem facing the agent is to maximise

$$\mathbb{E} \left[\int_0^\infty U(t, c_t) dt \right], \quad (21)$$

or more especially to determine optimal investment and consumption pairs $(\theta_t, c_t)_{t \geq 0}$. Again this problem can be attacked via primal or dual methods.

It should be noted that (21) is an unsatisfactory formulation in a couple of ways. Firstly (21) does not arise as the continuous time limit of a realistic situation in which consumption occurs in discrete lumps, and secondly, the value function depends only on the marginal distributions of the consumption process $(c_t)_{t \geq 0}$, and not on the joint distribution. Duffie and Epstein (1992) introduced stochastic differential utilities to address this second issue.

2.8 The successes and failures of the Black-Scholes model

The Black-Scholes model has the property that it is possible to define a unique fair price, the replication price, for any contingent claim. This price is given as the discounted expectation of the option payoff under the unique risk-neutral or martingale measure. The model can be extended to include dividends and to other types of underlyings, such as forwards, futures, indices and foreign exchange rates. Above all the Black-Scholes model has provided a language for the pricing of derivatives and a reference against which modifications of the model can be compared.

In principle, in the Black-Scholes paradigm the option pricing problem is solved, and the solution given in (13), but on occasion it may be difficult to evaluate this stochastic expression and give an analytic pricing formula. Instead practitioners sometimes resort to solving the pde (17), or approximate the price via Monte-Carlo simulation or even solve a multi-period extension of the Cox-Ross-Rubinstein binomial model. In such cases the issue is to execute any of these approaches efficiently and accurately, particularly in high dimensions.

Unfortunately the assumptions of the Black-Scholes model are never satisfied, a theme we return to in Section 4. (It is clear that something must be wrong since the traded prices of different derivatives are frequently, by which we mean invariably, consistent with different values of the volatility parameter.) Continuous trading is impossible, there are taxes, interest rates differ for borrowing and lending, agents are never price takers and face a bid/ask spread, and the prices of underlyings never quite follow exponential Brownian motion with constant known parameters. Understanding and accommodating some of these market frictions and imperfections is one of the main remaining goals of mathematical finance and one of the subjects of the remaining sections.

3 The General Theory

Our aim in this section is to review the analysis we gave in the Samuelson-Black-Scholes exponential Brownian case and to consider the extent to which the results and conclusions generalise to a wider class of models. At first sight it might appear that such generalisations are issues of idle mathematical curiosity. In fact, since the assumptions of the Black-Scholes model clearly fail in practice, it is crucial to understand which results are robust to model misspecification. Our brief survey is based on the discussion in Schachermayer (2003), and the reader who wishes to learn more about the background to the “théorie générale” is referred to that very readable text.

We begin with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T_\infty}, \mathbb{P})$ where T_∞ is a fixed horizon time which is greater than the maturity of any options of interest. We suppose that the discounted price process of the risky asset \tilde{X}_t is a (locally bounded, càdlàg) semi-martingale which is adapted to the filtration \mathcal{F}_t satisfying the usual conditions. The filtration \mathcal{F}_t captures the information available at time t . The process \tilde{X}_t may be vector valued, although our notation will not emphasise this. We have chosen to work with discounted price processes (in part this is just a choice of numéraire), so that although there is a bank account in the model, it does not appear in the analysis.

Already the perceptive reader may wonder whether it is necessary to assume that the price \tilde{X}_t of the risky asset is a semi-martingale. This assumption is very convenient because the well developed theory of stochastic integration is based upon semi-martingales. Further, according to Theorem 7.2 of Delbaen and Schachermayer (1994) if the model is to be consistent with no-arbitrage then the price process must be a semi-martingale, at least when the set of admissible trading strategies is sufficiently large. This rules out certain candidate families of models very quickly. For example fractional Brownian motion is not a semi-martingale. Rogers (1997) gives a direct proof that fractional Brownian motion models admit arbitrage. On the other hand, we cannot take models which are too simple: if the discounted price process is of finite variation then there is also arbitrage.

Our first task is to define the class of admissible portfolios and the associated value functions. Let θ_t be an adapted process which represents the purchases of the risky asset

and define \tilde{V}^θ , the associated self-financing value process with initial wealth \tilde{V}_0 , via

$$\tilde{V}_t^\theta = \tilde{V}_0 + \int_0^t \theta_s d\tilde{X}_s. \quad (22)$$

As before the investment ϕ in the bank account is implicit rather than explicit.

The integral on the right-hand-side of (22) is an Itô stochastic integral. In one sense the choice of the Itô integral is arbitrary — we could equally use the Stratanovich integral, for example, provided we include all the appropriate correction terms. But in another sense the Itô stochastic integral is the only stochastic integral which makes economic sense. To see this observe that if the portfolio θ_t is a simple (piecewise constant) strategy then the discounted gains from trade from investment in the risky asset are given by

$$\tilde{G}_t = \sum_i \theta_{t_i} (\tilde{X}_{t_{i+1} \wedge t} - \tilde{X}_{t_i \wedge t}).$$

In particular the gains process is obtained by multiplying the increments of the price process by the number of units of risky asset held at the beginning of the relevant time interval. The Itô integral shares this non-anticipatory property — it is the integral of the integrand against the forward increments of the integrator.

We now define an admissible strategy as an adapted portfolio process θ_t for which the associated value function is such that the Itô stochastic integral $\int_0^T \theta_t d\tilde{X}_t$ is well defined and \tilde{V}_t^θ defined via (22) is bounded below: $\tilde{V}_t^\theta \geq -M$ for some constant M . This definition is sufficient to rule out doubling strategies, but does not prevent suicide strategies.

The key idea which underpinned pricing in the Black-Scholes model was the notion of an equivalent martingale measure. In general it is too much to expect the underlying to become a martingale under a change of measure, and all we really need is that the discounted traded asset process, and hence the discounted wealth process, becomes a local martingale. We have the following tautological but important definition: a measure \mathbb{Q} , equivalent to \mathbb{P} , under which the discounted asset price is a local martingale is called an equivalent local martingale measure.

Before we discuss option pricing in general we would like to know whether the model we have makes economic sense, and in particular whether it is consistent with no-arbitrage. (If there are arbitrage opportunities in the model — loosely described to be ways of making profits at zero risk — then the model is unsustainable. Some or indeed all agents would want to follow these profit making strategies and the current market prices would not survive in equilibrium.) It turns out that the ‘right’ concept to work with is the idea of ‘no-free-lunch with vanishing-risk’ (NFLVR). Roughly speaking there is a free lunch with vanishing risk if, when you look at the class of contingent claims which can be replicated by an admissible portfolio, and then look at the limits of sequences of such claims, there is a limit random variable which is non-negative almost surely and positive with positive probability. The key result is due to Delbaen and Schachermayer (1994, Corollary 1.2), but see also Harrison and Pliska (1981) for the finite case, Kreps (1981) and Delbaen and Schachermayer (1998).

Theorem 3.1 (First Fundamental Theorem of Asset Pricing.) *Suppose \tilde{X} is a locally bounded semi-martingale. Then there exists an equivalent local martingale measure if and only if the model satisfies NFLVR.*

This theorem is one of the triumphs of the theory of mathematical finance in the abstract semi-martingale setting. It was clear that one side of the if and only if condition

should be the existence of an equivalent (local) martingale measure, since this is a powerful assumption from which many natural and useful properties follow easily. Thus the difficult part of the theorem involved finding the appropriate definitions of admissible strategy and no-arbitrage which would give the martingale measure condition an economically meaningful interpretation.

Since we want to work with economically meaningful models we assume that the model satisfies NFLVR. Hence we are entitled to assume that there exists an equivalent local martingale measure. Set $Z_T = d\mathbb{Q}/d\mathbb{P}$ and $Z_t = \mathbb{E}_t[Z_T]$. Then Z_t and $Z_t\tilde{X}_t$ are both \mathbb{P} -local martingales.

In the general setting we say that a pair (v, θ) is a super-replicating strategy for H if the strategy is admissible and if the associated value process $\tilde{V}^{v, \theta}$ satisfies (22) subject to $\tilde{V}_0^\theta = v$ and $\tilde{V}_T^{v, \theta} \geq \tilde{H}$, the discounted payoff of the claim. Then by the same analysis as before, if (v, θ) is a super-replicating strategy, then $Z_t\tilde{V}_t^\theta$ is a \mathbb{P} -super-martingale and

$$v \geq \mathbb{E}[Z_T\tilde{V}_T^\theta] \geq \mathbb{E}[Z_T\tilde{H}].$$

Hence $\mathbb{E}[Z_T\tilde{H}]$ is a lower bound on the replication price of the option.

This raises the question as to whether there is a super-replicating strategy for the option with initial wealth v . In the one-dimensional Brownian context we have seen how the Brownian martingale representation theorem can be used to produce a replicating strategy. In general it is not always the case that this is possible. The condition under which replicating strategies can be found for all options can again be related to a condition on the equivalent martingale measures, and is again given in Delbaen and Schachermayer (1994).

Theorem 3.2 (Second Fundamental Theorem of Asset Pricing.) *Every bounded claim can be replicated if and only if there is only one equivalent local martingale measure.*

This is the subject of the next section.

4 Incomplete markets

Our analysis of the Samuelson-Black-Scholes model relied on two results from the theory of stochastic processes and Brownian motion. Firstly, the Cameron-Martin-Girsanov Theorem guarantees the existence of an equivalent martingale measure \mathbb{Q} under which the discounted price process X_t is a martingale (or equivalently the existence of a state-price-density ζ_t with the property that $\zeta_t R_t$ and $\zeta_t X_t$ are martingales.) Secondly, the Brownian martingale representation theorem says that any random variable whose value is known at time T can be written as its expected value plus a stochastic integral against Brownian motion. In the Black-Scholes market setting this translates into the result that any option payoff can be written as the price plus the gains from trade from a dynamic investment strategy in the underlying asset.

In the previous section we saw that the existence of a martingale measure is related to the question of whether a model makes economic sense. In this section we consider the role of the martingale representation theorem, and especially the situation in which it is no longer possible to write every claim as the terminal value of a trading strategy.

Recall that R_T , which we no longer assume to be deterministic, is the value of R_0 units of cash invested in the bank account. We say that a contingent claim H is replicable if it

can be written

$$H = \frac{R_T}{R_0} \left(v + \int_0^T \theta_s d\tilde{X}_s \right)$$

for an admissible trading strategy θ , or equivalently if the option payoff can be replicated via a dynamic hedging strategy. In this case there is a unique fair replication price for the option

$$v = \mathbb{E}^{\mathbb{Q}} \left[\frac{R_0}{R_T} H \right] = \mathbb{E}[\zeta_T H]$$

where \mathbb{Q} is any martingale measure and ζ_T is the related state-price density. An option which can be replicated in this way is said to be redundant in the sense that adding the option to the (perfect frictionless) economy has no impact since its payoff can be created synthetically through dynamic trading. If every claim is redundant then the market is complete.

In an incomplete market it is not possible to replicate every contingent claim. For such claims there is no replication price, and the Black-Scholes theory we have introduced has nothing to say about the fair price of the option. Instead we have reached what Hakansson (1979) calls the ‘‘Catch 22 of option pricing’’: the claims we can price are redundant, and the claims that are not redundant we cannot price. The problem facing economists (and financial mathematicians) is to determine a method for pricing non-redundant options which is consistent with the Black-Scholes methodology for those derivatives which can be replicated.

It is clear that if there is more than one state-price density then there exists a claim for which it is possible to define more than one price (via expectation) and hence that that option cannot be replicated. The converse is also true, so that if there exists a unique equivalent local martingale measure then the model is complete and every claim can be replicated. This is the Second Fundamental Theorem of Asset Pricing.

Incompleteness can arise from many sources, for example transaction costs, (Hodges and Neuberger (1989), Davis et al (1993), Soner et al (1995)), jump models (Merton (1976), Bardhan and Chao (1996)), constraints on the trading strategies (Soner and Touzi (2001), Cvitanić and Karatzas (1993)) or stochastic volatility (Hull and White (1987), Heston (1993), Fouque et al (2000)) and to some extent the best approach to pricing and hedging must depend on the context. However, fundamentally, one has to answer the question of how to price and hedge a contingent claim H which is completely independent of the remainder of the model. Our goal is to analyse two simple models which exhibit incompleteness.

4.1 Non-traded Assets

As a first and simple example of an incomplete market (see Davis (2000), Henderson and Hobson (2002a, 2002b), Henderson (2002)) consider an economy with a deterministic bond $R_t = R_0 e^{rt}$ and a single risky asset with dynamics

$$\frac{dX_t}{X_t} = \sigma(dW_t + \lambda dt) + r dt \tag{23}$$

For simplicity we assume that all parameters σ , λ and r are constants. All contingent claims on X can be replicated. Now introduce a second risky asset Y_t with price process

$$dY_t = a_t dW_t' + b_t dt \tag{24}$$

where W' is correlated to W with $dW_t dW'_t = \rho dt$. Suppose that Y is not traded and consider the problem of pricing a contingent claim $H = H(Y_T)$.

The situation we are trying to model is one where an agent has a random endowment H whose payoff depends on an asset Y , but that asset cannot be used for hedging. This may be because of legal reasons (consider an executive who receives compensation in the form of stock options, but who is contractually forbidden from actively trading in stock on his own company (Henderson (2003)) or simply liquidity issues (trading in the asset Y may be so thin as to make hedging with Y impractical). However the agent can use the correlated asset X for hedging.

The Black-Scholes theory tells us that for pricing purposes we should switch to a martingale measure under which the discounted prices of traded assets are martingales, but it does not tell us how to determine the drifts on non-traded assets.

4.2 Stochastic Volatility Models

Consider a market consisting of a bond paying constant rate of interest r and a single risky asset with price process X_t . Suppose that under the physical measure \mathbb{P} the dynamics of the risky asset are given by

$$\frac{dX_t}{X_t} = \sigma(Y_t, t)(dW_t + \lambda(Y_t, t)dt) + rdt \quad (25)$$

where the process driving the volatility is an autonomous diffusion process

$$dY_t = a(Y_t, t)dW'_t + b(Y_t, t)dt \quad (26)$$

where W' is correlated to the Brownian motion W . The problem is to price an option with payoff $H = H(X_T)$.

Stochastic volatility models were introduced to model the empirical fact that historical time series for volatility reveals patterns which indicate that volatility changes randomly over time. Examples include modeling the volatility $\sigma(Y_t, t)$ as a shifted Ornstein-Uhlenbeck process (Stein and Stein (1991)), a square-root or Cox-Ingersoll-Ross process (Hull and White (1988) and Heston (1993)) and an exponential Brownian motion (Hull and White (1987)). There are also jump models for Y , see for example the model popularised by Barndorff-Nielsen and Shephard (2000).

Which model of stochastic volatility should one choose? A good model should be tractable, realistic (for example a shifted Ornstein-Uhlenbeck process can go negative which is an undesirable property) and it should be straightforward to estimate the parameters. Moreover, as well as providing a fit to historic price data the model should also have the ability to explain option price smiles both over strike and over maturity. Finally, the model should give superior hedging performance to the Black-Scholes model.

4.3 Incomplete markets and martingale measures

It is clear from the form of the models in both the non-traded asset and the stochastic volatility cases that these models are incomplete. In a frictionless diffusion model the rule of thumb is that a model is incomplete if the number of sources of randomness is greater than the number of traded assets.

We begin by describing the space of equivalent martingale measures. It is convenient to introduce a Brownian motion B which is independent of W and such that $W'_t = \rho W_t + \bar{\rho} B_t$

where $\bar{\rho}^2 = 1 - \rho^2$. Define

$$Z_t = \exp \left(- \int_0^t \lambda_u dW_u - \frac{1}{2} \int_0^t \lambda_u^2 du - \int_0^t \xi_u dB_u - \frac{1}{2} \int_0^t \xi_u^2 du \right). \quad (27)$$

Provided that $\mathbb{E}[Z_T] = 1$ we can define a (local) martingale measure \mathbb{Q} via a process similar to (14), see Frey (1997). (The first moment condition guarantees that \mathbb{Q} is a probability measure). Then $\zeta_t = e^{-rt} Z_t$ is a state-price density and $\zeta_t X_t$ is a \mathbb{P} (local) martingale. Under \mathbb{Q} , $W_t^{\mathbb{Q}} = W_t + \int_0^t \lambda_u du$ and $B_t^{\mathbb{Q}} = B_t + \int_0^t \xi_u du$ are Brownian motions. Note that the change of drift on W_t is enforced by the requirement that $W_t + \int_0^t \lambda_u du$ is a martingale, whereas the change of drift on B_t is undetermined. The class of changes of measure is parameterised by the process ξ , and we write \mathbb{Q}^ξ and $(W^{\mathbb{Q}^\xi}, B^{\mathbb{Q}^\xi}) \equiv (W^\xi, B^\xi)$ to emphasise this.

It remains to check that \mathbb{Q}^ξ is equivalent to \mathbb{P} , and hence that there exists an equivalent (local) martingale measure and thus there is no arbitrage. The task of checking that a general stochastic exponential such as (27) is a true martingale is a difficult one (the Novikov condition rarely applies), but in the Markovian setting other approaches have recently been developed (see Hobson and Rogers (1998) and Heyde and Wong (2004)) which reduce to checking that certain processes are non-explosive.

It remains to decide if the model is complete. By the (multidimensional) Brownian martingale representation theorem, given the measure \mathbb{Q}^ξ , the discounted option payoff $R_0 H_T / R_T$ can be written as a stochastic integral with respect to the two-dimensional Brownian motion (W_t^ξ, B_t^ξ) :

$$\begin{aligned} \frac{R_0}{R_T} H_T &= v + \int_0^T \psi_t^\xi dW_t^\xi + \int_0^T \chi_t^\xi dB_t^\xi \\ &= v + \int_0^T \frac{\psi_t^\xi}{\bar{X} \sigma(Y_t, t)} d\tilde{X}_t + \int_0^T \bar{\chi}_t^\xi dB_t^\xi \end{aligned}$$

The first two terms correspond to the initial wealth and discounted gains from trade respectively, of a dynamic hedging strategy involving investments in the traded asset and bank account. However it is not possible to trade on the second asset and in general the claim cannot be replicated.

5 Option Pricing in Incomplete Markets

In a complete market the fair prices of options are uniquely determined by the replication price. These prices can be calculated as the discounted expected values under the equivalent martingale measure. In an incomplete market there is no unique fair price and no universal pricing algorithm. Instead there are several alternative methodologies which have been proposed as pricing mechanisms.

The first approach is to finesse the problem by writing down the dynamics of assets under a pricing measure. This approach bypasses the physical measure. A second and related idea (see for example Heston (1993)) is to choose (essentially arbitrarily) a market price of risk for the non-traded assets. For example, the Föllmer-Schweizer (1990) minimal martingale measure corresponds to a choice of a zero market price of risk for the non-traded Brownian motion, or equivalently in our setting $\xi = 0$.

Another idea which has sometimes been exploited in the stochastic volatility literature (see Scott (1987)) is to assume that there is a call option which is liquidly traded. The

introduction of a second traded asset completes the market. Hence, given the traded price of a call option it is possible to price and hedge any other contingent claim. Of course this approach does not explain the price of the original traded call. This idea has been extended by Dupire (1994) to create an elegant (though not very robust) theory for the pricing of exotic options. Suppose that calls with all possible maturities and strikes are traded on the market. Then, under the assumption that the price process possesses the Markov property, it is possible to infer the dynamics of the underlying process. In this approach prices for vanilla options are taken from the market and then used to give prices for path-dependent exotic options. For a more robust version of the idea see Brown *et al* (2001).

The remaining approaches we shall discuss all acknowledge the incompleteness of the market and price options accordingly. Respectively they involve pricing via a hedging criteria, super-replication pricing, minimal distance martingale measures, convex risk measures and utility indifference pricing.

5.1 Hedging criteria

In an incomplete market perfect hedging is impossible. Instead one might aim to minimise some functional of the hedging error. Föllmer and Sondermann (1986) suggest minimising

$$\mathbb{E}[(H - V_T^{v,\theta})^2]$$

over initial wealths v and trading strategies θ . The resulting optimal values are the mean-variance price and hedge respectively. It turns out that in markets with zero interest rates $v = \mathbb{E}[H\zeta_T^{(2)}]$ where $\zeta_T^{(2)}$ is the variance-optimal state-price density which is independent of the choice of derivative H , see Schweizer (1996). For extensions of this idea see Gouriéroux *et al* (1998, stochastic interest rates) and Grandits and Krawczyk (1998, L^p norms on the hedging error.)

An alternative criterion is proposed by Föllmer and Leukert (2000). They propose minimising the shortfall $\mathbb{E}[(H - V_T^{v,\theta})^+]$. This overcomes the disadvantage of the quadratic hedging condition which penalises super-replication, but at the cost of tractability.

5.2 Super-replication pricing

In the discussion on the complete market we introduced the idea of super-replication. In an incomplete market we can use the same notion to define the super-replication price as the smallest initial fortune which is needed to super-hedge the option payoff with probability one. The super-replication price can be thought of as an extreme hedging criteria in which the agent is not willing to accept any risk.

The super-replication price is the supremum of the possible prices which are consistent with no-arbitrage. As such it often gives a price which is unrealistically high. In the non-traded assets model the super-replication price of a call option on Y is infinite (Hubalek and Schachermayer, 2001) whilst in a stochastic volatility model the super-replication price of a call on X is the cost of buying one unit of the underlying (Frey and Sin, 1999).

A key alternative characterisation of the super-replication price is given in El Karoui and Quenez (1995), see also Delbaen and Schachermayer (1994), Föllmer and Kramkov (1997) and Föllmer and Kabanov (1998), as

$$\sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}[R_0 H / R_T]$$

where the supremum is taken over the set of martingale measures. Thus the super-replication price is the price under the worst case martingale measure.

5.3 Minimal Distance Martingale Measures

Rather than choosing a state-price density arbitrarily, one approach is to choose the state-price density which is smallest in an appropriate sense. Given a convex function $f : \mathbb{R}^+ \mapsto \mathbb{R}$ the problem is to minimise $\mathbb{E}[f(\zeta_T)]$ over choices of state-price-density. When interest rates are deterministic and f is homogeneous, this minimisation problem is equivalent to finding the minimal distance martingale measure, the (local) martingale measure \mathbb{Q} which minimises

$$\mathbb{E}[f(Z_T)] \quad (28)$$

where $Z_T = d\mathbb{Q}/d\mathbb{P}$. (Some care is needed in this minimisation procedure as the optimising element may not itself belong to the class of equivalent martingale measures.)

As we pointed out earlier the class of martingale measures depends on the choice of numéraire. However, since incomplete markets involve unhedgeable risks, choice of almost any pricing criterion involves a decision about the units to be used to measure these risks. It seems most natural to use cash for this purpose. Alternatively, if we minimise $\mathbb{E}[f(\zeta_T)]$ then the problem is numéraire independent, and this is another argument for focusing on the state-price density. To date however the mathematical literature has concentrated on the problem of minimising (28). In any case, for the examples we consider, interest rates are deterministic and there is no distinction between the problems of determining the minimal distance state-price density and the minimal distance martingale measure for a cash numéraire.

The problem of finding minimal distance measures has been studied by many authors, but see especially Goll and Rüschendorf (2001) who give various characterisations which determine the optimal \mathbb{Q} in terms of f . One minimal distance measure which has been the subject of particular attention in the literature (for example Rouge and El Karoui (2000) and Frittelli (2000)) is the minimal entropy martingale measure.

Consider now our canonical models of incomplete markets. Suppose, following Hobson (2003a), that we have a representation of the mean-variance trade-off process of the form

$$\frac{1}{2} \int_0^T \lambda^2 du = c + \int_0^T \eta_u (dW_u + \lambda_u) du + \int_0^T \chi_u dB_u + \frac{1}{2} \int_0^T \chi_u^2 du \quad (29)$$

Note that this is an identification of random variables and not of processes, and that the solution consists of a constant c and integrands η and ξ . This equation can be viewed as an example of a Backward Stochastic Differential Equation (BSDE), see Mania *et al* (2003). BSDEs provide a general framework for many characterisation problems in finance (El Karoui *et al*, 1997).

Now consider $f(z) = z \ln z$, and $\mathbb{E}[f(Z_T^\xi)]$ for martingale measures Z_T^ξ given by (27). We have

$$\mathbb{E}[Z_T^\xi \ln Z_T^\xi] = \mathbb{E}^{\mathbb{Q}^\xi} [\ln(d\mathbb{Q}^\xi/d\mathbb{P})]$$

and using the representation (27)

$$\begin{aligned} \ln(d\mathbb{Q}^\xi/d\mathbb{P}) &= - \int_0^T \lambda_u dW_u^\xi + \frac{1}{2} \int_0^T \lambda^2 du - \int_0^T \xi_u dB_u^\xi + \frac{1}{2} \int_0^T \chi^2 du \\ &= c + \int_0^T (\eta_u - \lambda_u) dW_u^\xi + \int_0^T (\chi_u - \xi_u) dB_u^\xi + \frac{1}{2} \int_0^T (\chi_u - \xi_u)^2 du \end{aligned} \quad (30)$$

where we have used (29) and the fact that under \mathbb{Q}^ξ , W^ξ given by $dW_t^\xi = dW_t + \lambda_t dt$ and B^ξ given by $dB_t^\xi = dB_t + \xi_t dt$ are Brownian motions. Then, assuming that the stochastic integrals in (30) are true martingales we have

$$\mathbb{E}[Z_T^\xi \ln Z_T^\xi] = c + \frac{1}{2} \mathbb{E}^{\mathbb{Q}^\xi} \left[\int_0^T (\chi_u - \xi_u)^2 du \right] \geq c$$

with equality for $\xi = \chi$. Hence the problem of finding the minimal entropy martingale measure reduces to finding the solution of (29). More generally, (29) is the special case, corresponding to $q = 1$, of a more general formula which covers distance metrics of the form $f(x) = x^q / (q(q-1))$.

In the non-traded assets model described in Section 4.1 the left-hand side of (29) is constant, and there is a trivial solution corresponding to $\eta \equiv 0 \equiv \chi$. (In this case all the minimal distance measures are identical and equal to the Föllmer-Schweizer minimal martingale measure.) Alternatively, in the stochastic volatility model, if ρ is constant and Y is an autonomous diffusion, then there is a stochastic representation of the solution to (29) given in Hobson (2003a).

Once a minimal distance martingale measure \mathbb{Q}^* has been identified it can be used for pricing in the sense that we can define the option price to be

$$\mathbb{E}^{\mathbb{Q}^*} \left[\frac{R_0}{R_T} H \right] = \mathbb{E}[\zeta_T^* H]$$

where ζ_T^* is the state-price-density associated to the pricing measure \mathbb{Q}^* . The resulting prices are linear in the number of units of claim sold, and as we shall see later they are related to the marginal price of the claim for a utility maximising agent. Further, if we can solve the analogue of (29) for a variety of q , then we can begin to compare option prices under different martingale measures, see Henderson *et al* (2003).

5.4 Convex risk measures

Coherent risk measures were introduced by Artzner *et al* (1999), in an attempt to axiomatise measures of risk (and also to prove that *Value at Risk* was ‘incoherent’). In order to be consistent with the rest of this section we talk about coherent pricing measures for claims rather than measures of risks.

Let $H \in \mathcal{H}$ be a contingent claim. Then $\phi : \mathcal{H} \mapsto \mathbb{R}$ is a coherent pricing measure if it has the properties

Subadditivity	$\phi(H_1 + H_2) \leq \phi(H_1) + \phi(H_2)$
Positive homogeneity	for $\lambda \geq 0$, $\phi(\lambda H) = \lambda \phi(H)$
Monotonicity	$H_1 \leq H_2 \Rightarrow \phi(H_1) \leq \phi(H_2)$
Translation invariance	$\phi(H + m) = \phi(H) + m$

The idea is that ϕ represents the amount of compensation which an agent would demand in order to agree to sell the claim H (or the size of the reserves he should hold if he has outstanding obligations amounting to H). The key result of Artzner *et al* (1999) is that, at least for finite sample spaces, there is a representation of a coherent pricing measure of the form

$$\phi(H) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[H],$$

where \mathcal{Q} is a set of measures. For example the super-replication price is obtained by taking the set \mathcal{Q} to be the set of all martingale measures.

Subsequently, Föllmer and Schied (2002) introduced the notion of a convex risk measure. Convex risk measures attempt to model situations in which the ask price of a claim depends on the number of units sold. The subadditivity and positive homogeneity properties are replaced by a convexity property; for $\mu \in [0, 1]$,

$$\phi(\mu H_1 + (1 - \mu)H_2) \leq \mu\phi(H_1) + (1 - \mu)\phi(H_2).$$

Convex pricing measures are associated with a pricing mechanism which is non-linear in the number of units of the claim. Again there is a representation of a convex pricing measure of the form

$$\phi(H) = \sup_{\mathbb{Q} \in \mathcal{P}} \left\{ \mathbb{E}^{\mathbb{Q}}[H] - \alpha(\mathbb{Q}) \right\},$$

where now \mathcal{P} is the set of all probability measures, and α is a penalty function. For example, to recover the super-replication price we may take $\alpha(\mathbb{Q}) = 0$ if \mathbb{Q} is a martingale measure, and $\alpha(\mathbb{Q}) = \infty$ otherwise.

5.5 Utility Indifference Pricing

Utility indifferent option prices (Hodges and Neuberger (1989)) can be considered as a dynamic version of the notion of a certainty equivalent price in economics. The utility indifference (ask) price is the unique price p at which the agent is indifferent (in the sense that his expected utility under optimal trading is unchanged) between not selling the claim and receiving p now in return for agreeing to make the random payout H at time T .

Consider the problem with k units of the claim. (We take k to be positive if the agent is buying units of claim, and k negative if the agent is short the contingent claim.) Assume that initially the agent has wealth v and zero endowment of the claim. Define

$$u(v, k) = \sup \mathbb{E}[U(V_T + kH_T)]$$

where the supremum is taken over attainable terminal wealths which satisfy the budget constraint $\mathbb{E}[\zeta_T V_T] \leq v$ for all state-price densities ζ_T . Then the utility indifference price $p(k)$ is the solution to

$$u(v, 0) = u(v - p(k), k)$$

Note that if the claim can be replicated then $p(k) = k\mathbb{E}[\zeta_T H]$ for any state-price density ζ_T .

In order to solve for the utility indifference price we need to solve the agent's utility maximisation problem both with and without the claim. In the absence of the claim, the problem is the classical Merton problem in an incomplete market. By analogy with (20) we have an inequality, which holds for all state-price densities, of the form

$$\sup_{V_T} \mathbb{E}[U(V_T)] \leq \inf_{\mu} \inf_{\zeta_T} \left\{ \mu v + \mathbb{E}\hat{U}(\mu\zeta_T) \right\} \quad (31)$$

where \hat{U} is the Legendre transform of U . There is equality in (31) (subject to regularity conditions) if $U'(V_T^0) = \mu^0 \zeta_T^0$ for some optimal target wealth V_T^0 , Lagrange multiplier μ^0 and state-price density ζ_T^0 (the superscript zero corresponds to zero units of the claim). Note that if \hat{U} is a power law, then ζ_T^0 corresponds to a minimal distance state-price-density.

In the case with the option, see Cvitanic *et al* (2001), we have

$$\begin{aligned} U(V_T + kH) - \mu(\zeta_T V_T - v) &= U(V_T + kH) - \mu\zeta_T(V_T + kH) + \mu v + \mu k\zeta_T H \\ &\leq \hat{U}(\mu\zeta_T) + \mu(v + k\zeta_T H) \end{aligned}$$

and then

$$u(v, k) = \sup_{V_T} \mathbb{E}[U(V_T + kH)] \leq \inf_{\mu} \inf_{\zeta_T} \mathbb{E}[\hat{U}(\mu\zeta_T) + \mu(v + k\zeta_T H)]. \quad (32)$$

It follows that if μ^0 and ζ_T^0 are as above

$$u(v - k\mathbb{E}[\zeta_T^0 H], k) \leq \mathbb{E}[\hat{U}(\mu^0 \zeta_T^0) + \mu^0(v - k\mathbb{E}[\zeta_T^0 H]) + k\mu^0 \zeta_T^0 H] = u(v, 0) = u(v - p(k), k)$$

and the bid price for k units satisfies $p(k) \leq k\mathbb{E}[\zeta_T^0 H]$.

With further work, and under further assumptions (see Henderson and Hobson (2002a) Hobson (2003b) and also Hugonnier *et al* (2004)) it is possible to show that for positive claims

$$\lim_{k \downarrow 0} \frac{p(k)}{k} = \mathbb{E}[\zeta_T^0 H]$$

so that the marginal bid price is the discounted expected payoff under a minimal distance state-price density. For small claim amounts it is also possible to consider the total price as an expansion in k , see Henderson and Hobson (2002b) or Henderson (2002).

As an explicit example in the stochastic volatility model suppose $r = 0$ and $U(v) = -e^{-v}$ so that $\hat{U}(y) = y \ln y$. Then, when we take the infimum over μ we find that

$$\inf_{\mu} \inf_{\zeta_T} \mathbb{E}[\hat{U}(\mu\zeta_T) + \mu(v + k\zeta_T H)] = \exp \left(-1 - v - \inf_{\zeta_T} \{k\mathbb{E}[\zeta_T H] + \mathbb{E}[\zeta_T \ln \zeta_T]\} \right)$$

and the option price becomes (see Delbaen *et al* (2002))

$$p(k) = \inf_{\zeta_T} \{k\mathbb{E}[\zeta_T H] + \mathbb{E}[\zeta_T \ln \zeta_T]\} - \inf_{\zeta_T} \{\mathbb{E}[\zeta_T \ln \zeta_T]\}. \quad (33)$$

The problem of minimising the entropy was discussed in Section 5.3, but in general the problem of finding the first infimum in (33) is hard. There are however explicit solutions in the non-traded asset model, see Henderson and Hobson (2002a).

The expression in (33) shows that the utility indifference price for exponential utility corresponds to a convex risk measure. Note that exponential utility is unique in that wealth factors out of the problem, to leave option prices which are independent of wealth. This is a necessary condition for a risk measure.

6 Interest rate modeling

To date we have concentrated on markets in which the underlying is a risky asset which can be modelled by a diffusion process. Now we want to consider an interest rate market in which the characteristics of the traded assets are different. Three canonical texts on the subject are Musiela and Rutkowski (1997), Björk (1998) and Cairns (2004).

Consider a frictionless market in which there is a bank account and a family of zero-coupon bonds. A zero-coupon bond with maturity date T (a T -bond) is a contract which

guarantees to make a unit payment to the holder at time T . A T -bond makes no intermediate payments and is typically a mathematical ideal rather than a genuinely traded instrument. Let the time- t price of the T -bond be denoted by $p(t, T)$, and then $p(T, T) = 1$.

From the bond prices it is possible to deduce the instantaneous forward rates $f(t, T)$ which solve $f(t, T) = -(\partial/\partial T) \ln p(t, T)$ or equivalently $p(t, T) = \exp\{-\int_t^T f(t, s)ds\}$, and the instantaneous short rate $r_t = f(t, t)$. The assumption is that the bank account pays the instantaneous short rate as a stochastic rate of interest, and if so this is equivalent to investing in a portfolio of ‘just maturing’ bonds. Given the relationships between the short-rate, the bond prices and the forward prices we can choose to model any of these.

6.1 Short rate models

Models based on the short rate provide an important subclass of interest rate models. We suppose that the short rate r_t follows dynamics (under \mathbb{P})

$$dr_t = \sigma(t, r_t)(dW_t + \lambda(t, r_t)dt).$$

Examples include taking r_t to be a shifted Ornstein-Uhlenbeck process (Vasicek, 1977) or the sum of squares of OU processes (Cox *et al*, 1985). In a short rate model a zero-coupon bond plays the role of a derivative which is to be priced.

In the light of our previous discussion it is useful to know if the model is arbitrage-free and complete. In fact the discounted price of the traded asset is

$$\frac{R_0}{R_t} R_t = R_0$$

which is constant under any equivalent measure. Thus there exist equivalent martingale measures and every equivalent measure is an equivalent martingale measure. To put this another way, if we fix an equivalent measure \mathbb{Q} , then we can define bond prices via

$$p(t, T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s ds} | \mathcal{F}_t],$$

but these prices are not the only ones consistent with no-arbitrage.

We return to the problem we faced in the previous section: how do we choose an appropriate measure \mathbb{Q} . The two most popular solutions are to finesse the issue by writing down the dynamics under \mathbb{Q} , or to choose a market risk premium γ_t , whence, under \mathbb{Q}

$$dr_t = \sigma(t, r_t)(dW_t + (\lambda(t, r_t) - \gamma_t)dt).$$

Given a martingale measure \mathbb{Q} we can price bonds and more complicated derivatives such as options on bonds and interest rate swaps, and in simple cases we can often find analytical formulae for these quantities. However these instruments cannot be replicated, although, as in a stochastic volatility model, once it is assumed that one bond is traded, all other zero-coupon bonds with shorter maturity can be hedged through dynamic trading in that bond.

6.2 Forward rate models

Short rate models have the feature that the entire interest rate market is governed by a single explanatory variable. It is possible to overcome this drawback, perhaps by including

other interest rates in the model such as the long rate. However short rate models have largely been supplanted in the academic literature and the industry by a paradigm shift in which the fundamental modeling objects become the forward rates. This leads to interesting new mathematics, not least because the state-variable is now a yield curve which is an infinite-dimensional object.

The method we outline was first proposed by Heath, Jarrow and Morton (1992). Let W be a d -dimensional Brownian motion and suppose that for each fixed T the forward rates satisfy

$$df(t, T) = \sigma(t, T)(dW_t + \alpha(t, T)dt). \quad (34)$$

The initial condition $\{f(0, T)\}_{T \geq 0}$ can be specified by the initial market of bond prices and forward rates.

When we switch to the martingale measure \mathbb{Q} , under which the discounted traded quantities (the discounted T -bonds) are martingales, we find that the forward rates satisfy

$$df(t, T) = \sigma(t, T) \left(dW_t^{\mathbb{Q}} + \left(\int_t^T \sigma(s, T) ds \right) dt \right)$$

and that, although the no-arbitrage conditions fix the drifts in (34) there is almost complete freedom in modeling the volatility structure. Once the volatility co-efficients have been specified under \mathbb{P} or \mathbb{Q} the market is complete and any derivative can be priced and replicated using d zero-coupon bonds as hedging instruments.

6.3 Market Models

A more tractable alternative to the class of forward-rate models are the market models of Miltersen, Sandmann and Sondermann (1997) and Brace, Gatarek and Musiela (1997). Instead of concentrating upon the unobservable forward rates a market model takes quoted interest rates such as LIBOR as the fundamental modelling objects. Moreover, these key objects are assumed to have a log-normal distribution. One of the main benefits of this assumption is that it is possible to derive closed form expressions for simple derivatives such as caps and floors.

7 Credit and Default Risk

Financial risks occur in many forms. To date in this article we have been concerned with market risk — the adverse effects of changes in the values of underlying assets or interest rates on the market value of a portfolio. But there are other risks facing agents in financial markets including credit risk, the risk that a counterparty will fail to meet its obligations. Given the recent high profile failures of Enron and WorldCom, these risks have claimed a prominent position in the market psyche.

In a fairly general setting the issue of credit risk can be synthesised into the pricing of bonds issued by a company. In this case the valuation problems inherent in interest rate products are compounded by the risk of default by the issuing company.

There are two main classes of models for credit risk. The first class of models, called structural models, were introduced by Merton (1974) in an attempt to model default via a microeconomic description of the assets and liabilities of the firm. The firm defaults the first time that the assets fall below some threshold. If the assets are described by a diffusion process then this means that default is a predictable event, and it follows

that credit spreads of very short term bonds should be close to zero. Unfortunately this property is not a feature of credit data. There have been various attempts to modify the class of structural models to overcome this failing, for instance by making the price process a jump-diffusion (Zhou, 2001), or allowing for imperfect information (Duffie and Lando, 2001).

The second class of credit risk models are the reduced form or intensity based models. In this class credit events are specified exogenously and default arrives according to a Poisson process with intensity γ_t . These models are somewhat arbitrary, but they provide a good match to data, they are flexible and tractable, and they can be made to fit smoothly into an interest rate framework. For example, if default events happen at rate γ_t then the probability of no default by time t is $\exp(-\int_0^t \gamma_u du)$ and the value of a T -bond (assuming zero recovery on default) is given by

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T (r_u + \gamma_u) du} \right],$$

where expectations are taken with respect to an equivalent martingale measure.

The above descriptions have concentrated on the modeling of default events for a single company, but one of the main problems in credit is to price portfolios of corporate debt, in which case it is necessary to model correlated and dependent default. Schönbucher (2003) gives a full review of credit modeling.

8 Final Thoughts

Mathematical finance is concerned with the related problems of quantifying risk, pricing risk and mitigating the impact of risk via hedging. In general we think of these risks as arising from changes in the prices of underlying assets — stock prices, exchange rates, interest rates — which are specified exogenously to the model. (But one can ask where these prices come from, see for example Bick (1987) or Cox *et al* (1985), and what, if any, are the rational explanations of bubbles and market crashes.) Given the prices of underlyings the beautiful Black-Scholes-Merton theory gives powerful insights into the way derivatives are priced, and leads us to the conclusion that in perfect markets the prices of derivatives are fully determined.

In imperfect markets option prices are not fully determined. Market imperfections can arise in many ways, some of which we have discussed in the article above, and the first challenge facing mathematicians is to model these imperfections in a way which is amenable to analysis. In some markets, such as energy or weather derivatives (Brody *et al*, 2002), exponential Brownian motion is a poor descriptor of the price process. In some markets liquidity issues mean that delta-hedging is infeasible (Cetin *et al*, 2004). In some markets agents may have differential information (Amendinger *et al* (1998), Föllmer *et al* (1999)). In all markets the ways that agents interact and their relative market power (Cvitanic and Ma (1996), Platen and Schweizer (1998), Bank and Baum (2004)) can have a fundamental impact. These problems require careful and sympathetic modeling.

The second challenge facing financial mathematics is to relate the conclusions from these models to real world financial practice. This means that questions of model fit and parameter estimation become crucial together with an acknowledgement that often the behaviour of agents is as much influenced by factors outside the model, such as tax considerations or regulatory issues, as the predictions of a sophisticated mathematical theory.

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A Stochastic Calculus

In this section we review, briefly, the essentials of stochastic calculus that are needed for the derivation of the Black-Scholes formula. Standard texts on Itô processes include Revuz and Yor (1998) and Rogers and Williams (2000), or for more basic treatments motivated solely by the applications to finance consider Mikosch (1998) or Steele (2001).

If $Z_t = f(W_t, t)$ then Itô's formula (Rogers and Williams, 2000, IV.32.8) tells us that (provided the various derivatives exist)

$$Z_t = Z_0 + \int_0^t f'(W_s, s) dW_s + \int_0^t \left[\frac{1}{2} f''(W_s, s) + \dot{f}(W_s, s) \right] ds \quad (35)$$

where the first integral is an Itô stochastic integral and the second is Lebesgue-Stieltjes. Sometimes it is convenient to abbreviate this expression to a stochastic differential equation

$$dZ_t = df(W_t, t) = f'(W_t, t) dW_t + \left[\frac{1}{2} f''(W_t, t) + \dot{f}(W_t, t) \right] dt \quad (36)$$

but this differential version should be interpreted via the stochastic integral representation (35). Itô's formula can be extended to cover functions of semi-martingales $Z_t = f(Y_t, t)$ and to functions of more than one stochastic variable $Z_t = f(Y_t^1, Y_t^2, t)$.

The Cameron-Martin-Girsanov theorem (Rogers and Williams, 2000, IV.38.5) says that if $(\Omega, \mathcal{F}, \mathbb{P})$ is the canonical probability space supporting a Brownian motion W (such that the filtration \mathcal{F}_t satisfies the usual conditions), and if $(Z_t)_{0 \leq t \leq T}$ defined via

$$Z_t = \exp \left(\int_0^t \eta_s dW_s - \frac{1}{2} \int_0^t \eta_s^2 ds \right) \quad (37)$$

is a uniformly integrable martingale then \mathbb{Q} defined via

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_T} = Z_T \quad (38)$$

is equivalent to \mathbb{P} and under \mathbb{Q} , $B_s = W_s - \int_0^s \eta_u du$ is a Brownian motion. Moreover the converse is also true, in the sense that if \mathbb{Q} is equivalent to \mathbb{P} , then \mathbb{Q} has a representation via (38) and (37).

The Brownian martingale representation theorem (Rogers and Williams, 2000, IV.36.5) says that if M_t is a martingale with respect to a filtration \mathcal{F}_t generated by a Brownian motion W_t then M_t can be written

$$M_t = M_0 + \int_0^t \psi_s dW_s$$

for some integrand ψ .

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