Pricing American options with stochastic volatility and model uncertainty

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**Problem: solving the optimal stopping problem** We want to find the payoff (and stopping time) for the following (stochastic volatility) optimal stopping problem:

\[
v(x, y, T) = \sup_{\tau \leq T} \mathbb{E}_{x,y}[e^{-q \tau} g(X_\tau)]
\]

or

\[
v(x, y, T) = \sup_{\tau \leq T} \mathbb{E}_{x,y}[e^{-r \tau} g(e^{r \tau} X_\tau)]
\]

where

\[
X_t = x + \int_0^t \sigma(X_s) Y_s dB_s,
\]

\(Y\) is independent of \(B\) and either

\[
Y_t = y + \int_0^t \eta(Y_s) dW_s + \int_0^t \mu(Y_s) ds
\]

or \(Y\) is a skip-free Markov chain on \(E\), a countable subset of \((0, \infty)\)
Motivated by Jobert and Rogers (2006), where they show the optimal continuation region in the perpetual American put/infinite problem is of the form

\[ C = \{ (x, y) \in \mathbb{R} \times E : x > b(y) \} \quad (1) \]

and give an algorithm to find \( b \).

When \( E \) is large, the algorithm can become very intensive if the ordering of the values of \( \{ b(e) : e \in E \} \) is not known.
Our aim is first to show that, under fairly general conditions, 
\( \nu(x, \cdot, T) \) is increasing and hence if (1) holds then \( b \) is decreasing.

We do this by a coupling argument.

Hobson makes very similar arguments for comparison in the European case.

From now on specialise to stochastic volatility case.

The idea: timechange \( X \) to \( G \) which solves the sde

\[
G_t = x + \int_0^t \sigma(G_s) d\tilde{B}_s
\]

using the timechange \( \Gamma^y = (A^y)^{-1} \) where \( A^Y_t = \int_0^t (Y_s^y)^2 ds \).

Notice that, since \( Y \) is skip-free, \( y' > y \) implies \( A^{y'} \geq A^y \) and \( \Gamma^{y'} \leq \Gamma^y \).
It follows that

\[ v(x, y, t) = \sup_{\rho \leq \Lambda_T^y} \mathbb{E}_x \left[ e^{-q \Gamma^y_\rho} g(G_\rho) \right] \]  \hspace{1cm} (2)

or

\[ v(x, y, t) = \sup_{\rho \leq \Lambda_T^y} \mathbb{E}_x \left[ e^{-r \Gamma^y_\rho} g(e^{r \Gamma^y_\rho} G_\rho) \right]. \]  \hspace{1cm} (3)

In the first case, increasing \( y \) increases the index set and decreases the discount. In the second case we need \( g \) decreasing since the argument of \( g \) increases when \( y \) increases.

The correct coupling argument starts the construction in reverse, by first constructing \( G \) and time-changed versions of \( Y_\rho^y \) and \( Y_\rho^y' \).
Recall that $Y$ satisfies

$$Y_t = y + \int_0^t \eta(Y_s) dW_s + \int_0^t \mu(Y_s) ds.$$  

Drift rates are hard to estimate, so suppose we only know $\mu_\ast \leq \mu \leq \mu^\ast$ and we wish to price the American option. The superhedging price will be

$$V^s(x, y, T) = \sup_{\mu \in \mathcal{M}, \tau \leq T} \mathbb{E}_{x,y}[e^{-q\tau} g(X_\tau)]$$

where

$$\mathcal{M} = \{ \text{adapted processes } \mu \text{ such that } \mu_\ast(Y_t) \leq m_t \leq \mu^\ast(Y_t) \}.$$  

Conversely, the client’s price will be

$$V^b(x, y, T) = \inf_{\mu \in \mathcal{M}} \sup_{\tau \leq T} \mathbb{E}_{x,y}[e^{-q\tau} g(X_\tau)]$$
Point is that as soon as we know that $V$ is increasing in $y$ the candidate drift control is obvious: choose maximum drift to achieve supremum and minimal drift for infimum!

**Sketch proof** (superhedging case): look at HJB equation for stochastic control + optimal stopping problem

$$\max \left( \sup_{m \in [\mu^*, \mu^*]} \left[ \frac{1}{2} y^2 \sigma^2(x) V_{xx}^s + \frac{1}{2} \eta^2(y) V_{yy}^s + m V_y^s - V_t^s - q V^s \right], \right)$$

$$g - V^s = 0$$

(4)

If we take $V^s$ to be the corresponding value of $v$ with $\mu = \mu^*$ then, since $v$ is increasing in $y$, $V_y^s \geq 0$ and so the sup in (4) is attained at $m = \mu^*(y)$. 

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So, since $v$ solves the optimal stopping problem, $e^{-qt}v(X_t, Y_t, T - t)$ is a martingale on the continuation region and equals $g$ on the stopping region.

It follows that

$$\frac{1}{2}y^2\sigma^2(x)V_{xx}^s + \frac{1}{2}\eta^2(y)V_{yy}^s + \mu^*V_y^s - V_t^s - qV_s^s = 0$$

on the continuation region and $g = V^s$ on the stopping region so that $V^s$ satisfies the HJB equation.
Now, what happens if we are only 95% certain that $\mu$ lies in the interval $[\mu_*, \mu^*]$?

If we assume that the payoff is zero when this constraint is broken and denote the stopping time at which the constraint is broken is $\sigma$, then the Lagrangian for the superhedging/pricing problem is

$$V(x, y, T) = \sup_{m \in \mathcal{M}} \sup_{\tau \leq T} \sup_{\sigma} \mathbb{E}_{x, y} [e^{-q\tau} g(X_\tau)1_{\tau < \sigma} + \lambda 1_{\sigma \leq \tau}].$$

It’s (fairly) obvious that this means that

$$V^s(x, y, T) = \sup_{m \in \mathcal{M}} \sup_{\tau \leq T} \mathbb{E}_{x, y} [\max(e^{-q\tau} g(X_\tau), \lambda)].$$
Similarly, get

$$V^b(x, y, T) = \inf_{m \in M} \sup_{\tau \leq T} \mathbb{E}_{x, y} \left[ \min \left( e^{-q\tau} g(X_\tau), \lambda \right) \right].$$

In either case, presence of max or min does not affect monotonicity argument for $V$ and hence for optimal choice of $m$. Continuity of $V$ in $\lambda$ allows calibration in $\lambda$ to obtain the appropriate constrained optimum.