# ON REPRESENTING AND HEDGING CLAIMS FOR COHERENT RISK MEASURES

SAUL JACKA AND ABDELKAREM BERKAOUI

Abstract. We consider the problem of representing claims for coherent risk measures. For this purpose we introduce the concept of (predictable and optional ) time-consistency with respect to a portfolio of assets, generalizing the one defined by Delbaen.

In a similar way we extend the notion of multiplicative, or m-stability, by introducing predictable and optional versions for a portfolio of assets. We then prove that the two concepts of m-stability and time-consistency are equivalent, thus giving necessary and sufficient conditions for a coherent risk measure to be represented by a market with proportional transaction costs. Finally we show that any market with proportional transaction costs is equivalent to a market priced by a coherent risk measure, essentially establishing the equivalence of the two concepts.

August 29, 2008

## 1. Introduction

The biggest practical success of Mathematical Finance to date is in explaining how to hedge against contingent claims (and thus how to price them uniquely) in the context of a complete and frictionless market.

Two relatively recent developments in Mathematical Finance are the introduction of the concept of coherent risk measure and work on trading with (proportional) transaction costs. Both of these developments seek to deal with deviations from the idealised situation decribed above.

Coherent risk measures were first introduced by Artzner, Delbaen, Eber and Heath [1], in order to give a broad axiomatic definition for monetary measures of risk.

In their fundamental theorem, Artzner *et al.* showed that such a coherent risk measure can be represented as the infimum of expectation over a set of test probabilities. Thus the setup includes superhedging under the class of all EMMs (in an incomplete, frictionless market).

---

*Date*: August 29, 2008.

**Key words:** reserving; hedging; representation; coherent risk measure, transaction costs; pricing mechanism; time-consistency; m-stability.

**AMS 2000 subject classifications:** Primary 91B24; secondary 60E05; 91B30; 60G99; 90C48; 46B09; 91B30.

The authors are grateful for many fruitful discussions with Jon Warren on the topics of this paper.

This work was initiated while the second author was a Research Fellow at Warwick University.

This research was partially supported by the grant 'Distributed Risk Management' in the Quantitative Finance initiative funded by EPSRC and the Institute and Faculty of Actuaries.

Recent work on trading with transaction costs by Kabanov, Stricker, Rasonyi, Jouini, Kallal, Delbaen, Valkeila and Schachermayer, amongst others ([14], [15], [13], [6], [19]), lead to a necessary and sufficient condition for the closure of the set of claims attainable for zero endowment to be arbitrage-free (Theorem 1.2 of [12]) and a characterisation of the 'dual' cone of pricing measures (the so-called consistent price processes—see [19]).

In this paper[1], we consider a coherent risk measure as a pricing mechanism (by changing the sign on the argument): in other words we assume that an economic agent is making a market in (or at least reserving for) risk according to a coherent risk measure, $\rho$ say, and they charge or reserve for a random claim $X$ the price $\rho(-X)$.

So, we consider the risk value of a financial claim as the basic price for the associated contract. Unfortunately such a pricing mechanism is not closely linked to the notion of hedging, and so the price evolution from trading time to maturity time is not well-defined. For example, taking the obvious definition for $\rho_t$—the price of risk at time $t$—it is not necessarily true that $\rho = \rho \circ \rho_t$ (see Delbaen [5]). Indeed, Delbaen has given a necessary and sufficient condition for $\rho$ to be time-consistent in this way: the m-stability property ([5]), and this condition is easily violated. Notice that, in the absence of m-stability, reserving is not possible (without 'new business strain'), since the time 0 price of (reserving for) the time-$t$ reserve for a claim $X$ may (and sometimes will) be greater than the time 0 reserve for $X$.

In this paper we first:

(1) define a $v$-denominated risk measure with the same acceptance set as $\rho$, where $v$ is the final value of a positive claim (which we might think of as a different currency).

Then we pursue the idea of pricing using several currencies/commodities/denominations.

If we do this, then the possibility of creating reserves in several currencies becomes available. Moreover, the possibility of trading between currencies or commodities in order to hedge a contingent claim also appears.

Our main results are as follows:

(2) in Theorem 3.1 we give a necessary and sufficient condition (which generalises Delbaen's m-stability property) for time-consistency with respect to a portfolio of assets (we term this condition *predictable m-stability*);
(3) in Theorem 3.5, we give necessary and sufficient conditions (akin to Schachermayer's description of the cone of consistent price processes) for the attainability of all acceptable claims purely by trading in a portfolio of assets;
(4) finally, we show, in Theorem 4.3, that every arbitrage-free market corresponding to trading with transaction costs in fact corresponds (when restricted to $L^\infty$) to the representation of a coherent risk measure using a set of commodities or numéraires .

[1]An earlier version of this paper, [11], discussed some topics which, in the interests of brevity and readability, we here omit, and also contained proofs of several results which we now leave to the reader.

## 2. Pricing measures, time consistency and multiplicative stability

2.1. **Conditional coherent risk measures and pricing measures.** In the paper we will be dealing with pricing monetary risks in the future and, in general, in the presence of partial information. Accordingly, we recall the definition and the main result on the characterization of a *conditional* coherent risk measure. This concept was introduced by Wang [20] and has been further elaborated upon within different formal approaches by Artzner *et al.* [2], Riedel [17], Weber [21], Engwerda et al. [8], Scandolo [18], Detlefsen and Scandolo [7].

We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, \mathbb{P})$. We denote the corresponding appropriate Lebesgue spaces by $\mathcal{L}^p(\mathcal{F}_t, \mathbb{R}^d)$ $(0 \le p \le \infty)$ or just $\mathcal{L}^p$ where no confusion as to the $\sigma$-algebra may arise. Positive (non-negative) elements are denoted by $\mathcal{L}^p_+$. We reserve the notation $L^p$ for the case $\mathcal{L}^p(\mathcal{F}_t, \mathbb{R})$. Throughout this paper we use the term *arbitrage-free* of a subset $S$ of $\mathcal{L}^0(\mathcal{F}, \mathbb{R}^d)$ to mean that $S \cap \mathcal{L}^0_+(\mathcal{F}, \mathbb{R}^d) = \{0\}$.

Throughout this subsection we consider the mapping $\rho_0 : L^\infty(\mathcal{F}) \to L^\infty(\mathcal{F}_0)$.

**Definition 2.1.** *(See Detlefsen and Scandolo* [7]*) We say that the mapping $\rho_0$ is a relevant, conditional coherent risk measure with the Fatou property if it satisfies the following axioms:*

(1) *Monotonicity: For every $X, Y \in L^\infty(\mathcal{F})$,*
$$X \le Y \text{ a.s } \Rightarrow \rho_0(X) \ge \rho_0(Y) \text{ a.s.}$$
(2) *Subadditivity: For every $X, Y \in L^\infty(\mathcal{F})$,*
$$\rho_0(X+Y) \le \rho_0(X) + \rho_0(Y) \text{ a.s.}$$
(3) *$\mathcal{F}_0$-Translation invariance: For every $X \in L^\infty(\mathcal{F})$ and $y \in L^\infty(\mathcal{F}_0)$,*
$$\rho_0(X+y) = \rho_0(X) - y \text{ a.s.}$$
(4) *$\mathcal{F}_0$-Positive homogeneity: For every $X \in L^\infty(\mathcal{F})$ and $a \in L^\infty_+(\mathcal{F}_0)$, we have*
$$\rho_0(a\,X) = a\,\rho_0(X) \text{ a.s.}$$
(5) *The Fatou property: a.s $\rho_0(X) \le \liminf \rho_0(X_n)$, for any sequence $(X_n)_{n\ge1}$ uniformly bounded by 1 and converging to $X$ in probability.*
(6) *Relevance: for each set $F \in \mathcal{F}$ with $\mathbb{P}[F|\,\mathcal{F}_0] > 0$ a.s, $\rho_0(1_F) < 0$ a.s.*

In accordance with our aim of interpreting $\rho_0$ (a risk measure) as a *pricing* mechanism, we introduce a change of sign in Definition 2.1 to define a *pricing measure*; so

**Definition 2.2.** *we define $p_0$ as a* conditional pricing measure *if $X \mapsto p_0(-X)$ satisfies Definition 2.1.*

We give the corresponding properties for a pricing mechanism derived from this sign change. So

**Proposition 2.3.** *(See Detlefsen and Scandolo* [7]*) Let the mapping $\rho_0$ be a conditional pricing mechanism. Then*

(1) *The* acceptable risk set

$$\mathcal{A}_0 \stackrel{def}{=} \{X \in L^\infty(\mathcal{F})\ ;\ \rho_0(X) \leq 0\ a.s\}$$

*is a weak*-closed convex cone, arbitrage-free, stable under multiplication by bounded positive $\mathcal{F}_0$-measurable random variables and contains $L^\infty_-(\mathcal{F})$.*

(2) *There exists a convex set of probability measures $\mathcal{Q}$, all of them being absolutely continuous with respect to $\mathbb{P}$ and containing at least one equivalent probability measure, such that for every $X \in L^\infty(\mathcal{F})$:*

$$\rho_0(X) = \textit{ess-sup}\,\{\mathbb{E}_\mathbb{Q}(X|\,\mathcal{F}_0)\ ;\ \mathbb{Q} \in \mathcal{Q}\}\,. \tag{2.1}$$

**Remark 2.4.** *If one is being very careful one might note that the essential supremum in (2.1) is a little ambiguous, since $E_\mathbb{Q}[\cdot|\mathcal{F}_0]$ is only defined up to $\mathbb{Q}$-a.s. equality. To be more careful but less transparent, we could replace $\mathbb{E}_\mathbb{Q}(X|\,\mathcal{F}_0)$ by $\mathbb{E}_\mathbb{P}(X\frac{d\mathbb{Q}}{d\mathbb{P}}|\,\mathcal{F}_0)$ in (2.1) and elsewhere in this paper where such essential suprema occur.*

**Definition 2.5.** *Given a conditional pricing mechanism $\rho_0$, we define $\mathcal{Q}^{\rho_0}$ as follows:*

$$\mathcal{Q}^{\rho_0} = \left\{\textit{probability measures } \mathbb{Q} \ll \mathbb{P}\ :\ \frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathcal{A}_0^*\right\}, \tag{2.2}$$

*where $\mathcal{A}_0^*$ is the polar cone of $\mathcal{A}_0$ (we take this in $L^1$). Conversely, given $\mathcal{Q}$, a collection (not necessarily closed, or convex) of probability measures absolutely continuous with respect to $\mathbb{P}$, we define*

$$\rho_0^{\mathcal{Q}}(X) = \textit{ess-sup}\,\{\mathbb{E}_\mathbb{Q}(X|\,\mathcal{F}_0)\ :\ \mathbb{Q} \in \mathcal{Q}\}\,. \tag{2.3}$$

*The set $\mathcal{Q}^{\rho_0}$ is the largest set $\mathcal{Q}$ for which $\rho_0 = \rho_0^{\mathcal{Q}}$.*

2.2. **Change of numéraire.** First, we do the following:

(1) we fix a pricing mechanism, $\rho : L^\infty \to \mathbb{R}$ with acceptable risk set $\mathcal{A}$ and test probabilities $\mathcal{Q}$, a maturity time $T$ and a unit of account $\mathbf{1}$. The unit of account $\mathbf{1}$ is interpreted as a contract that pays one pound at time $T$, i.e. a zero coupon bond with redemption value of one pound.

(2) we suppose that trading is frictionless at time $T$ and then for any claim or asset $\hat{X}$, we denote by $X$ its value in terms of the unit of account $\mathbf{1}$ at time $T$.

**Definition 2.6.** Numéraires. *We* define *$\mathcal{N}$, the set of all numéraires by:*

$$\mathcal{N} = \{v \in L^\infty_+ :\ \frac{1}{v} \in L^\infty_+\}, \tag{2.4}$$

*and refer to any element of $\mathcal{N}$ as a numéraire.*

Now we make the natural extensions to acceptable risk sets and pricing mechanisms. We define the *acceptable risk set at time $t$*:

$$\mathcal{A}_t \stackrel{def}{=} \{X \in L^\infty\ :\ \mathbb{E}_\mathbb{Q}(X|\mathcal{F}_t) \leq 0\ \mathbb{P}\text{ a.s. for all } \mathbb{Q} \in \mathcal{Q}\},$$

and we define the *time-$t$ price* (associated with $\rho$), $\rho_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$, by

$$\rho_t(X) = \text{ess-inf}\,\{m \in L^\infty(\mathcal{F}_t) :\ X - m \in \mathcal{A}_t\}\,.$$

**Remark 2.7.** *It is easy to show that*

$$\mathcal{A}_t = \{X : \alpha X \in \mathcal{A} \text{ for all } \alpha \in L^\infty(\mathcal{F}_t)\}$$

*and that*

$$\rho_t(X) = \underset{\mathbb{Q}\in\mathcal{Q}}{\text{ess-sup}}\, \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t).$$

Finally we give the change of numéraire "recipe" for a pricing mechanism. Given a numéraire, we may, of course use it as a new unit of account. First we define a measure that prices claims (expressed in units of account **1**), in terms of contracts which pay the new numéraire. We might, for example, think of the original numéraire as paying sterling at time $T$ and the new one as paying US dollars at time $T$.

**Definition 2.8.** *Let $v \in L^\infty_+$. The mapping $\tau : L^\infty(\mathcal{F}) \to L^\infty(\mathcal{F}_0)$ is said to be a $v$-*denominated, conditional pricing mechanism, with respect to $\mathcal{F}_0$ *if it satisfies properties 1, 2, 4, 5 and 6 of Definition 2.1 and $\mathcal{F}_0$-translation invariance with respect to $v$, i.e for every $X \in L^\infty(\mathcal{F})$ and $y \in L^\infty(\mathcal{F}_0)$, we have*

$$\tau(X + yv) = \tau(X) + y \text{ a.s.} \tag{2.5}$$

We may define

**Definition 2.9.** *For all $v \in \mathcal{N}$ and $t = 0, 1, \ldots, T$, we define the mapping $\rho^v_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ by*

$$\rho^v_t(X) = \text{ess-inf}\{m \in L^\infty(\mathcal{F}_t) : X - mv \in \mathcal{A}_t\}.$$

Then it is easy to show

**Lemma 2.10.** *For all $v \in \mathcal{N}$ and $t = 0, 1, \ldots, T$, the mapping $\rho^v_t$ as defined in Definition 2.9, is a $v$-denominated, conditional pricing mechanism, given by*

$$\rho^v_t(X) = \underset{\mathbb{Q}\in\mathcal{Q}}{\text{ess-sup}} \left( \frac{\mathbb{E}_{\mathbb{Q}}(X|\,\mathcal{F}_t)}{\mathbb{E}_{\mathbb{Q}}(v|\,\mathcal{F}_t)} \right),$$

*and*

$$\mathcal{A}_t = \{X \in L^\infty : \rho^v_t(X) \le 0 \ \mathbb{P} \text{ a.s}\}.$$

To conclude this subsection, we state (without proof) the following

**Lemma 2.11.** *Suppose that $0 \le t \le t + s \le T$ and $v, w^1, ..., w^d \in \mathcal{N}$. Then whenever $X_1, \ldots, X_d \in L^\infty$ with $X = X_1 + ... + X_d$ we have:*

$$\rho^v_t(X) \le \rho^v_t \left( \sum_{i=1}^d \rho^{w^i}_{t+s}(X_i) w^i \right).$$

*In particular, if $v \in \mathcal{N}$, then*

$$\rho^v_t(X) \le \rho^v_t \left( \rho^v_{t+1}(X) v \right).$$

2.3. **Time-consistency properties.** As we discussed in the introduction, the essential element of pricing or hedging in a financial market is to build a financing strategy that starts with the price of a claim and ends with a value equal to the claim itself at maturity. Speaking loosely (for now), if this strategy is built by trading in a specific set of assets $V = (v^1, \dots, v^d)$, we shall say that the claim is *represented* by the vector $V$. Notice that if we allow ourselves a large enough collection of assets, then representation is always possible: to hedge the bounded claim $X$ we need only buy (and hold from time 0 onwards) an asset whose value at time $T$ is $X$. Indeed, even if we require assets to be numéraires, we may simply buy and hold an asset worth, say $X - \text{ess-inf}\, X + 1$, and then hold $\text{ess-inf}\, X - 1$ in cash. Interest, therefore, should be focused on choosing a parsimonious collection of representing numéraires, $V$.

Delbaen gave the following definition in [5]:

**Definition 2.12.** *A coherent risk measure $\rho$ is said to be* time-consistent *if for all $0 \leq t \leq t+s \leq T$ we have $\rho_t \circ \rho_{t+s} = \rho_t$.*

We generalize this concept in the following

**Definition 2.13.** Predictable time-consistency *Let $v_1, \dots, v_d \in \mathcal{N}$ and $V = (v_1, \dots, v_d)$. We say that $\mathcal{A}$ is* predictably $V$-time-consistent *if for each $i$, $t \in \{0, 1, \dots, T-1\}$ and $X \in L^\infty$, there exist sequences $X_n \in L^\infty$ and $Y^n_{t+1} \in L^\infty(\mathcal{F}_{t+1}; \mathbb{R}^d)$*

(i) *the sequence $X_n$ converges weakly* to $X$ in $L^\infty$,*
(ii) *$X_n - Y^n_{t+1}.V \in \mathcal{A}_{t+1}$,*

*and*

(iii) *$\rho^v_t(X) = \liminf_{n\to+\infty} \rho^v_t\left(Y^n_{t+1}.V\right)$.*

*In particular, we say that $\mathcal{A}$ is predictably time-consistent if $\mathcal{A}$ is predictably $\{1\}$-time-consistent.*

**Definition 2.14.** Optional time-consistency *Let $V$ be as above. We say that $\mathcal{A}$ is* optionally $V$-time-consistent *(or optionally time-consistent with respect to $V$) if for each $i$, $t = 0, 1, \dots, T-1$ and $X \in L^\infty$, there exist sequences $X_n \in L^\infty$ and $Y^n_t \in L^\infty(\mathcal{F}_t; \mathbb{R}^d)$ such that*

(i) *the sequence $X_n$ converges weakly* to $X$ in $L^\infty$,*
(ii) *$X_n - Y^n_t.V \in \mathcal{A}_{t+1}$,*

*and*

(iii) *$\rho^v_t(X) = \liminf_{n\to+\infty} \rho^v_t\left(Y^n_t.V\right)$.*

*We say that $\mathcal{A}$ is optionally time-consistent if $\mathcal{A}$ is optionally $\{1\}$-time-consistent.*

These definitions are actually equivalent to the following, apparently stronger, ones as Theorem 2.17 will show.

Given a cone $D$ in $L^\infty$ and a vector $V$ of numéraires, we define $D(V)$, the collection of portfolios attaining $D$, by

$$D(V) = \{X \in \mathcal{L}^\infty(\mathcal{F};\mathbb{R}^d) : X.V \in D\}.$$

**Definition 2.15.** Predictable representation. *We say that the cone of acceptable risks, $\mathcal{A}$, is* predictably represented *by the $\mathbb{R}^d$-valued vector of numéraires $V$ if the cone $\mathcal{A}(V)$ is* predictably decomposable*, i.e*

$$\mathcal{A}(V) = \overline{\oplus_{t=0}^{T-1} K_t(\mathcal{A}, V)},$$

*where $K_t(\mathcal{A}, V) \stackrel{def}{=} \mathcal{A}_t(V) \cap \mathcal{L}^\infty(\mathcal{F}_{t+1};\mathbb{R}^d)$ and the closure is taken in the weak* topology.*

Thus *predictable representation* means that every element of $\mathcal{A}$ is attainable by a collection of one-period bets in units of $V$ at times $0, \ldots, T-1$ and trades at times $0, \ldots, T$.

**Definition 2.16.** Optional representation. *We say the cone $\mathcal{A}$ is* optionally represented *by the $\mathbb{R}^d$-valued vector of numéraires $V$ if the cone $\mathcal{A}(V)$ is* optionally decomposable*, i.e*

$$\mathcal{A}(V) = \overline{\oplus_{t=0}^{T} \mathcal{C}_t(\mathcal{A}, V)},$$

*where $\mathcal{C}_t(\mathcal{A}, V) \stackrel{def}{=} \mathcal{A}_t(V) \cap \mathcal{L}^\infty(\mathcal{F}_t;\mathbb{R}^d)$ and the closure is taken in the weak* topology.*

Thus *optional representation* means that every element of $\mathcal{A}$ is attainable by a collection of trades in units of $V$ at times $0, \ldots, T$.

Now we give the equivalence between representation of the cone $\mathcal{A}$ by a finite portfolio $V$ and $V$-time consistency.

**Theorem 2.17.** *Let $v_1, \ldots, v_d \in \mathcal{N}$ and $V = (v_1, \ldots, v_d)$. Then $\mathcal{A}$ is optionally (respectively predictably ) $V$-time-consistent if and only if it's optionally (respectively if it's predictably ) represented by $V$.*

*Proof.* See Appendix A.1 □

**Remark 2.18.** *The following assertions are obviously equivalent:*

(1) *$\mathcal{A}$ is predictably represented (respectively, optionally represented) by $V$ and the cone $\oplus_{t=0}^{T-1} K_t(\mathcal{A}, V).V$ (respectively $\oplus_{t=0}^{T-1} \mathcal{C}_t(\mathcal{A}, V).V$) is closed.*
(2) *For all $t \in \{0, \ldots, T-1\}$ (respectively, for all $t \in \{0, \ldots, T\}$), $v \in V$ and $X \in L^\infty$, there exists some $Y \in L^\infty(\mathcal{F}_{t+1};\mathbb{R}^d)$ (respectively, in $L^\infty(\mathcal{F}_t;\mathbb{R}^d)$) such that $X - Y.V \in \mathcal{A}_{t+1}$ and*

$$\rho_t^v(X) = \rho_t^v(Y.V).$$

*In particular, if either of the statements (1) or (2) holds in the case of predictable representation, then, we have that for all $t \in \{0, \ldots, T-1\}$, $v \in V$ and $X \in L^\infty$, there exists some $X_1, \ldots, X_d \in L^\infty(\mathcal{F})$ such that $X = \sum_{i=1}^d X_i$ and*

$$\rho_t^v(X) = \rho_t^v\left(\sum_{i=1}^d \rho_{t+1}^{v_i}(X_i) v_i\right).$$

**Remark 2.19.** *Since the cone $\oplus_{t=0}^{T-1} K(\mathcal{A}, \{1\})$ is closed, the time-consistency property introduced by Delbaen is equivalent to the predictable time-consistency property.*

2.4. **Stability properties.** We first define our generalisation of m-stability in a vector-valued context. Henceforth, we use the following notational convention: if $Y \in \mathcal{L}^1(\mathcal{F}, \mathbb{R}^k, \mathbb{P})$ for some $k \geq 1$ then

$$Y_t \stackrel{def}{=} \mathbb{E}_{\mathbb{P}}[Y|\mathcal{F}_t].$$

**Definition 2.20.** Predictable m-stability. *Let $D \subset \mathcal{L}^1_+(\mathcal{F}, \mathbb{R}^d)$. We say that $D$ is* predictably m-stable *if whenever there exist a stopping time $\tau \leq T$, $Z = (Z^1, \ldots, Z^d)$ and $W = (W^1, \ldots, W^d)$ in $D$ with*

$$\alpha = \frac{Z^i_\tau}{W^i_\tau} \text{ being the same for each } i, \tag{2.6}$$

*then $X$, given by*

$$X = \alpha W,$$

*is also in $D$.*

We define optional m-stability in a similar fashion.

**Definition 2.21.** Optional m-stability. *We say that $D$ is* optionally m-stable *if, whenever $\tau \leq T-1$ is a stopping time, and $Z$ and $W$ are in $D$ and there exist $R^i_{\tau+1} \in L^1_+(\mathcal{F}_{\tau+1})$ $(1 \leq i \leq d)$ such that $\mathbb{E}[R^i_{\tau+1}|\mathcal{F}_\tau] = 1$ and*

$$\alpha = \frac{Z^i_\tau R^i_{\tau+1}}{W^i_{\tau+1}} \text{ is the same for each } i, \tag{2.7}$$

*then $X$, given by*

$$X = \alpha W,$$

*is also in $D$.*

Now we extend these definitions to our context as follows:

**Definition 2.22.** $V$-m-stability. *Let $V$ be a $d$-dimensional vector of elements of $\mathcal{N}$ and $D \subset L^1_+$. We say that $D$ is* predictably (respectively optionally )) $V$-m-stable *if $DV \stackrel{def}{=} \{YV : Y \in D\}$ is predictably (respectively) optionally m-stable.*

In particular we say that $D$ is predictably m-stable when $D$ is predictably $\{1\}$-m-stable. This agrees with the definition of Delbaen [5], since condition (2.6) is then automatically satisfied.

Note that optional m-stability implies predictable m-stability.

**Example 2.23.** *Denoting by $\mathbb{M}(S)$ the set of all EMMs of a strictly positive bounded $\mathbb{R}^d$-valued process $(S_t)_{t=0,1,\ldots,T}$ with $S^1_\cdot \equiv 1$, the set $\mathbb{M}(S)$ is strongly $S_T$-m-stable, on identifying probability measures with their densities with respect to $\mathbb{P}$.*

*Proof.* Fix $\Lambda$ and $M$ in $\mathbb{M}(S)$ and $\tau$ and $R_{\tau+1}$ such that

$$\alpha = \frac{\Lambda_\tau R^i_{\tau+1} S^i_\tau}{M_{\tau+1} S^i_{\tau+1}} \text{ is the same for each } i.$$

We claim that $X$, given by

$$X = \alpha M$$

is in $\mathbb{M}(S)$.

Fix $t \in \{0, 1, \ldots, T\}$, then $X_t = \mathbb{E}[X|\mathcal{F}_t] = \mathbb{E}[\frac{\Lambda_\tau R^1_{\tau+1} S^1_\tau}{M_{\tau+1} S^1_{\tau+1}} M|\mathcal{F}_t] = \alpha M_t 1_{t \geq \tau+1} + \Lambda_t 1_{t \leq \tau}$, whilst

$$\mathbb{E}[\alpha M S^i_T|\mathcal{F}_t] = \alpha M_t S^i_t 1_{t \geq \tau+1} + \Lambda_t S^i_t 1_{t \leq \tau} = X_t S^i_t$$

as required. □

## 3. MAIN RESULTS

We state the equivalence between representation of the cone $\mathcal{A}$ by the finite portfolio $V$ and $V$-m-stability of its polar cone. Note that throughout this paper, when we have a weak*-closed cone, $\mathcal{B}$, in $\mathcal{L}^\infty$ we take its polar cone in $\mathcal{L}^1$ and denote it by $\mathcal{B}^*$. In this way we obtain the desirable reflexivity property that $(\mathcal{B}^*)^* = \mathcal{B}$.

**Theorem 3.1.** *$\mathcal{A}$ is predictably (respectively optionally ) represented by $V$ if and only if $\mathcal{A}^*$ is predictably (respectively optionally ) $V$-m-stable.*

The result will follow from the following more abstract version of the theorem:

**Theorem 3.2.** *Suppose that $\mathcal{B}$ is a weak*-closed convex cone in $\mathcal{L}^\infty(\mathcal{F};\ \mathbb{R}^d)$ which is arbitrage-free. Then $\mathcal{B}$ is predictably decomposable (respectively optionally decomposable)$\Leftrightarrow$ $\mathcal{B}^*$ is predictably stable (respectively optionally stable).*

*Proof.* See Appendix A.2 □

*Proof of Theorem 3.1:* To prove Theorem 3.1 assuming Theorem 3.2, all we need do is establish

**Lemma 3.3.** *Under our assumptions on $V$, if $D$ is a convex cone in $L^\infty(\mathcal{F})$ then we have*

$$D(V)^* = D^* V \stackrel{def}{=} \{ZV :\ Z \in D^*\}.$$

*In particular, the polar of the cone of portfolios $\mathcal{A}(V)$ is given by:*

$$\mathcal{A}(V)^* = \mathcal{A}^* V \stackrel{def}{=} \{ZV :\ Z \in \mathcal{A}^*\}, \tag{3.1}$$

since then the result is an immediate consequence of Theorem 3.2 and (3.1).

*Proof of Lemma 3.3:* First take $Z \in D^*$; then, for any $X \in D(V)$, $\mathbb{E}ZV.X \leq 0$ since $X.V \in D$. It follows that $ZV \in D(V)^*$, and so we conclude that

$$D(V)^* \supset VD^*.$$

To prove the reverse inclusion, denote the $i$th canonical basis vector in $\mathbb{R}^d$ by $e_i$. Now note first that, since $V.(\alpha(v_ie_j - v_je_i)) = 0$, $\alpha(v_ie_j - v_je_i) \in D(V)$ for any $\alpha \in L^\infty$. It follows that if $Z \in D(V)^*$ then $Z.(v_ie_j - v_je_i) = 0$ and so any $Z \in D(V)^*$ must be of the form $WV$ for some $W \in L^1(\mathcal{F}_T)$. Now given $C \in D$, take $X$ such that $X.V = C$ (which implies that $X \in D(V)$), then $0 \geq \mathbb{E}WV.X = \mathbb{E}WC$ and, since $C$ is arbitrary, it follows that $W \in D^*$. Hence $D(V)^* \subset VD^*$. □

For the sake of completeness, we give a converse to Lemma 3.3 in Appendix A.4.

The following example explicitly gives the predictable representation of an element of $\mathcal{A}$ for the given pricing mechanism.

**Example 3.4.** *Consider a binary branching tree with two branches. So $\Omega = \{1, 2, 3, 4\}$, $\mathcal{F}_0$ is trivial, $\mathcal{F}_1 = \sigma(\{1, 2\}, \{3, 4\})$ and $\mathcal{F}_2 = 2^\Omega$. Equating each probability measure $\mathbb{Q}$ on $\Omega$ with the vector of probability masses $(\mathbb{Q}(\{1\}), \mathbb{Q}(\{2\}), \mathbb{Q}(\{3\}), \mathbb{Q}(\{4\}))$, take $\mathcal{Q} = conv(\mathbb{Q}_1, \mathbb{Q}_2)$, where $\mathbb{Q}_1 = (\frac{1}{3}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4})$ and $\mathbb{Q}_2 = (\frac{1}{2}, \frac{1}{8}, \frac{3}{16}, \frac{3}{16})$ (here and henceforth, conv denotes the convex hull). Let $\rho$ be the associated pricing mechanism.*

*Denoting $X \in L^\infty$ by the corresponding (lower case) vector $(x_1, x_2, x_3, x_4)$ (so that $X(i) = x_i$) we see that*

$$\rho(X) = \max\left(\frac{1}{3}x_1 + \frac{1}{6}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4, \frac{1}{2}x_1 + \frac{1}{8}x_2 + \frac{3}{16}x_3 + \frac{3}{16}x_4\right),$$

*and*

$$\rho_1(X)(\omega) = \begin{cases} \max(\frac{2}{3}x_1 + \frac{1}{3}x_2, \frac{4}{5}x_1 + \frac{1}{5}x_2) : & \omega \in \{1, 2\} \\ \frac{1}{2}x_3 + \frac{1}{2}x_4 : & \omega \in \{3, 4\} \end{cases}$$

*Take $X = (3, 4, 0, 0)$ to see that $\rho \circ \rho_1 \neq \rho$:*

$$\rho_1(X) = \begin{cases} \max(\frac{10}{3}, \frac{16}{5}) = \frac{10}{3} : & \omega \in \{1, 2\} \\ 0 : & \omega \in \{3, 4\}, \end{cases}$$

*and $\rho(\rho_1(X)) = \max(\frac{5}{3}, \frac{25}{12}) = \frac{25}{12}$, whereas $\rho(X) = \frac{5}{3}$.*

*Now, setting $v = 1 + 1_{\{1\}}$ and $\tilde{x} = \frac{1}{2}(x_3 + x_4)$, it is easy to check that*

$$X = W_X + Z_X + \Delta_X,$$

*where*

$$W_X = 2x_2 - x_1,\ \Delta_X = \frac{1}{2}(x_3 - x_4)(1_{\{3\}} - 1_{\{4\}}),\ Z_X = ((x_1 - x_2)1_{\{1,2\}} + (x_1 + \tilde{x} - 2x_2)1_{\{3,4\}})v.$$

*We claim that $\rho$ is predictably $(1, v)$-time-consistent.*

*Proof.* To check this, first take $V = (1, v)$ and $Y_2^n = Y_2 = (X, 0)$ for each $n$, then $X - Y_2.V = 0 \in \mathcal{A}_2$. Now take $Y_1^n = Y_1 = (W_X, \frac{Z_X}{v})$ for each $n$, then $X - Y_1.V = \Delta_X$ and $\rho_1(\Delta_X) = 0$, so $\Delta_X \in \mathcal{A}_1$. Finally, $\rho_1(X) = \rho_1(Y_2.V)$ (obviously) while it is easy to see that $\rho(Y_1.V) = \rho((x_1, x_2, \tilde{x}, \tilde{x})) = \rho(X)$ so the result follows. □

Now we state the main condition for representation in the optional case.

**Theorem 3.5.** *$\mathcal{A}$ is optionally represented by $V$ if and only if there exists a collection $(G_t)_{t=0}^T$, with each $G_t$ being an $\mathcal{F}_t$-measurable random, closed, convex cone in $\mathbb{R}^d$ such that:*

$$\mathcal{A}^* = \bigcap_{t=0}^{T} \{Z \in \mathcal{L}^1(\mathcal{F});\ \mathbb{E}(ZV|\mathcal{F}_t) \in G_t \ a.s.\}.$$

*Proof.* This is an immediate consequence of Theorem 3.6 below. □

**Theorem 3.6.** *Suppose that $\mathcal{B}$ is a weak$^*$-closed convex cone in $\mathcal{L}^\infty(\mathcal{F}, \mathbb{R}^d)$. Then $\mathcal{B}$ is optionally decomposable if and only if there exists a collection $(G_t)_{t=0}^T$, with each $G_t$ being an $\mathcal{F}_t$-measurable random, closed, convex cone in $\mathbb{R}^d$ such that:*

$$\mathcal{B}^* = \bigcap_{t=0}^{T} \{Z \in \mathcal{L}^1(\mathcal{F}; \mathbb{R}^d);\ Z_t \in G_t \ a.s.\}. \tag{3.2}$$

*Proof.* See Appendix A.2 □

**Remark 3.7.** *It is natural to ask whether we may extend the results in this section to the case of a countable collection of numéraires. This is done in subsection 7.3 of* [10]*. We also gave a proof there that, under a separability assumption on $\mathcal{L}^0(\mathcal{F}, \mathbb{P})$, there always exists a countable collection $U = \{v_1, \ldots\}$ such that $\mathcal{A}$ is optionally represented by $U$.*

## 4. Associating a pricing mechanism to a trading cone.

As promised, we now show how to represent a trading cone as (essentially) the acceptable risk set of a pricing mechanism[2].

Let $\mathcal{B}$ be an arbitrage-free, closed convex cone in $\mathcal{L}^\infty(\mathcal{F}, \mathbb{R}^d)$, given by $\mathcal{B} = K_0 + ... + K_T$ where, as described below, each $K_t$ is generated by positive $\mathcal{F}_t$-measurable multiples of the vectors $-e_i$, $e_j - \pi_t^{ij}\, e_i$ for $1 \le i, j \le d$.

Now recall the setup from Schachermayer's paper [19]: we may trade in $d$ assets at times $0, \ldots, T$. We may burn or freely dispose of any asset and otherwise trades are given by a bid-ask process $\pi$ taking values in $\mathbb{R}^{d\times d}$, with $\pi$ adapted to $(\mathcal{F}_t)_{t=0}^T$. The bid-ask process gives the (time $t$) price for one unit of each asset in terms of each other asset, so that

$$\pi_t^{i,i} = 1,\ \forall i,$$

and $\pi_t^{i,j}$ is the (random) number of units of asset $i$ which can be traded for one unit of asset $j$ at time $t$. We assume (with Schachermayer) that we have "netted out" any advantageous trading opportunities, so that, for any $t$ and any $i_0, \ldots, i_n$:

$$\pi_t^{i_0,i_n} \le \pi_t^{i_0,i_1} \ldots \pi_t^{i_{n-1},i_n}.$$

The time $t$ trading cone, $K_t$, consists of all those random trades (including the free disposal of assets) which are available at time $t$. Thus we can think of $K_t$ as consisting

[2]A version of the results in this section originally appeared in [12]. Since they are only distantly related to the main results in that paper, we have removed them from the version of that paper which is to appear.

of all those random vectors which live (almost surely) in a random closed convex cone $K_t(\omega)$, where, denoting the $i$th canonical basis vector of $\mathbb{R}^d$ by $e_i$, $K_t(\omega)$ is the finitely-generated convex (hence closed) cone with generators $\{e_j - \pi_t^{i,j}(\omega)e_i, 1 \leq i \neq j \leq d;$ and $-e_k, 1 \leq k \leq d\}$. We shall say that $\eta$ is a self-financing process if $\eta_t - \eta_{t-1} \in K_t$ for each $t$, with $\eta_{-1} \stackrel{def}{=} 0$.

It follows that the cone of claims attainable from zero endowment is $K_0 + \ldots + K_T$ and we denote this by $\mathcal{B}(\pi)$. Note that $-K_t$ is the time-$t$ solvency cone of claims, i.e. all those claims which may be traded to 0 at time $t$. Note also that, following Kabanov *et al.* [16], Schachermayer uses "hat" notation (which we have dropped) to stress that we are trading physical assets and uses $-K$ where we use $K$.

Recall that null strategies are elements $(\xi_0, \ldots, \xi_T)$ of $K_0 \times \ldots \times K_T$ satisfying $\sum_0^T \xi_t = 0$, and, from Theorem 1.2 of [12], that we may suppose without loss of generality that the null strategies of this decomposition form a vector space. Our aim in this section is to transform trading with transaction costs to a partially frictionless setting by adding a new period on the time axis and then to show that the revised trading cone is (essentially) the acceptable risk set of a pricing mechanism.

We introduce some notation. For $i \in \{1, \ldots, d\}$ we define the random variables $B^i \stackrel{def}{=} \pi_T^{1i}$ and $S^i \stackrel{def}{=} 1/\pi_T^{i1}$ We define the random, closed, convex sets $H = \{1\} \times \prod_{i=2}^{d} [S^i, B^i]$ and, for $\varepsilon > 0$,

$$H_\varepsilon = \{1\} \times \prod_{i=2}^{d} [(1-\varepsilon)S^i, (1+\varepsilon)B^i].$$

Let $\Psi_\varepsilon$ be the (finite) set of extreme points (in $\mathbb{R}^d$) of the set $H_\varepsilon$, i.e the $2^{d-1}$ random vectors of the form $(1, X_2, ..., X_d)$ where each $X_i = (1-\varepsilon)S^i$ or $(1+\varepsilon)B^i$. Let $\tilde{\Omega} = \{0,1\}^{d-1}$ and enumerate the elements of $\Psi_\varepsilon$ as follows:

$$\Psi_\varepsilon = \{Y(\omega, \tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}\},$$

where
(4.1)

$$Y(\omega, \tilde{\omega}_1, \ldots, \tilde{\omega}_{d-1}) \stackrel{def}{=} e_1 + \sum_{j=1}^{d-1} \left\{(1-\tilde{\omega}_j)(1-\varepsilon)\, S^{j+1}(\omega) + \tilde{\omega}_j(1+\varepsilon)\, B^{j+1}(\omega)\right\} e_{j+1}.$$

Define $\mathcal{B}^o$ to be the collection of consistent price processes for $\mathcal{B}$. Recall from [19] that this means that

$$\mathcal{B}^o = \{Z \in \mathcal{L}^1_+(\mathcal{F}_T, \mathbb{R}^d) : Z > 0 \text{ a.s. and } Z_t \in K_t^{(*)} \text{ a.s.}\},$$

where $K_t^{(*)} \stackrel{def}{=} \{X \in \mathcal{L}^1(\mathcal{F}_t, \mathbb{R}^d) : X.Y \leq 0\, \mathbb{P} \text{ a.s. for all } Y \in K_t\}$.

**Proposition 4.1.** *Let $Z \in \mathcal{B}^o$. Then there exist strictly positive random variables $\lambda^{(Z_T)}(\cdot, \tilde{\omega})$ defined for each $\tilde{\omega} \in \tilde{\Omega}$ such that $\sum_{\tilde{\omega}\in\tilde{\Omega}} \lambda^{(Z_T)}(\omega, \tilde{\omega}) = 1$ and*

$$\frac{Z_T}{Z_T^1} = \sum_{\tilde{\omega}\in\tilde{\Omega}} \lambda^{(Z_T)}(\omega, \tilde{\omega}) Y(\omega, \tilde{\omega}).$$

*Proof.* We know from the properties of consistent price processes that for $i, j = 1, ..., d$,

$$\frac{Z_T^j}{Z_T^i} \leq \pi_T^{ij} \leq \pi_T^{i1}\, \pi_T^{1j} = \frac{B^j}{S^i}.$$

In consequence, for every $i = 2, \cdots, d$ we get

$$\bar{Z}_T^i \stackrel{def}{=} \frac{Z_T^i}{Z_T^1} \in [S^i, B^i],$$

and so

$$\bar{Z}_T = (\bar{Z}_T^1, ..., \bar{Z}_T^d) \in H \subset H_\varepsilon.$$

Now, for $2 \leq i \leq d$, let

$$\theta(\omega, i) \stackrel{def}{=} \frac{\bar{Z}_T^i - S^i(1-\varepsilon)}{B^i(1+\varepsilon) - S^i(1-\varepsilon)},$$

and then define

$$\lambda^{(Z_T)}(\omega, \tilde{\omega}) \stackrel{def}{=} \prod_1^{d-1} \theta(\omega, i+1)^{\tilde{\omega}_i} (1 - \theta(\omega, i+1))^{1-\tilde{\omega}_i}$$

Since $\theta(\omega, i)$ is exactly the co-efficient $\theta$ such that

$$\bar{Z}_T^i = \theta B^i(1+\varepsilon) + (1-\theta) S^i(1-\varepsilon)$$

the result follows. □

To set up the new probability space, let $\tilde{\mathcal{F}}$ be the power set of $\tilde{\Omega}$ and let $\tilde{\mathbb{P}}$ be the uniform measure on $\tilde{\Omega}$, then define $\hat{\Omega} = \Omega \times \tilde{\Omega}$, $\hat{\mathcal{F}} = \mathcal{F} \otimes \tilde{\mathcal{F}}$ and $\hat{\mathbb{P}} = \mathbb{P} \otimes \tilde{\mathbb{P}}$.

Now we define the frictionless bid-ask prices at time $T+1$ by

$$\pi_{T+1}^{ij} \stackrel{def}{=} \frac{Y_j}{Y_i}$$

(where the random vector $Y$ is defined in (4.1)), and so $K_{T+1}$ is the convex cone generated by positive $\mathcal{F}_{T+1}$-measurable multiples of the vectors $-e_i$ and $e_j - \pi_{T+1}^{ij}\, e_i$, where $\mathcal{F}_{T+1} = \mathcal{F}_T \otimes \tilde{\mathcal{F}}$. We define the new trading cone by $\mathcal{B}_{T+1} \stackrel{def}{=} \mathcal{B} + K_{T+1}$. Here we assume the obvious embedding of $\mathcal{B}$ in $L^0(\mathcal{F}_{T+1}; \mathbb{R}^d)$.

From now on, closedness and arbitrage-free properties are to be understood to hold with respect to the vector space $\mathcal{L}^0(\mathcal{F}_{T+1})$.

**Proposition 4.2.** *The cone $\mathcal{B}_{T+1}$ is closed and arbitrage-free.*

*Proof.* We prove first that a consistent price process for the cone $\mathcal{B}$ can be extended to (be the trace of) a consistent price process for the cone $\mathcal{B}_{T+1}$.

Let $Z_T \in \mathcal{B}^o$ and define

$$Z_{T+1} \stackrel{def}{=} 2^{d-1}\, Z_T^1\, \lambda^{(Z_T)}\, Y,$$

where the random variable $\lambda^{(Z_T)}$ is given in Proposition 4.1. Then $Z_{T+1} > 0$, $Z_{T+1} \in K^*_{T+1}$ and for $X_T \in \mathcal{L}^\infty_+(\mathcal{F}_T)$ we have, by Fubini's Theorem,

$$\mathbb{E}_{\hat{\mathbb{P}}}\left(X_T.Z_{T+1}\right) = \sum_{\tilde{\omega}\in\tilde{\Omega}} \mathbb{E}_{\mathbb{P}}\left(X_T.\left(Z_T^1\,\lambda^{(Z_T)}(\cdot,\tilde{\omega})Y(\cdot,\tilde{\omega})\right)\right) = \mathbb{E}_{\mathbb{P}}\left(X_T.\left(Z_T^1\bar{Z}_T\right)\right) = \mathbb{E}_{\mathbb{P}}\left(X_T.Z_T\right).$$

Consequently $Z_{T+1} \in \mathcal{L}^1$ with $Z_T = \mathbb{E}_{\hat{\mathbb{P}}}\left(Z_{T+1}|\mathcal{F}_T\right)$, therefore $(Z_0,\ldots,Z_T,Z_{T+1})$ is a consistent price process for the cone $\mathcal{B}_{T+1}$ and so we conclude from Theorem 4.10 of [12] that $\bar{\mathcal{B}}_{T+1}$ is arbitrage-free. We shall now show that

$$K_{T+1} \cap \mathcal{L}(\mathcal{F}_T) \subset \mathcal{B}. \tag{4.2}$$

Indeed, let $X \in K_{T+1} \cap \mathcal{L}(\mathcal{F}_T)$, so for every $n \geq 1$, we have

$$X^n \stackrel{def}{=} X\, 1_{(|X|\leq n)} \in K_{T+1} \cap L^\infty(\mathcal{F}_T);$$

therefore, for any consistent price process, $Z$,

$$\mathbb{E}(Z_T.X^n) = \mathbb{E}(Z_{T+1}.X^n) \leq 0\, \mathbb{P} \text{ a.s.}$$

It follows from Theorem 4.14 of [12] that $X^n \in \mathcal{B}$ and thus, by closure, $X \in \mathcal{B}$.

Now we prove that the cone $\mathcal{B}_{T+1}$ is closed. We do this by showing that $\text{Null}(K_0 \times \ldots \times K_{T+1})$, the collection of null strategies of the decomposition $K_0 + \ldots + K_{T+1}$, is a vector space.

Let

$$(x_0,\ldots,x_{T+1}) \in \text{Null}(K_0 \times \ldots \times K_{T+1})$$

and define

$$x = x_0 + ... + x_T$$

so that $x + x_{T+1} = 0$. Then it follows (since $x \in \mathcal{L}(\mathcal{F}_T)$) that $x_{T+1} \in \mathcal{L}(\mathcal{F}_T)$ and so we conclude from (4.2) that $x_{T+1} \in \mathcal{B}$. We deduce that there exist $y_0 \in K_0,\ldots,y_T \in K_T$ such that $x_{T+1} = y_0 + \ldots + y_T$. We conclude that each $-(x_t + y_t) \in K_t$ and then, by adding $x_t$, respectively $y_t$, we conclude that both $-x_t$ and $-y_t$ are contained in $K_t$ for $0 \leq t \leq T$.

Observe that since the time $T+1$ bid-ask prices are frictionless, it follows that every element, $u \in K_{T+1}$ can be written as $u = u_1 - u_2$, where $u_1 \in lin(K_{T+1})$, the lineality space of $K_{T+1}$, and $u_2 \geq 0$. If we express $x_{T+1}$ like this, we then have that

$$0 \leq u_2 = u_1 - x_{T+1} = u_1 + x \in \mathcal{B}_{T+1},$$

so, since $\mathcal{B}_{T+1}$ is arbitrage-free, $u_2 = 0$ and therefore

$$-x_{T+1} = -u_1 \in K_{T+1}$$

(since $u_1 \in lin(K_{T+1})$). It follows that $\text{Null}((K_0 \times \ldots \times K_{T+1})$ is a vector space. □

Now define the subset of probabilities

$$\mathcal{Q} \stackrel{def}{=} \left\{ \mathbb{Q} : \frac{d\mathbb{Q}}{d\hat{\mathbb{P}}} = 2^{d-1} \frac{Z_T^1}{Z_0^1} \lambda^{(Z_T)} \text{ for some } Z \in \mathcal{B}^o \text{ with } \mathbb{E}Z_T^1 = 1 \right\},$$

and denote by $\rho$ the associated pricing mechanism.

**Theorem 4.3.** *For every $X \in \mathcal{L}^\infty(\mathcal{F}_T; \mathbb{R}^d)$ we have:*

$$\rho(Y.X) = \sup\{\mathbb{E}(Z_T.X) : \ Z \in \mathcal{B}^o, \ \mathbb{E}Z_T^1 = 1\}. \tag{4.3}$$

*In particular*

$$\begin{aligned} &\mathcal{B} \cap \mathcal{L}^\infty(\mathcal{F}_T; \mathbb{R}^d) \\ &= \left\{ X \in \mathcal{L}^\infty(\mathcal{F}_T; \mathbb{R}^d) : \mathbb{E}_{\mathbb{Q}}(Y.X) \le 0 \ \textit{for all} \ \mathbb{Q} \in \mathcal{Q} \right\} \\ &= \left\{ X \in \mathcal{L}^\infty(\mathcal{F}_T; \mathbb{R}^d) : \rho(Y.X) \le 0 \right\}. \end{aligned} \tag{4.4}$$

*Proof.* Equality (4.3) is immediate from the definition of $\mathcal{Q}$; the second equality in (4.4) follows from (4.3), while the first follows from Theorem 4.14 of [12] and the fact that, as in the proof of Proposition 4.2, $\mathbb{E}_{\mathbb{Q}}(Y.X) = \mathbb{E}_{\mathbb{P}}(Z_T.X)$ □

**Remark 4.4.** *If we define $\rho_t : \mathcal{L}^\infty(\mathcal{F}_{T+1}) \to \mathcal{L}^\infty(\mathcal{F}_t)$ by*

$$\rho_t(X) = \textit{ess inf}\{\lambda \in \mathcal{L}^\infty(\mathcal{F}_t) : \ \rho(c(X - \lambda)) \le 0 \ \textit{for all} \ c \in \mathcal{L}^\infty_+(\mathcal{F}_t)\},$$

*then it is easy to show that*

(i) $\{X : \ cX \in \mathcal{B}_{T+1} \cap \mathcal{L}^\infty(\mathcal{F}_{T+1}; \mathbb{R}^d)$ *for all* $c \in \mathcal{L}^\infty_+(\mathcal{F}_t)\} = \{X : \ \rho_t(Y.X) \le 0 \, \mathbb{P}$*-a.s.*$\}$.

(ii) $\mathcal{C}_t^\infty \stackrel{def}{=} \mathcal{C}_t(\mathcal{B}_{T+1}) \cap \mathcal{L}^\infty(\mathcal{F}_{T+1}; \mathbb{R}^d) = \{X \in \mathcal{L}^\infty(\mathcal{F}_t; \mathbb{R}^d) : \ \rho_t(Y.X) \le 0 \, \mathbb{P}$*-a.s.*$\}$.

*It follows directly from Theorem 4.16 of* [12] *that $\mathcal{C}_t^\infty$ is $\sigma(\mathcal{L}^\infty(\mathbb{P}), \mathcal{L}^1(\mathbb{P}))$-closed and hence we may apply Corollary 4.7 of* [12] *to deduce that, for each $t$, there exists a random, $\mathcal{F}_t$-measurable, closed cone $G_t$ such that*

$$\mathcal{C}_t^\infty = \{Z \in \mathcal{L}^\infty(\mathcal{F}_t, \mathbb{R}^d) : \ Z \in G_t \ \textit{a.s.}\}.$$

## References


[1] Artzner, Delbaen, Eber and Heath (1999): Coherent measures of risk. *Math. Finance* **9**(3), 203–228.

[2] P. Artzner, F. Delbaen, J.M. Eber, D. Heath, H. Ku (2004): Coherent multiperiod risk adjusted values and Bellmans principle. Working Paper, ETH Zurich.

[3] P. Cheridito, F. Delbaen, M. Kupper (2006): Dynamic monetary risk measures for bounded discrete-time processes. *Electronic Journal of Probability*, **11**, pages 57–106.

[4] F. Delbaen (2002): Coherent risk measures on general probability spaces. *Advances in Finance and Stochastics, 1–37, Springer, Berlin.*

[5] F. Delbaen: The structure of m-stable sets and in particular the set of risk neutral measures. Preprint http://www.math.ethz.ch/ delbaen/

[6] F. Delbaen, Y. M. Kabanov and E. Valkeila (2002): Hedging under transaction costs in currency markets: a discrete-time model. *Math. Finance* **12**(1), 45–61.

[7] K. Detlefsen and G. Scandolo (2005): Conditional and dynamic convex risk measures. *Fin. & Stochastics* **9**, 539–561.

[8] J. Engwerda, B. Roorda, H. Schumacher (2004): Coherent acceptability measures in multiperiod models. Working Paper, University of Twente, Math. Finance.
[9] S. D. Jacka (1992): A martingale representation result and an application to incomplete financial markets. *Math. Finance* **2**, 23–34.
[10] S. D. Jacka and A. Berkaoui (2006): On representing claims for coherent risk measures http://arxiv.org/abs/math.PR/0603041
[11] S. D. Jacka and A. Berkaoui (2007): On decomposing risk in a financial-intermediate market and reserving. http://arxiv.org/abs/0708.0512
[12] S. D. Jacka, A. Berkaoui and J. Warren (2006): No arbitrage and closure results for trading cones with transaction costs. To appear in *Fin. & Stochastics*
[13] E. Jouini and H. Kallal (1995): Arbitrage in securities markets with short-sales constraints. *Math. Finance* **5**(3), 197–232.
[14] Yu. M. Kabanov (1998): Hedging and liquidation under transaction costs in currency markets, *Fin. & Stochastics* **3**(2), 237–248.
[15] Yu. M. Kabanov, M. Rasonyi and C. Stricker (2002): No-arbitrage criteria for financial markets with efficient friction, *Fin. & Stochastics* **6**(3), 371–382.
[16] Yu. M. Kabanov, M. Rasonyi and C. Stricker (2003): On the closedness of sums of convex cones in $\mathcal{L}^0$ and the robust no-arbitrage property, *Fin. & Stochastics* **7**(3), 403–411.
[17] F. Riedel (2004): Dynamic conditional coherent risk measures. *Stochast. Proc. Appl.* **112**, 185–200.
[18] G. Scandolo (2003): Risk measures in a dynamic setting. PhD Thesis, Università di Milano.
[19] W. Schachermayer (2004): The fundamental theorem of asset pricing under proportional transaction costs in discrete time. *Math. Finance* **14**(1), 19–48.
[20] T. Wang (1999): A class of dynamic risk measures. Working Paper, University of British Columbia.
[21] S. Weber (2003): Distribution-invariant dynamic risk measures. Working Paper, HU Berlin.

## Appendix A. Some proofs and technical lemmas

Nearly all the results proved in this appendix have both an optional and a predictable version. We can usually give a common proof by defining a parameter $\eta$ which takes the value 1 in the predictable case and 0 in the optional case.

### A.1. **Equivalence of representation and time-consistency.** We now give the

*Proof of Theorem 2.17:* suppose that $\mathcal{A}$ is $\eta$-$V$-time-consistent. Fix $t \in \{0, \dots, T-\eta\}$ and $X \in \mathcal{A}_t$, so there exists two sequences $X_n \in L^\infty$ and $Y^n_{t+\eta} \in L^\infty(\mathcal{F}_{t+\eta}; \mathbb{R}^d)$ such that

$$X_n - Y^n_{t+\eta}.V \in \mathcal{A}_{t+1}$$

and the sequence $X_n$ converges weakly$^*$ to $X$ in $L^\infty$ with

$$\rho_t(X) = \liminf \rho_t\left(Y^n_{t+\eta}.V\right).$$

Therefore, for all $\varepsilon > 0$, there exists some $N \geq 1$ such that for all $n \geq N$

$$\rho_t(X) + \varepsilon \geq \rho_t\left(Y^n_{t+\eta}.V\right).$$

Now we can write $X_n - \varepsilon = (X_n - Y^n_{t+\eta}.V) + (Y^n_{t+\eta}.V - \varepsilon)$ with $X_n - Y^n_{t+\eta}.V \in \mathcal{A}_{t+1}$ and $Y^n_{t+\eta}.V - \varepsilon \in K^\eta_t(\mathcal{A}, V).V$. So $X_n \in K^\eta_t(\mathcal{A}, V).V + \mathcal{A}_{t+1}$. By taking the limit we see that $X \in \overline{K^\eta_t(\mathcal{A}, V).V + \mathcal{A}_{t+1}}$ and then $\mathcal{A}_t = \overline{K^\eta_t(\mathcal{A}, V).V + \mathcal{A}_{t+1}}$. By induction on $t \in \{0, \dots, T\}$ we get $\mathcal{A} = \overline{\oplus_{t=0}^{T-\eta} K^\eta_t(\mathcal{A}, V).V}$.

Conversely, fix $t \in \{0, \dots, T-1\}$ and $X \in \mathcal{A}_t$ with $\rho_t(X) = 0$, then there exists a sequence

$$X_n \in \oplus_{s=t}^{T-\eta} K_s^\eta(\mathcal{A}, V).V,$$

which converges weakly* to $X$ in $L^\infty$. So there exists $Y^n \in K_t^\eta(\mathcal{A}, V)$ such that $Z^n = X_n - Y^n.V \in \mathcal{A}_{t+1}$. We conclude that

$$0 = \rho_t(X) \leq \liminf \rho_t(X_n) = \liminf \rho_t\left(Y^n.V + Z^n\right) \leq \liminf \rho_t\left(Y^n.V\right) \leq 0 \, \mathbb{P} \text{ a.s.},$$

so $\rho_t(X) = \liminf \rho_t\left(Y^n.V\right)$. Now for all $X \in L^\infty$ we have $X - \rho_t(X) \in \mathcal{A}_t$ and from previously

$$\rho_t(X - \rho_t(X)) = \liminf \rho_t\left(Y^n.V\right),$$

therefore

$$\rho_t(X) = \liminf \rho_t\left(Y^n.V + \rho_t(X)\right) = \liminf \rho_t\left((Y^n + \rho_t(X)e_1).V\right).$$

□

A.2. **Proof of Theorem 3.2.** Recall that the parameter $\eta$ takes the value 0 in the optional case and 1 in the predictable case. To prove Theorem 3.2 we shall need the following two lemmas whose proofs we defer:

**Lemma A.1.** *Let $D \subset \mathcal{L}^1_+$ be an $\eta$-stable cone, then, defining*

$$\mathcal{M}_t^\eta(D) = \{Z \in \mathcal{L}^1(\mathcal{F}; \mathbb{R}^d) : \exists \alpha_t \in L^0_+(\mathcal{F}_t) \text{ and } Z' \in D \text{ such that } Z_{t+\eta} = \alpha_t \, Z'_{t+\eta}\},$$

*we have*

$$D = \cap_{t=0}^{T-\eta} \mathcal{M}_t^\eta(D).$$

**Lemma A.2.** *For each $D \subset \mathcal{L}^1_+(\mathcal{F}, \mathbb{R}^d)$ and each $t = 0, 1, \dots, T$ we define:*

$$D_{(t)} = \overline{conv}\left\{\alpha \, Z : \, Z \in D, \, \alpha \in L^\infty_+(\mathcal{F}_t)\right\},$$

*where the closure is in the $\mathcal{L}^1$ topology. Now we define*

$$[D] \stackrel{def}{=} \bigcap_{t=0}^{T-\eta} \mathcal{R}_t^\eta(D)$$

*where*

$$\mathcal{R}_t^\eta(D) \stackrel{def}{=} \{Z \in \mathcal{L}^1 : \, Z_{t+\eta} = Z'_{t+\eta} \text{ for some } Z' \in D_{(t)}\} = \overline{conv}\big(\mathcal{M}_t^\eta(D)\big).$$

*Then*

(1) *$[D]$ is the smallest $\eta$-stable closed convex cone in $\mathcal{L}^1$, containing $D$.*
(2) *$D = [D]$ if and only if $D$ is an $\eta$-stable closed convex cone in $\mathcal{L}^1$.*

*Proof of Theorem 3.2:* Remark that $\mathcal{B}^*$ is a closed convex cone in $\mathcal{L}^1$. We claim that if we can prove that for all $t = 0, \dots, T-\eta$ we have $K_t^\eta(\mathcal{B}) = (M_t^\eta(\mathcal{B}^*))^*$, the result will follow thanks to Lemmas A.1 and A.2. To see this, note first that Lemma A.1 tells us that $\mathcal{B}^*$ being $\eta$-stable implies that $\mathcal{B}^* = \cap \mathcal{M}_t^\eta$. Thus if $K_t^\eta = (\mathcal{M}_t^\eta)^*$ then $\mathcal{B} = \mathcal{B}^{**} = (\cap \mathcal{M}_t^\eta)^* = \overline{\oplus(\mathcal{M}_t^\eta)^*} = \overline{\oplus(K_t^\eta)}$, establishing the reverse implication.

Conversely, $(\mathcal{M}_t^\eta)^* = (\overline{conv}(\mathcal{M}_t^\eta))^* = (\mathcal{R}_t^\eta(D))^*$, so, if $\mathcal{B} = \overline{\oplus(K_t^\eta)} = \overline{\oplus(\mathcal{M}_t^\eta)^*}$, then $\mathcal{B} = \overline{\oplus(\mathcal{R}_t^\eta(D))^*}$ so $\mathcal{B}^* = \cap\mathcal{R}_t^\eta(D) = [\mathcal{B}^*]$ and then, by Lemma A.2, $\mathcal{B}^*$ is $\eta$-stable.

First we prove that $\mathcal{M}_t^\eta(\mathcal{B}^*) \subset (K_t^\eta(\mathcal{B}))^*$. Let $Z \in \mathcal{M}_t^\eta(\mathcal{B}^*)$, then there exists some $Z' \in \mathcal{B}^*$ and $\alpha \in \mathcal{L}_+^0(\mathcal{F}_t)$ with $\alpha\, Z' \in \mathcal{L}^1$ such that $Z_{t+\eta} = \alpha\, Z'_{t+\eta}$. Take $X \in K_t^\eta(\mathcal{B})$, then

$$\mathbb{E}(Z.X) = \mathbb{E}(Z_{t+\eta}.X) = \mathbb{E}(\alpha_t\, Z'_{t+\eta}.X),$$

since $X \in \mathcal{L}^\infty(\mathcal{F}_{t+\eta}, \mathbb{R}^d)$. We then obtain:

$$\mathbb{E}(Z.X) = \mathbb{E}(Z'_{t+\eta}.\alpha_t\, X) = \lim_{n\to\infty} \mathbb{E}(Z'.\alpha_t\, 1_{(\alpha_t \le n)}\, X) \le 0,$$

since $\alpha_t\, 1_{(\alpha_t\le n)}\, X \in \mathcal{B}$ and $Z' \in \mathcal{B}^*$.

Now we prove that $(\mathcal{M}_t^\eta(\mathcal{B}^*))^* \subset K_t^\eta(\mathcal{B})$. We remark that $\mathcal{B}^* \subset \mathcal{M}_t^\eta(\mathcal{B}^*)$ and also that $L_+^\infty(\mathcal{F}_t)\, \mathcal{M}_t^\eta(\mathcal{B}^*) \subset \mathcal{M}_t^\eta(\mathcal{B}^*)$, so $(\mathcal{M}_t^\eta(\mathcal{B}^*))^* \subset \mathcal{B}_t$. Let $X \in (\mathcal{M}_t^\eta(\mathcal{B}^*))^*$, we want to prove that $X \in \mathcal{L}^\infty(\mathcal{F}_{t+\eta}, \mathbb{R}^d)$. Let $Z \in \mathcal{L}^1(\mathcal{F}, \mathbb{R}^d)$, we remark that $Z - Z_{t+\eta} \in \mathcal{M}_t^\eta(\mathcal{B}^*)$ and consequently $\mathbb{E}((Z - Z_{t+\eta}).X) \le 0$. We deduce then that $\mathbb{E}((X - X_{t+\eta}).Z) = \mathbb{E}((Z - Z_{t+\eta}).X) \le 0$ for all $Z \in \mathcal{L}^1$. Therefore $X = X_{t+\eta}$ $\mathbb{P}$-a.s. □

**Remark A.3.** *From Lemma A.1 we see that if a subset $D \subset \mathcal{L}^1$ is $\eta$-stable, then its polar cone $D^*$ in $\mathcal{L}^\infty$ is $\eta$-decomposable.*

We need one more result before we give the proofs of Lemmas A.1 and A.2:

**Lemma A.4.** *Let $D \subset \mathcal{L}_+^1(\mathcal{F}; \mathbb{R}^d)$. The following are equivalent:*

(i) *for each $t \in \{0, 1, \dots, T-1+\eta\}$, whenever $Y, W \in D$ are such that there exists $Z \in D$, a set $F \in \mathcal{F}_t$, $\alpha, \beta \in \mathcal{L}_+^0(\mathcal{F}_{t+1-\eta})$ with $\alpha Y, \beta W \in \mathcal{L}^1$ and*

$$X \stackrel{def}{=} 1_F \alpha Y + 1_{F^c} \beta W \tag{A.1}$$

*satisfies $X_t = \mathbb{E}(X|\mathcal{F}_t) = \mathbb{E}(Z|\mathcal{F}_t) = Z_t$ then we have $X \in D$;*

(ii) *for each $t \in \{0, 1, \dots, T-1+\eta\}$, whenever $Y, W \in D$ are such that there exists $Z \in D$, a set $F \in \mathcal{F}_t$, and for each $i \in \{1, \dots, d\}$ there is an $R^i_{t+1-\eta} \in L_+^1(\mathcal{F}_{t-\eta+1})$ with $R^i_{t+1-\eta} Y^i, R^i_{t+1-\eta} W^i \in \mathcal{L}^1$, $\mathbb{E}[R^i_{t+1-\eta}|\mathcal{F}_t] = 1$ and such that*

$$R^i_{t+1-\eta} Z^i_t \left( 1_F \frac{1}{Y^i_{t+1-\eta}} + 1_{F^c} \frac{1}{W^i_{t+1-\eta}} \right)$$

*is the same for each $i$, then $X$, given by*

$$X^i = Z^i_t R^i_{t+1-\eta} (1_F \frac{Y^i}{Y^i_{t+1-\eta}} + 1_{F^c} \frac{W^i}{W^i_{t+1-\eta}}) \tag{A.2}$$

*is in $D$.*

(iii) *$D$ is $\eta$-stable, i.e. for each stopping time $\tau \le T-1+\eta$, whenever there exist $Z$ and $W$ in $D$ and $R^i_{\tau+1-\eta} \in L_+^1(\mathcal{F}_{\tau+1-\eta})$, such that $\mathbb{E}[R^i_{\tau+1-\eta}|\mathcal{F}_\tau] = 1$ and*

$$\frac{Z^i_\tau R^i_{\tau+1-\eta}}{W^i_{\tau+1-\eta}}$$

*is the same for each $i$, then $X$, defined by*

$$X^i = W^i \frac{Z^i_\tau R^i_{\tau+1-\eta}}{W^i_{\tau+1-\eta}}, \tag{A.3}$$

*is in $D$.*

*Proof. ((i)⇔ (ii))*: Assume that (ii) holds. Observe that (A.1) implies that

$$1_F X^i_{t+1-\eta} = 1_F \alpha Y^i_{t+1-\eta} \text{ and } 1_{F^c} X^i_{t+1-\eta} = 1_{F^c} \beta W^i_{t+1-\eta},$$

for each $i$. It follows that

$$1_F \alpha = 1_F \frac{X^i_{t+1-\eta}}{Y^i_{t+1-\eta}} \text{ and } 1_{F^c}\beta = 1_{F^c} \frac{X^i_{t+1-\eta}}{W^i_{t+1-\eta}} \text{ for each } i.$$

Setting

$$R^i_{t+1-\eta} = \frac{X^i_{t+1-\eta}}{X^i_t} = \frac{X^i_{t+1-\eta}}{Z^i_t},$$

we see that (A.2) holds establishing (i).

Conversely, assuming (i), if (A.2) is satisfied, then

$$1_F \frac{X^i}{Y^i} = 1_F \frac{Z^i_t R^i_{t+1-\eta}}{Y^i_{t+1-\eta}},$$

and

$$1_{F^c} \frac{X^i}{Y^i} = 1_{F^c} \frac{Z^i_t R^i_{t+1-\eta}}{W^i_{t+1-\eta}},$$

and both are independent of $i$. Setting these common values to $\alpha$ and $\beta$ respectively, we see that (A.1) holds and $X^i_t = Z^i_t$ for each $i$, so that $X \in D$, establishing (ii).

*((ii)⇔ (iii))*: suppose that (iii) holds, then, setting

$$\tau = t1_{F^c} + (T - 1 + \eta)1_F$$

in (A.3) we see that

$$X = 1_{F^c} Z^i_t R^i_{t+1-\eta} \frac{W}{W^i_{t+1-\eta}} + 1_F Z^i_{T-1+\eta} R^i_T, \tag{A.4}$$

and $X \in D$. Now take $Z = X$, $W = Y$ and $\tilde{\tau} = t1_F + (T-1+\eta)1_{F^c}$ in (A.3). We obtain

$$X = Z^i_t R^i_{t+1-\eta} (1_F \frac{Y}{Y^i_{t+1-\eta}} + 1_{F^c} \frac{W}{W^i_{t+1-\eta}})$$

and $X \in D$, establishing (ii).

Conversely, suppose that (ii) holds. We prove (iii) by backwards induction on a lower bound for $\tau$. The property is immediate for $\tau = T - 1 + \eta$. Now suppose that (iii) holds whenever $\tau \geq k + 1$ a.s., and that the stopping time $\tilde{\tau}$ satisfies $\tilde{\tau} \geq k$ a.s. Define $F = (\tilde{\tau} \geq k + 1)$ (so that $F^c = (\tilde{\tau} = k)$) and set

$$\tau^* = \tilde{\tau} 1_F + (T - 1 + \eta)1_{F^c}.$$

Notice that $\tau^* \geq k + 1$.

Suppose that $W, Z \in D$ and $R_{\tilde{\tau}+1-\eta} \in \mathcal{F}_{\tilde{\tau}+1-\eta}$ satisfy the hypotheses of (iii) then

$$W \frac{Z^i_{\tilde{\tau}} R^i_{\tilde{\tau}+1-\eta}}{W^i_{\tilde{\tau}+1-\eta}} 1_F,$$

is independent of $i$, so $Y$, defined by

$$Y = W \frac{Z^i_{\tau} R^i_{\tau+1-\eta}}{W^i_{\tau+1-\eta}} 1_F + W 1_{F^c},$$

is also independent of $i$. Now

$$Y = W \frac{Z^i_{\tau^*} R^i_{\tau^*+1-\eta}}{W^i_{\tau^*+1-\eta}},$$

where $R^i_{\tau^*+1-\eta} = R^i_{\tau+1-\eta} 1_F + \frac{Z^i}{Z^i_{T-1+\eta}} 1_{F^c}$. It is easy to check that $\mathbb{E}[R^i_{\tau^*+1-\eta} | \mathcal{F}_{\tau^*}] = 1$ so, by the inductive hypothesis, $Y \in D$

Now substitute $Y, W, Z$ and $F$ in (A.3), with $t = k$ and $R^i_{k+1-\eta} = \frac{Z^i_{k+1-\eta}}{Z^i_k} 1_F + R^i_{\tilde{\tau}+1-\eta} 1_{F^c}$ to see that $X = W \frac{Z^i_{\tilde{\tau}} R^i_{\tilde{\tau}+1-\eta}}{W^i_{\tilde{\tau}+1-\eta}}$ and (by (ii)) $X \in D$, which establishes the inductive step. □

*Proof of Lemma A.1:* The inclusion $D \subset \bigcap_{t=0}^{T-\eta} \mathcal{M}^\eta_t(D)$ is trivial.

Now we let $Y \in \cap_{t=0}^{T-\eta} \mathcal{M}^\eta_t(D)$ and seek to prove that $Y \in D$. So, for all $t \in \{0, \ldots, T-\eta\}$, there exists some $Z^t \in D$, $\beta_t \in \mathcal{L}^0_+(\mathcal{F}_t)$ with $\beta_t Z^t \in \mathcal{L}^1$ such that $Z_{t+\eta} = \beta_t \, Z^t_{t+\eta}$. Define $\xi^{T-\eta} = Z^{T-\eta}$ and for $t \in \{0, \ldots, T-\eta-1\}$:

$$\xi^t = 1_{F_t} \kappa_t \, \xi^{t+1} + 1_{F^c_t} \, Z^t,$$

where $F_t = (\beta_t > 0)$ and $\kappa_t = \beta_{t+1}/\beta_t$. Remark that

$$Z_{t+1+\eta} = \beta_{t+1} \, Z^{t+1}_{t+1+\eta},$$

and

$$Z_{t+\eta} = \beta_t \, Z^t_{t+\eta}.$$

Thus

$$\mathbb{E}[\beta_{t+1} Z^{t+1} | \mathcal{F}_{t+\eta}] = \mathbb{E}[Z_{t+1+\eta} | \mathcal{F}_{t+\eta}] = Z_{t+\eta} = \beta_t \, Z^t_{t+\eta},$$

which leads us to deduce that

$$\mathbb{E}((1_{F_t} \kappa_t \, Z^{t+1} + 1_{F^c_t} Z^t) | \mathcal{F}_{t+\eta}) = Z^t_{t+\eta}.$$

Remark also that, since $D \subset \mathcal{L}^1_+(\mathcal{F}; \mathbb{R}^d)$,

$$Z = \beta_0 \, \kappa_0 \times \ldots \times \kappa_{T-\eta-1} \, Z^{T-\eta}.$$

We deduce that $Z = \beta_0 \, \xi^0$.

Now we prove by backwards induction on $t$ that $Z^t_{t+\eta} = \xi^t_{t+\eta}$ and $\xi^t \in D$. For $t = T-\eta$, we have $\xi^{T-\eta} = Z^{T-\eta} \in D$ by definition. Suppose that for all $s = T-\eta, \ldots, t+1$, we have $Z^s_{s+\eta} = \xi^s_{s+\eta}$ and $\xi^s \in D$ and show that $Z^t_{t+\eta} = \xi^t_{t+\eta}$ and $\xi^t \in D$. First $\xi^t = 1_{F_t} \kappa_t \, \xi^{t+1} + 1_{F^c_t} \, Z^t$ so we get

$$\xi^t_{t+\eta} = \mathbb{E}((1_{F_t} \kappa_t \, \xi^{t+1} + 1_{F^c_t} \, Z^t) | \mathcal{F}_{t+\eta}) = \mathbb{E}((1_{F_t} \kappa_t \, Z^{t+1} + 1_{F^c_t} \, Z^t) | \mathcal{F}_{t+\eta}) = Z^t_{t+\eta}.$$

Then

$$Z^t_{t+\eta} = \mathbb{E}((1_{F_t}\kappa_t\,\xi^{t+1} + 1_{F_t^c}\,Z^t)|\mathcal{F}_{t+\eta}),$$

with $Z^t, \xi^{t+1} \in D$. By the $\eta$-stability of the subset $D$ and Lemma A.4 we deduce that $\xi^t \in D$ and consequently $Z = \beta_0\,\xi^0 \in D$ since $\xi^0 \in D$ and $\beta_0$ is a positive scalar. □

*Proof of Lemma A.2:* we remark that $[D]$ is a closed convex cone in $\mathcal{L}^1$, containing $D$. Now we prove that $[D]$ is $\eta$-stable. Fix $t \in \{0, 1, \ldots, T-\eta\}$ and suppose that $Z^1$ and $Z^2$ are in $[D]$, and are such that there exist $Z \in [D]$, sets $F^1, F^2 \in \mathcal{F}_t$ ( with $F^2 = (F^1)^c$) and $\alpha^1, \alpha^2 \in \mathcal{L}^0_+(\mathcal{F}_{t-\eta+1})$ with each $\alpha^i\,Z^i \in \mathcal{L}^1$ and

$$\mathbb{E}(Z|\,\mathcal{F}_t)1_{F^i} = \mathbb{E}(\alpha^i\,Z^i|\mathcal{F}_t)1_{F^i},$$

for $i = 1, 2$. We want to prove that

$$Y \stackrel{def}{=} \sum_{i=1}^{2} 1_{F_t^i}\alpha^i\,Z^i \in [D].$$

Now for $s \geq t-\eta+1$ we have

$$Y_{s+\eta} = \sum_{i=1}^{2} 1_{F_t^i}\alpha^i\,Z^i_{s+\eta},$$

with $Z^i_{s+\eta} = W^i_{s+\eta}$ for some $W^i \in D_{(s)}$. Therefore

$$Y_{s+\eta} = \left(\sum_{i=1}^{2} 1_{F_t^i}\alpha^i\,W^i\right)_{s+\eta},$$

with $\sum_{i=1}^{2} 1_{F_t^i}\alpha^i\,W^i \in D_{(s)}$.

Now for $s \leq t-\eta$ we have $Y_{s+\eta} = (Y_t)_{s+\eta}$ and

$$Y_t = \sum_{i=1}^{2} 1_{F_t^i}\mathbb{E}(\alpha^i\,Z^i|\mathcal{F}_t) = Z_t,$$

and then $Y_{s+\eta} = Z_{s+\eta} = W_{s+\eta}$ for some $W \in D_{(s)}$. Therefore $Y \in [D]$.

Now, we prove that $D = [D] \Leftrightarrow D$ is an $\eta$-stable closed convex cone in $\mathcal{L}^1$.

For the reverse implication, thanks to Lemma A.1, we have:

$$D = D^{**} = \cap_{t=0}^{T-\eta}\overline{conv}(\mathcal{M}^\eta_t(D)) = \cap_{t=0}^{T-\eta}\overline{\mathcal{R}^\eta_t(D)},$$

with $D \subset [D] \subset \cap_{t=0}^{T-\eta}\overline{\mathcal{R}^\eta_t(D)}$. Then $D = [D]$. The direct implication is trivial from the first assertion.

Finally, to prove that $[D]$ is the smallest $\eta$-stable closed convex cone in $\mathcal{L}^1$ which contains $D$, simply let $D'$ be an $\eta$-stable closed convex cone in $\mathcal{L}^1$, containing $D$. Then $[D] \subset [D'] = D'$. □

A.3. **Proof of Theorem 3.6.** In order to prove the optional representation result, we need

**Definition A.5.** *For a fixed $t \in \{0, \dots, T\}$, we say that a closed convex cone $\mathcal{H}$ in $\mathcal{L}^1(\mathcal{F}_t; \mathbb{R}^d)$ is an $\mathcal{F}_t$-cone (or a $t$-cone) if $\alpha\mathcal{H} \subset \mathcal{H}$ for each $\alpha \in L^\infty_+(\mathcal{F}_t)$.*

**Remark A.6.** *The property of being a $t$-cone is like an $\mathcal{F}_t$-measurable version of the convex cone property.*

*It follows from Theorem 4.5 and Corollary 4.7 of* [12] *that $\mathcal{H}$ is a $t$-cone if and only if there is a random closed convex cone, $C$ in $\mathbb{R}^d$ such that*

$$\mathcal{H} = \{X \in \mathcal{L}^1(\mathcal{F}_t; \mathbb{R}^d) : \ X \in C \ \ a.s.\}.$$

To prove Theorem 3.6 we shall also need the following useful characterisation of $\mathcal{C}_t(\mathcal{B})$:

**Lemma A.7.** *Let $X \in \mathcal{L}^\infty(\mathcal{F}; \mathbb{R}^d)$. Then the following assertions are equivalent:*

(1) $X \in \mathcal{C}_t(\mathcal{B})$,
(2) $X \in \mathcal{L}^\infty(\mathcal{F}_t; \mathbb{R}^d)$ *and* $Z_t.X \leq 0$ $\mathbb{P}$*-a.s. for all* $Z \in \mathcal{B}^*$.
(3) $\mathbb{E}[(W.X)|\, \mathcal{F}_t] \leq 0$ *for all* $W \in \mathcal{L}^1$ *such that* $W_t = Z_t$ *for some* $Z \in \mathcal{B}^*$.

*Proof.* (1) $\Leftrightarrow$ (2): for $Z \in \mathcal{B}^*$ and $f_t^+ \in L^\infty_+(\mathcal{F}_t)$ we have:

$$\mathbb{E} f_t^+ (Z_t.X) = \mathbb{E} Z.(f_t^+ X) \leq 0,$$

since $f_t^+ X \in \mathcal{B}$. Then $Z_t.X \leq 0\, \mathbb{P}$ a.s. Conversely, letting $f_t^+ \in L^\infty_+(\mathcal{F}_t)$, we want to prove that $f_t^+ X \in \mathcal{B}$. Suppose that $Z \in \mathcal{B}^*$, then

$$\mathbb{E} Z.(f_t^+ X) = \mathbb{E} Z_t.(f_t^+ X) = \mathbb{E} f_t^+ (Z_t.X) \leq 0.$$

Since $Z$ is arbitrary it follows that $f_t^+ X \in \mathcal{B}$.

(2) $\Leftrightarrow$ (3): suppose (2) and take $W \in \mathcal{L}^1$ such that $W_t = Z_t$ for some $Z \in \mathcal{B}^*$. Then for any $f_t^+ \in L^\infty_+(\mathcal{F}_t)$, we have

$$\mathbb{E}(f_t^+ (W.X)) = \mathbb{E}[\mathbb{E}[(f_t^+ (W.X))|\mathcal{F}_t]] = \mathbb{E}(f_t^+ (W_t.X)) = \mathbb{E}(f_t^+ (Z_t.X)) \leq 0,$$

the second equality following since $X \in \mathcal{L}^\infty(\mathcal{F}_t)$. This establishes (3) since $W$ is arbitrary.

Conversely suppose (3). We prove first that $X \in L^\infty(\mathcal{F}_t)$. Remark that for every $W \in \mathcal{L}^1$ we have $\mathbb{E}(W - W_t|\, \mathcal{F}_t) = 0$ so, setting $W' = W - W_t$ we see that $W'_t = Z_t$ with $Z = 0 \in \mathcal{B}^*$! It follows that, by hypothesis, $\mathbb{E}[(W - W_t).X] \leq 0$. Consequently, for every $W \in \mathcal{L}^1$ we get

$$\mathbb{E}(W.(X - X_t)) = \mathbb{E}(W.X) - \mathbb{E}(W_t.X_t) = \mathbb{E}((W - W_t).X) \leq 0.$$

And so, clearly, $\mathbb{E} W.(X - X_t) = 0$ for every $W \in \mathcal{L}^1$. Thus $X = X_t$ a.s. To conclude, let $Z \in \mathcal{B}^*$, then

$$Z_t.X = \mathbb{E}(Z.X|\, \mathcal{F}_t) \leq 0,$$

as required $\square$

*Proof of Theorem 3.6:* Suppose that $\mathcal{B}$ is optionally decomposable and define

$$\mathcal{H}_t = \overline{\{\alpha Z_t : \alpha \in L^\infty_+(\mathcal{F}_t),\, Z \in \mathcal{B}^*\}}.$$

Thanks to Lemma A.7 and Remark A.6, we have (3.2).

Conversely, remark that

$$\{Z \in \mathcal{L}^1;\, Z_t \in \mathcal{H}_t\}^* \subset \mathcal{C}_t(\mathcal{B}).$$

Indeed define $N_t = \{Z \in \mathcal{L}^1;\, Z_t \in \mathcal{H}_t\}$, and let $X \in \mathcal{L}^\infty$ be such that $\mathbb{E}(Z.X) \leq 0$ for all $Z \in N_t$. Since $W - W_t \in N_t$ for all $W \in \mathcal{L}^1$ (because $(W - W_t)_t = 0$), we may deduce that $X \in \mathcal{L}^\infty(\mathcal{F}_t)$ using Lemma lem5.3. Now for all $Z \in \mathcal{B}^*$ and $\alpha \in L^\infty_+(\mathcal{F}_t)$, we have $\mathbb{E}(Z.\alpha X) = \mathbb{E}(\alpha Z.X) \leq 0$ since $\alpha Z \in N_t$. The result follows. □

## A.4. **A converse to Lemma 3.3.**

**Lemma A.8.** *Suppose that $\mathcal{B}$ is a weak*-closed convex cone in $\mathcal{L}^\infty(\mathcal{F};\mathbb{R}^d)$ and $V$ is a $d$-dimensional vector of elements of $\mathcal{N}$.*

*Define the cone $\mathcal{B}.V$ by*

$$\mathcal{B}.V = \{X.V : X \in \mathcal{B}\}.$$

*Then*

$$\mathcal{B} \subset (\mathcal{B}.V)(V)$$

*and*

$$\mathcal{B} = (\mathcal{B}.V)(V) \text{ if and only if } \alpha(v_j\, e_i - v_i\, e_j) \in \mathcal{B} \text{ for all } \alpha \in L^\infty_+(\mathcal{F}).$$

*In this case the set $\mathcal{B}.V$ is weak*-closed.*

*Proof.* The first inclusion is immediate from the definition of $(\mathcal{B}.V)(V)$.

Define $w_{ij} \stackrel{def}{=} v_j\, e_i - v_i\, e_j$ and suppose that $\mathcal{B} = (\mathcal{B}.V)(V)$, then for all $\alpha \in L^\infty_+(\mathcal{F})$ we have $\alpha w_{ij}.V = 0 = 0.V$. So $\alpha w_{ij} \in (\mathcal{B}.V)(V) = \mathcal{B}$.

Conversely, suppose that $\alpha w_{ij} \in \mathcal{B}$ for all $\alpha \in L^\infty_+(\mathcal{F})$ and all pairs $(i, j)$. It follows by the same argument as in the proof of Lemma 3.3 that $\mathcal{B}^* = VC$, for some closed convex cone $C \subset L^1(\mathcal{F})$. Now suppose that $Z \in \mathcal{B}^*$, so that $Z = VW$ for some $W \in C$. It follows that $\mathbb{E}Z.X = \mathbb{E}WV.X \leq 0$ for all $X \in B$ and thus, since $X$ is arbitrary, that $W \in (\mathcal{B}.V)^*$. Hence $\mathcal{B}^* \subset V(\mathcal{B}.V)^*$.

Now we've already observed that

$$\mathcal{B} \subset (\mathcal{B}.V)(V)$$

so

$$\mathcal{B}^* \supset (\mathcal{B}.V)(V)^*.$$

But by Lemma 3.3, $(\mathcal{B}.V)(V)^* = V(\mathcal{B}.V)^*$ and so

$$\mathcal{B}^* = V(\mathcal{B}.V)^*.$$

Taking polar cones once more we see that, since $\mathcal{B}$ is weak*- closed,

$$\mathcal{B}^{**} = \mathcal{B} = (V(\mathcal{B}.V)^*)^* = (\mathcal{B}.V)(V)^{**} = \overline{(\mathcal{B}.V)(V)}.$$

Finally, since $\mathcal{B} \subset (\mathcal{B}.V)(V)$, we conclude that

$$\mathcal{B} = \overline{(\mathcal{B}.V)(V)} = (\mathcal{B}.V)(V).$$

To see that $\mathcal{B}.V$ is closed, let $x^n \in \mathcal{B}.V$ be a sequence which converges to $x$, then $\frac{x^n}{v_1}\, e_1 \in (\mathcal{B}.V)(V) = \mathcal{B}$ converges to $\frac{x}{v_1}\, e_1 \in \mathcal{B}$ (since $\mathcal{B}$ is closed). Hence $x = \frac{x}{v_1}\, e_1.V \in \mathcal{B}.V$. $\square$

Dept. of Statistics, University of Warwick, Coventry CV4 7AL, UK

*E-mail address*: s.d.jacka@warwick.ac.uk

College of Sciences, Al-Imam Mohammed Ibn Saud Islamic University, P.O. Box 84880, Riyadh 11681, Saudi Arabia.

*E-mail address*: berkaoui@yahoo.fr