Chapter 2

Analysis of Operators on Idempotent Semimodules

2.0. Introduction

In this chapter we study endomorphisms, or linear operators, on semimodules of functions that range in idempotent semirings. Here the specific nature of idempotent analysis exhibits itself in the fact that each linear operator on such a semimodule is an integral operator, that is, has the form

$$(Bh)(x) = \int^{\oplus} b(x,y) \odot h(y)\, d\mu(y) = \inf_y (b(x,y) \odot h(y))$$

for some idempotent integral kernel $b(x,y)$. In §2.1 we give necessary and sufficient conditions for this function to specify a continuous operator. We give a characterization of weak and strong convergence of operator families in terms of kernels and then, in §2.2, we describe two important operator classes—invertible and compact operators—in the same terms. Here another specific feature of the semialgebra of idempotent linear operators is important—the supply of invertible operators is very small; namely, the group of invertible operators is generated by the diagonal operators and by the homomorphisms of the base. Hence, this group consists of idempotent analogs of weighted translation operators. It follows that all automorphisms of the operator semialgebra are inner automorphisms.

In §2.3 we investigate the eigenvector equation

$$Bh = \lambda \odot h = \lambda + h$$

for a compact linear operator B. Here one encounters another specific feature of the idempotent situation: in contrast with the conventional linear theory, the spectrum of a generic compact idempotent operator contains a single eigenvalue. This makes it difficult to construct functional calculus of idempotent operators. So far, we have no reasonable answers to the questions as to when the root of an operator can be extracted or when an operator can be included in a one-parameter semigroup. The latter question is rather important in applied problems, since continuous operator semigroups can be studied by the methods of theory of differential equations with the help of infinitesimal generators, whereas the standard settings of optimization problems (say, in mathematical economics) deal with separate operators rather than continuous semigroups. Nevertheless, the unique eigenvalue of an idempotent operator bears an important economical interpretation of the mean profit per step in the multistep dynamic optimization problem specified by the corresponding idempotent Bellman operator; the eigenvector defines stationary strategies and, sometimes, turnpikes.

We give two proofs of the spectrum existence theorem and examine the simplest situation in which one can prove that the eigenfunction is unique or at least that the eigenspace is finite-dimensional. As a consequence, we obtain asymptotic formulas describing the behavior of iterations of idempotent linear operators and a criterion for convergence and finiteness of the Neumann series.

In §2.4 we present the concept of infinite extremals recently proposed by S. Yu. Yakovenko [?, ?] for deterministic control problems and based on the spectral analysis of idempotent linear operators. In the end of §2.4, we discuss the reduction of homogeneous models of economical dynamics (von Neumann–Gale models) to the general dynamic optimization problem, thus defining infinite extremals for these models. Then we discuss the projection turnpike theory for these models. The iterates P^k, $k \to \infty$, of usual matrices converge to the projection on the eigenspace of P corresponding to the eigenvalue highest in magnitude; quite similarly, in idempotent analysis the iterates $B^k v$, $k \to \infty$, converge, under some nondegeneracy assumption, to the eigenvector h of B (corresponding to the unique eigenvalue of B) and this fact is, in a sense, equivalent to the turnpike theorem. This consideration reduces the turnpike theory, well-known in mathematical economics, to a mere consequence of the general results of idempotent analysis. To show how the method works, we present a simple proof of the classical Radner turnpike theorem.

In the last section (§2.5) we start dealing with nonlinear idempotent analysis. Although the Bellman operator B that arises in the theory of controlled Markov processes and stochastic games is apparently not linear in any semiring with idempotent addition, it inherits the homogeneity with respect to the generalized multiplication $\odot = +$ in the number semiring from the deterministic operator. Hence, the "eigenvector" equation $Bh = \lambda \odot h = \lambda + h$ makes sense in this case as well. Under quite general assumptions (the existence of a class of related states), this equation is uniquely solvable, and the solution determines stationary optimal strategies and turnpike control modes. This provides a unified approach to the study of the properties of optimal trajectories on an infinite time horizon for general controlled Markov jump processes with discrete or continuous time, for stochastic multistep and differential games, and for controlled quantum systems with observations. Just as homogeneous models of economical dynamics can be reduced to the general multistep optimization problem (§2.4), so stochastic homogeneous models (stochastic Neumann–Gale models [?]) can be reduced to the general multistep stochastic optimization problem, which permits one to use the results of §2.5 so as to prove turnpike theorems in these models. It also seems to be of interest to apply this conception to the dynamic theory of market equilibrium, for example, to multicurrency models such as those proposed in [?].

2.1. The General Form of Endomorphisms of the Space of Continuous Functions Ranging in an Idempotent Semimodule. Weak and Strong Convergence

In this section we study the general properties of A-linear continuous operators on (or homomorphisms of) semimodules $C_0(X)$, that is, of continuous mappings $B: C_{\mathbb{0}}(Y) \to C_{\mathbb{0}}(X)$ such that

$$B(a \odot h \oplus c \odot g) = a \odot B(h) \oplus c \odot B(g)$$

for each $a, c \in A$ and $h, g \in C_{\mathbb{0}}(Y)$.

Recall that A is the idempotent number semiring introduced in Example I.1.1.

Throughout the section we assume that all topological spaces in question are separable and locally compact. All notation to be used is introduced in §1.4.

First, let us give a precise characterization of "integral kernels" of an operator.

Theorem 2.1 *Let $B: C_{\mathbb{0}}(Y) \to C_{\mathbb{0}}(X)$ (respectively, $B: C_{\mathbb{0}}^{\infty}(Y) \to C_{\mathbb{0}}^{\infty}(X)$) be a continuous A-linear operator. Then*

1. *There exists a unique function $b: X \times Y \to A$ lower semicontinuous with respect to the second argument and such that*

$$(Bh)(x) = \inf_y b(x, y) \odot h(y). \tag{2.1}$$

2. *The function $b(x, y)$ is jointly lower semicontinuous in (x, y).*

3. *For any $(x_0, y_0) \in X \times Y$ and any $\varepsilon > 0$ there exist arbitrarily small neighborhoods $U_{x_0} \subset X$ of x_0 and $U_{y_0} \subset Y$ of y_0 such that*

$$\sup_{x \in U_{x_0}} \inf_{y \in U_{y_0}} b(x, y) < b(x_0, y_0) + \varepsilon.$$

4. *For any compact set $K_y \subset Y$ there exists a compact set $K_x \subset X$ such that $b(x, y) = \mathbb{0}$ (respectively, $\rho(b(x, y), \mathbb{0}) < \varepsilon$ for any prescribed ε) whenever $x \notin K_x$ and $y \in K_y$.*

5. *For the operator $B: C_{\mathbb{0}}^{\infty}(Y) \to C_{\mathbb{0}}^{\infty}(X)$, the kernel $b(x, y)$ is bounded on $X \times Y$ (that is, $b(x, y) \geq M > -\infty$ everywhere).*

Conversely, if $b: X \times Y \to A$ possesses properties 1–5, *then Eq.* (2.1) *specifies a continuous A-linear operator.*

Proof. Necessity. Since for each fixed x the expression $(Bh)(x)$ specifies a continuous A-linear functional on $C_{\mathbb{0}}(X)$ (respectively, on $C_{\mathbb{0}}^{\infty}(X)$), it readily follows from Theorem I.1.4 that B can be uniquely represented in the form (2.1) with kernel $b: X \times Y \to A$ lower semicontinuous in the second argument. Moreover, $b(x, y) = B(\delta_y)(x)$ is the value of the continued operator B on the δ-function

$$\delta_y(z) = \begin{cases} \mathbb{1} & \text{for } z = y, \\ \mathbb{0} & \text{for } z \neq y. \end{cases}$$

Let us now prove assertions 2 and 3. To be definite, assume that $b(x_0, y_0) \neq \mathbb{0}$. Since $b(x, y) = B(\delta_y)(x)$, it follows that for each $\varepsilon > 0$ there exist arbitrarily small neighborhoods U_{y_0} and V_{y_0} of the point y_0 and a function $h \in C_{\mathbb{0}}(Y)$ such that $y_0 \in V_{y_0} \subset \overline{V}_{y_0} \subset U_0$, $h \equiv \mathbb{1}$ in $\overline{V}_{y_0}$, $h \geq \mathbb{1}$ everywhere, $\operatorname{supp}_{\mathbb{0}} h \subset U_{y_0}$, and

$$b(x_0, y_0) - \varepsilon < Bh(x_0) \leq b(x_0, y_0). \tag{2.2}$$

Then, obviously,

$$\inf_{y \in U_{y_0}} b(x, y) \leq \inf_{y \in U_{y_0}} (b(x, y) + h(y)) = Bh(x) \leq \inf_{y \in V_{y_0}} b(x, y) \tag{2.3}$$

for each x.

Furthermore, since the function Bh is continuous, it follows that there exists an arbitrarily small neighborhood $U_{x_0} \subset X$ of the point x_0 such that $|Bh(x) - Bh(x_0)| < \varepsilon$ for $x \in U_{x_0}$. Hence, inequalities (2.2) and (2.3) imply

$$\inf_{y \in V_{y_0}} b(x, y) \geq (Bh)(x) > (Bh)(x_0) - \varepsilon > b(x_0, y_0) - 2\varepsilon,$$
$$\inf_{y \in U_{y_0}} b(x, y) \leq (Bh)(x) < (Bh)(x_0) + \varepsilon \leq b(x_0, y_0) + 2\varepsilon.$$

The last inequalities prove assertions 2 and 3.

Assertion 4 readily follows from the observation that the image of a function $h \in C_{\mathbb{0}}(Y)$ identically equal to $\mathbb{1}$ on K_y is compactly supported (respectively, tends to $\mathbb{0}$ at infinity).

The proof of assertion 5 for the operator $B\colon C_{\mathbb{0}}^{\infty}(Y) \to C_{\mathbb{0}}^{\infty}(X)$ is by contradiction. Indeed, let the values $b(x_n, y_n)$ converge to $-\infty$ along some sequence $\{(x_n, y_n)\} \subset X \times Y$. Then there exists an increasing sequence $a_n \to +\infty$ such that $b_n(x_n, y_n) + a_n \to -\infty$. Consider a function $h \in C_{\mathbb{0}}^{\infty}(Y)$ such that $h(y_n) = a_n$ for each n (to construct h, one must consider the closure of the sequence $\{y_n\}$ in the one-point compactification of Y and then apply Urysohn's continuation lemma). Then $Bh \notin C_{\mathbb{0}}^{\infty}(X)$, which is a contradiction.

Sufficiency. Let us prove that if b is jointly lower semicontinuous, then the operator B (2.1) takes lower semicontinuous functions with compact support $\operatorname{supp}_{\mathbb{0}}$ on Y to lower semicontinuous functions on X. The other assertions are very easy to prove, and we omit them altogether.

Let $\operatorname{supp}_{\mathbb{0}} h = K \subset Y$, and let the functions b and h be lower semicontinuous. Then

$$Bh(x_0) = \inf_{y} \{b(x_0, y) + h(y)\} = b(x_0, y_0) + h(y_0)$$

for some y_0. To be definite, assume that $h(y_0) \neq \mathbb{0}$ and $b(x_0, y_0) \neq \mathbb{0}$. Since K is compact, it follows that there exist finitely many points $y_j \in K$, $j = 0, \ldots, m$, neighborhoods U_j of y_j such that $\bigcup_j U_j \supset K$, and a neighborhood $V_0 \subset X$ of x_0 such that

$$b(x, y) + h(y) > b(x_0, y_{0j}) + h(y_{0j}) - \varepsilon$$

for any $y \in U_j$ and any $x \in V_0$. Then

$$Bh(x) = \min_j \inf_{y \in U_j} (b(x, y) + h(y)) > (Bh)(x_0) - \varepsilon$$

for $x \in V_0$. The theorem is proved.

The function $b(x, y)$ in Eq. (2.1) specifying the operator B will naturally be called the (*idempotent*) *integral kernel* of B, since Eq. (2.1) is an idempotent analog of the standard integral representation

$$(Kh)(x) = \int k(x, y)h(y)\, dy$$

of usual linear operators in $L^2(X)$.

Let us derive a composition formula for integral kernels.

Let $B\colon C_\mathbb{0}(Y) \to C_\mathbb{0}(Z)$ and $D\colon C_\mathbb{0}(Z) \to C_\mathbb{0}(X)$ be continuous A-linear operators. Then the integral kernel $d \circ b$ of the product $D \circ B\colon C_\mathbb{0}(Y) \to C_\mathbb{0}(X)$ can be expressed via the kernels d and b of D and B by the formula

$$d \circ b(x, y) = \inf_z d(x, z) \odot b(z, y). \tag{2.4}$$

Indeed,

$$\begin{aligned} D \circ B(h)(x) = \inf_z d(x, z) \odot (Bh)(z) &= \inf_z d(x, z) \odot \inf_y (b(z, y) \odot h(y)) \\ &= \inf_y \Big(\inf_z d(x, z) \odot b(z, y) \Big) \odot h(y), \end{aligned}$$

and it remains to observe that the function (2.4) is lower semicontinuous in the second argument.

Let us now present a criterion for strong convergence of an operator sequence.

Proposition 2.1 *A sequence of A-linear continuous operators $B_n\colon C_\mathbb{0}(Y) \to C_\mathbb{0}(X)$ with integral kernels b_n is strongly convergent to an operator B (that is, $\lim_{n\to\infty} B_n h = Bh$ in the topology of $C_\mathbb{0}(X)$ for each $h \in C_\mathbb{0}(Y)$) with integral kernel b if and only if the following two conditions on the kernels are satisfied.*

(a) *For any $x_0 \in X$, $y_0 \in Y$, and $\varepsilon > 0$, there exist arbitrarily small neighborhoods $U_0 \subset X$ of x_0 and $V_0 \subset Y$ of y_0 and a number N such that $\inf_{y \in V_0} b_n(x, y)$ is ε-close to $\inf_{y \in V_0} b(x, y)$ in the metric ρ on A for any $x \in U_0$ and any $n > N$.*

(b) *For any compact set $K_y \subset Y$, there exists a compact set $K_x \subset X$ such that b and all b_n are simultaneously equal to $\mathbb{0}$ for all $x \notin K_x$ and $y \in K_y$.*

Proof. Necessity. Property (b) is quite obvious. Let us prove (a). We consider only the case in which $b(x_0, y_0) \neq \mathbb{0}$. It follows from Theorem 2.1 that there exist arbitrarily small neighborhoods $U \subset X$ of x_0 and $V \subset Y$ of y_0 such that

$$b(x_0, y_0) - \varepsilon < \inf_{y \in V} b(x, y) \leq b(x_0, y_0) + \varepsilon \tag{2.5}$$

for any $x \in U$. Consequently, there exist embedded neighborhoods of y_0 in Y,

$$y_0 \in V_0 \subset \overline{V}_0 \subset V_1 \subset \overline{V}_1 \subset V_2,$$

and a neighborhood U_0 of the point $x_0 \in X$ such that inequality (2.5) holds for any $x \in U_0$ and for each of the neighborhoods V_j, $j = 0, 1, 2$. Consider a function $h \in C_{\mathbf{0}}(Y)$ such that $\operatorname{supp}_{\mathbf{0}} h \subset V_2$, $h \equiv \mathbb{1}$ on $\overline{V}_1$, and $h \geq \mathbb{1}$ everywhere. Then, starting from some number n, $(B_n h)(x)$ differs from $Bh(x)$ at most by ε for all $x \in U_0$. It follows that

$$\begin{aligned}\inf_{y \in V_2} b_n(x,y) &\leq (B_n h)(x) \leq (Bh)(x) + \varepsilon \\ &\leq \inf_{y \in V_1} b(x,y) + \varepsilon \leq b(x_0, y_0) + 2\varepsilon \leq \inf_{y \in V_2} b(x,y) + 3\varepsilon\end{aligned}$$

and

$$\begin{aligned}\inf_{y \in V_1} b_n(x,y) &\geq (B_n h)(x) \geq (Bh)(x) - \varepsilon \\ &\geq \inf b(x,y) - \varepsilon \geq b(x_0, y_0) - 2\varepsilon \geq \inf_{y \in V_1} b(x,y) - 3\varepsilon.\end{aligned}$$

Similarly, by considering the pair (V_0, V_1), we obtain the first of the preceding inequalities for V_1, and so for this neighborhood we have a two-sided inequality.

Sufficiency. Just as in the proof of sufficiency in Theorem I.4.3, in the present case we find that for any $h \in C_{\mathbf{0}}(Y)$, $x_0 \in X$, and $\varepsilon > 0$, there exists a neighborhood $U_0 \subset X$ of x_0 such that $(B_n h)(x)$ is ε-close to $(Bh)(x)$ in the metric ρ on A for large n uniformly with respect to $x \in U_0$. It remains to use property (b), which says that $\left(\bigcup_{n=1}^{\infty} \operatorname{supp}_{\mathbf{0}} B_n h\right) \cup \operatorname{supp}_{\mathbf{0}} Bh$ lies in some compact subset of X. The theorem is proved.

Theorem I.4.4 readily implies a criterion for weak convergence of A-linear operators. By analogy with the conventional functional analysis, we say that a sequence $B_n \colon C_{\mathbf{0}}(Y) \to C_{\mathbf{0}}(X)$ of A-linear operators is weakly convergent to an operator B if for any $h \in C_{\mathbf{0}}(Y)$ and $g \in C_{\mathbf{0}}(X)$ we have

$$\lim_{n \to \infty} \langle B_n h, y \rangle_A = \langle Bh, g \rangle_A,$$

where the inner product $\langle\ ,\ \rangle_A$ is defined in Eq. (I.4.4).

Proposition 2.2 *A sequence of operators B_n with kernels b_n is weakly convergent to an operator B with kernel b if and only if the function sequence $b_n \colon X \times Y \to A$ is weakly convergent to $b \colon X \times Y \to A$ on the space $C_{\mathbf{0}}(X \times Y)$.*

Let us give two more propositions; we omit the elementary proofs.

Proposition 2.3 (a criterion for uniform convergence of an operator sequence $B_n \colon C_{\mathbf{0}}^{\infty}(Y) \to C_{\mathbf{0}}^{\infty}(X)$, i.e., convergence in the metric of the space of continuous mappings of metric spaces). *The sequence B_n is uniformly convergent to B if and only if the sequence of integral kernels $b_n(x,y)$ is uniformly convergent to the integral kernel of B.*

Proposition 2.4 (idempotent analog of the Banach–Steinhaus theorem on uniform boundedness). *Let $\{B_\alpha : C_0^\infty(Y) \to C_0^\infty(X)\}$ be an operator family such that for each $h \in C_0^\infty(Y)$ the function family $B_\alpha h$ is bounded, $B_\alpha h \geq M > -\infty$ for all α. Then the family $\{b_\alpha\}$ of their integral kernels is also bounded, $b_\alpha(x, y) \geq \widetilde{M} > -\infty$ for all x, y, and α.*

2.2. Invertible and Compact Operators

In this section we study two important classes of A-linear operators, namely, invertible and compact operators.

Theorem 2.2 ([?, ?] (structure of invertible operators)) *Let*

$$B\colon C_0(Y) \to C_0(X) \text{ and } D\colon C_0(X) \to C_0(Y)$$

or

$$B\colon C_0^\infty(Y) \to C_0^\infty(X) \text{ and } D\colon C_0^\infty(X) \to C_0^\infty(Y)$$

be mutually inverse A-linear operators. Then there exists a homeomorphism $\beta\colon X \to Y$ and continuous functions $\varphi\colon X \to A$ and $\psi\colon Y \to A$ nowhere assuming the value $\mathbb{0}$ such that $\varphi(x) \odot \psi(\beta(x)) \equiv \mathbb{1}$ and the operators B and D are given by the formulas

$$(Bh)(x) = \varphi(x) \odot h(\beta(x)), \tag{2.6}$$

$$(Dg)(y) = \psi(y) \odot g(\beta^{-1}(y)). \tag{2.7}$$

Proof. Theorem 2.1 and the composition law (2.4) permit us to write out the condition that D and B are inverses of each other in the form of two equations for the integral kernels b and d of B and D, respectively:

$$\begin{aligned} \inf_y b(x, y) \odot d(y, z) = b \circ d(x, z) &= \begin{cases} \mathbb{1}, & x = z, \\ \mathbb{0}, & x \neq z, \end{cases} \\ \inf_x d(y, x) \odot b(x, t) = d \circ b(y, t) &= \begin{cases} \mathbb{1}, & y = t, \\ \mathbb{0}, & y \neq t. \end{cases} \end{aligned} \tag{2.8}$$

It readily follows that for each $x \in X$ there exists a $y(x) \in Y$ such that $b(x, y(x)) \neq \mathbb{0}$ and $d(y(x), x) \neq 0$, and for each $y \in Y$ there exists an $x(y) \in X$ such that $d(y, x(y)) \neq \mathbb{0}$ and $b(x(y), y) \neq \mathbb{0}$. Moreover, $y(x)$ and $x(y)$ are uniquely determined, since the conditions $y_1 \neq y_2$, $d(y_i, x) \neq \mathbb{0}$ would imply $d \circ b(y_i, y(x)) \neq \mathbb{0}$, $i = 1, 2$, which contradicts (2.8). Thus, there exists a bijection $\beta\colon X \to Y$ such that $b(x, y) \neq 0 \iff d(y, x) \neq \mathbb{0} \iff y = \beta(x)$. It is now obvious that, by virtue of Eq. (2.8), the functions $\varphi(x) = b(x, \beta(x))$ and $\psi(y) = d(y, \beta^{-1}(y))$ satisfy the identity $\varphi(x) \odot \psi(\beta(x)) \equiv \mathbb{1}$ and Eqs. (2.6) and

(2.7). The continuity of φ, ψ, and β can readily be proved by contradiction. The theorem is proved.

Thus, each invertible operator is the composition of a diagonal operator (i.e., multiplication by a function nowhere equal to $\mathbb{0}$) with a "change of variables." Classical analogs of such operators are known as weighted translation operators.

Corollary 2.1 *If the topological semimodules $C_0(X)$ and $C_0(Y)$ are homeomorphic, then so are the topological spaces X and Y (a similar fact is known to be valid in conventional analysis).*

Corollary 2.2 *We say that an operator $B\colon C_0(X) \to C_0(X)$ is symmetric if $\langle B\varphi, h\rangle_A = \langle \varphi, Bh\rangle_A$ for each $\varphi, h \in C_0(X)$. It is clear that B is symmetric if and only if its integral kernel is symmetric. It follows from Theorem 2.2 that each invertible symmetric operator has the form*

$$(Bh)(x) = \varphi(x) \odot h(\beta(x)),$$

where $\beta = \beta^{-1}\colon X \to X$ is an involutive homeomorphism and $\varphi(x)$ is a continuous function $X \to A$ such that $\varphi(x) = \varphi(\beta(x))$.

Corollary 2.3 *We say that an invertible operator B is* orthogonal *if $B^{-1} = B'$ is the adjoint of B with respect to the inner product on $C_0(X)$. Each orthogonal operator on $C_0(X)$ is generated by a "change of variables" on X.*

Corollary 2.4 *If X is a finite set, $X = \{1, \dots, n\}$, then $C(X, A) = A^n$ and the operators $A^n \to A^n$ are represented by $n \times n$ matrices with entries in A. It follows from Theorem 2.2 that there are "very few" invertible operators on A^n; namely, these are the compositions of diagonal matrices with permutations of the elements of the standard basis (see §1.1).*

Remark 2.1 The study of the relationship between the properties of a topological space X and the properties of the linear space of continuous functions on X is very important in general topology. The notion of linear equivalence (l-equivalence) of topological spaces plays the central role in these studies. Idempotent analysis gives rise to a natural analog of this notion. Let X be a topological space, and let $C_p(X, A)$ denote the semimodule of continuous functions $X \to A$ equipped with the topology of pointwise convergence. We say that X is linearly equivalent (l-equivalent) to a topological space Y in the sense of the semiring A if the semimodules $C_p(X, A)$ and $C_p(Y, A)$ are isomorphic (the semimodules C_p are introduced in the end of §1.3). In the usual notion of l-equivalence, the semiring is the ring of real numbers. In that case, the problem of finding general criteria for l-equivalence is rather delicate. For example, it is worth noting that although any two compact sets with isomorphic spaces of continuous functions are homeomorphic, l-equivalence of compact sets does not imply that they are homeomorphic (e.g., a closed interval and a disjoint union of two closed intervals are equivalent). It turns out that the situation is simpler if l-equivalence in the sense of a semiring with idempotent addition is considered. Using Theorem I.3.6, we can generalize Theorem 2.2 to the spaces $C_p(X, A)$ so that if X and Y are completely regular topological spaces, then they are l-equivalent in the sense of the semiring A if and only if they are homeomorphic [**?**].

One more result following from Theorem 2.2 is that all automorphisms of the operator semialgebra are inner automorphisms. More precisely, the following theorem is valid.

Theorem 2.3 *Any automorphism F (i.e., a homeomorphism preserving the $\oplus$-addition and the composition of operators) of the semialgebra of linear operators on $C_{\mathbb{0}}(X)$ takes an operator B with integral kernel b to the operator $F(B)$ with integral kernel Fb according to the formula*

$$Fb(x, y) = \varphi(x) - \varphi(y) + b(\beta(x), \beta(y)), \tag{2.9}$$

where $\beta: X \to X$ is a homeomorphism and $\varphi: X \to A \setminus \{\mathbb{0}\}$ is a continuous function. In other words,

$$FB = C \circ B \circ C^{-1},$$

where C is an invertible operator, which is the composition of a diagonal operator (the operator of $\odot$-multiplication by φ) with the operator induced by a change of variables.

Proof. Since operators are in one-to-one correspondence with integral kernels, that is, functions $X \times X \to A$, we can derive the general form of isomorphisms of the semimodule of operators (neglecting the semialgebra structure determined by the composition of operators) from Theorem 2.2, which says that such an isomorphism must have the form

$$b(x, y) \to Fb(x, y) = \widetilde{\varphi}(x, y) \odot b(\beta_1(x, y), \beta_2(x, y)), \tag{2.10}$$

where $\tilde{\beta} = (\beta_1, \beta_2): X \times X \to X \times X$ is a homeomorphism. (Rigorously speaking, Theorem 2.2 does not apply directly to our case since the integral kernels need not be continuous but are only lower semicontinuous. However, by analyzing the proof of Theorem 2.2 and Theorem 2.1, one can observe that the statement concerning the general form of invertible transformations remains valid in this case.) Let $h_n^\xi(x)$ be a δ-shaped sequence in $C_{\mathbb{0}}(X)$, that is, a sequence of continuous functions such that $h_n^\xi(x) \geq \mathbb{1}$ for all $x \in X$, $h_n^\xi(\xi) = \mathbb{1}$, and $\operatorname{supp}_{\mathbb{0}} h_n^\xi \subset U_n(\xi)$, where $\{U_n(\xi)\}$ is a base of neighborhoods of the point ξ (if X does not satisfy the first countability axiom, then such a sequence may fail to exist, and we must take a net instead). Let ξ, η, and ζ be arbitrary points in X. Consider the sequences

$$B_n^{\xi,\eta}, B_n^{\eta,\zeta}: C_{\mathbb{0}}(X) \to C_{\mathbb{0}}(X)$$

of idempotent operators with integral kernels

$$b_n^{\xi,\eta}(x,y) = h_n^{\xi}(x) \odot h_n^{\eta}(y), \qquad b_n^{\eta,\zeta}(x,y) = h_n^{\eta}(x) \odot h_n^{\zeta}(y).$$

Then, by the composition formula, $B_n^{\xi,\eta} \circ B_n^{\eta,\zeta} = B_n^{\xi,\zeta}$, that is, this operator has the kernel

$$b_n^{\xi,\zeta}(x,y) = h_n^{\xi}(x) \odot h_n^{\zeta}(y).$$

Let us now apply Eq. (2.10) on both sides of the identity

$$FB_n^{\xi,\zeta} = FB_n^{\xi,\eta} \circ FB_n^{\eta,\zeta}$$

and pass to the limit as $n \to \infty$ pointwise. As a result, we obtain the identity

$$\begin{aligned} \widetilde{\varphi}(x,y) \odot \delta^{\xi}_{\beta_1(x,y)} \odot \delta^{\zeta}_{\beta_2(x,y)} = \inf_z \widetilde{\varphi}(x,z) \odot \widetilde{\varphi}(z,y) \odot \delta^{\xi}_{\beta_1(x,z)} \\ \odot \delta^{\eta}_{\beta_2(x,z)} \odot \delta^{\eta}_{\beta_1(z,y)} \odot \delta^{\zeta}_{\beta_2(z,y)}, \end{aligned} \tag{2.11}$$

where, as usual,

$$\delta^x_y = \begin{cases} \mathbb{1} & \text{for } x = y, \\ \mathbb{0} & \text{for } x \neq y. \end{cases}$$

Set $\xi = \beta_1(x,y)$ and $\zeta = \beta_2(x,y)$. Then the left-hand side in Eq. (2.11) is $\widetilde{\varphi}(x,y) \neq \mathbb{0}$. Thus, the right-hand side is also different from $\mathbb{0}$, and we find that for each η there necessarily exists a $z_0 = z(x,y,\eta)$ such that

$$\begin{aligned} &\beta_1(x,z_0) = \beta_1(x,y) = \xi, \\ &\beta_2(z_0,y) = \beta_2(x,y) = \zeta, \\ &\beta_1(z_0,y) = \beta_2(x,z_0) = \eta. \end{aligned} \tag{2.12}$$

Set $\eta = \xi$. Then the identities $\beta_1(z_0,y) = \xi$, $\beta_2(z_0,y) = \zeta$ and $\beta_1(x,y) = \xi$, $\beta_2(x,y) = \zeta$ together with the fact that $\tilde{\beta} = (\beta_1,\beta_2)$ is a self-bijection of $X \times X$ imply that $z_0 = z(x,y,\xi) = x$. Using Eq. (2.12) in this case once more, we find that $\beta_1(x,x) = \beta_1(x,y)$, that is, $\beta_1(x,y) = \beta_1(x)$ is independent of the second argument. Similarly, by setting $\eta = \zeta$ in Eq. (2.11) we find that $\beta_2(x,y) = \beta_2(y)$ is independent of the first argument. Finally, it follows from the last equation in (2.11) that $\beta_1(x) = \beta_2(x)\ \forall\, x \in X$. Thus, the homeomorphism $\tilde{\beta} = (\beta_1,\beta_2)$ is the product of two identical homeomorphisms of X, $\tilde{\beta}(x,y) = (\beta(x),\beta(y))$. Then it follows from Eq. (2.12) that

$$\widetilde{\varphi}(x,y) = \widetilde{\varphi}(x,z) \odot \widetilde{\varphi}(z,y)$$

for $z = \beta^{-1}(\eta)$, that is, for an arbitrary z, since η is arbitrary. The last functional equation implies that

$$\widetilde{\varphi}(x,y) = \varphi(x) - \varphi(y)$$

for some function $\varphi: X \to A \setminus \{\mathbb{0}\}$, and hence, Eq. (2.10) coincides with Eq. (2.9), as desired. Theorem 2.3 is proved.

Invertible operators on $C_{\mathbb{0}}(X)$ form a group. Operators B_1 and B_2 on $C_{\mathbb{0}}(X)$ are said to be *conjugate* if there exists an invertible A-linear operator C such that $B_1 = C^{-1}B_2C$. As is the case in conventional linear algebra, by invariant properties or characteristics of an operator we mean properties or characteristics that are the same for all conjugate operators. If invertible operators are conceived of as "changes of variables" in $C_{\mathbb{0}}(X)$, then conjugate operators become just various "coordinate representations" of the same operator. As in conventional analysis, the main invariant characteristic of an operator is its spectrum, which is considered in the next section.

Another important class of operators is compact or completely continuous operators.

Definition 2.1 A continuous A-linear operator $B: C_{\mathbb{0}}^{\infty}(Y) \to C_{\mathbb{0}}^{\infty}(X)$ is said to be *compact*, or *completely continuous*, if it carries each set bounded in the metric to a precompact set.

For simplicity, in what follows we assume that X is a separable space, so that its topology is determined by some metric d.

Theorem 2.4 ([?]) *An operator B is compact if and only if its integral kernel $b(x, y)$ is equicontinuous in x with respect to y (that is, the condition*

$$\forall \varepsilon > 0\ \exists \delta > 0 :\ d(x_1, x_2) < \delta \implies \forall y\ \ \rho(b(x_1, y), b(x_2, y)) < \varepsilon$$

is satisfied) and tends to $\mathbb{0}$ at infinity uniformly with respect to $y \in Y$ (that is,

$$\forall \varepsilon > 0\ \exists K_x \subset X :\ K_x \text{ is compact and } \forall x \notin K_x\ \forall y\ \ \rho(b(x, y), \mathbb{0}) < \varepsilon).$$

The proof is based on the Arzela–Ascoli theorem, which can be stated as follows for A-valued functions: a subset $M \subset C_{\mathbb{0}}^{\infty}(X)$ is precompact if and only if it is uniformly bounded ($\varphi(x) \geq c > -\infty\ \forall \varphi \in M$), equicontinuous, and uniformly tends to $\mathbb{0}$ at infinity.

Here we only demonstrate the necessity of the first condition in Theorem 2.4; the remaining part of the proof is quite simple and we omit it altogether. Suppose that this condition is violated. Then there exists an $\varepsilon > 0$, a sequence $\delta_n \to 0$, and sequences $\{x_n^1\}, \{x_n^2\} \subset X$ and $\{y_n\} \subset Y$ such that $d(x_n^1, x_n^2) < \delta_n$ and

$$\rho(b(x_n^1, y_n), b(x_n^2, y_n)) > 2\varepsilon. \tag{2.13}$$

It follows from the lower semicontinuity of b and y that there exists a sequence of neighborhoods $U_n \ni y_n$ in Y such that $b(x_n^i, y) > b(x_n^i, y_n) - \varepsilon_n$ for $y \in U_n$, $i = 1, 2$, where ε_n is an arbitrary positive sequence, $\varepsilon_n \to 0$ (to simplify the notation, we assume that the sequences $b(x_n^i, y_n)$ are bounded away from zero). Furthermore, let us construct a sequence of continuous functions $h_n \geq 1$, $h_n(y_n) = \mathbb{1}$, with support $\operatorname{supp}_{\mathbb{0}} h_n \subset U_n$ such that

$$b(x_n^i, y_n) \geq (Bh_n)(x_n^i) \geq b(x_n^i, y_n) - \varepsilon_n.$$

By choosing the sequence ε_n so that

$$\rho(b(x_n^i, y_n), b(x_n^i, y_n) - \varepsilon_n) < \varepsilon/2 \quad \text{for all } n,$$

from the last estimate and from (2.13) we obtain

$$\rho((Bh_n)(x_n^1), (Bh_n)(x_n^2)) > \varepsilon. \tag{2.14}$$

Since B is compact, it follows that there exists a subsequence $\{Bh_{n_k}\}$ convergent to some $v \in C_{\mathbb{0}}^\infty(X)$; consequently, for large n_k we have

$$\begin{aligned}\rho(Bh_{n_k}(x_{n_k}^1), Bh_{n_k}(x_{n_k}^2)) &\leq \rho(Bh_{n_k}(x_{n_k}^1), v(x_{n_k}^1)) \\ &\quad + \rho(Bh_{n_k}(x_{n_k}^2), v(x_{n_k}^2)) + \rho(v(x_{n_k}^1), v(x_{n_k}^2)) < \varepsilon,\end{aligned}$$

which contradicts (2.14).

Corollary 2.5 *Each compact operator $B: C_{\mathbb{0}}^\infty(Y) \to C_{\mathbb{0}}^\infty(X)$ can be continued to the space of all bounded A-valued functions on Y and takes this space to $C_{\mathbb{0}}^\infty(X)$.*

2.3. Spectra of Compact Operators and Dynamic Programming

The idea of using the spectral characteristics of A-linear operators for estimating the behavior of their iterations, which arise in solving optimization problems by Bellman's dynamic programming and its various modifications, goes back to V. I. Romanovskii [**?**, **?**]. In [**?**, **?**], methods of nonstandard analysis were used to obtain a spectrum existence theorem for integral operators with continuous kernel nowhere equal to $\mathbb{0}$ acting on the space of continuous functions from a compactum into an idempotent semiring with multiplication $\odot$ satisfying the cancellation law. Here we prove the spectrum existence theorem for general compact A-linear operators on the spaces $C_{\mathbb{0}}^\infty(X)$ for the case in which A is the number semiring.

We start from the classical Frobenius–Perron theorem in linear algebra, since the proof of this theorem given below serves as a model for all other proofs in this section.

Let $\mathbb{R}_+^n$ be the nonnegative orthant in $\mathbb{R}^n$, i.e., the set

$$\mathbb{R}_+^n = \{v = (v_1, \dots, v_n) \in \mathbb{R}^n : v_j \geq 0 \ \forall j\},$$

and let $\operatorname{int} \mathbb{R}_+^n$ be the interior of $\mathbb{R}_+^n$, i.e., the set of vectors all of whose coordinates are positive.

Theorem 2.5 (Frobenius–Perron) *If M is a nondegenerate real $n \times n$ matrix with nonnegative entries, then it has a positive eigenvalue λ with the corresponding eigenvector $v \in \mathbb{R}^n_+$. If all entries of M are positive, then $v \in \operatorname{int} \mathbb{R}^n_+$ and the eigenvalue $\lambda > 0$ for which the corresponding eigenvector lies in $\operatorname{int} \mathbb{R}^n_+$ is unique.*

Proof. Let us consider the following equivalence relation on $\mathbb{R}^n_+$: $v \sim w$ if and only if $v = \lambda w$ for some $\lambda > 0$. Obviously, the quotient space $K = [\mathbb{R}^n_+ \setminus \{0\}]/\sim$ is homeomorphic to the standard simplex $\{v \in \mathbb{R}^n_+ : v_1 + \cdots + v_n = 1\}$. Since M is nondegenerate and its entries are nonnegative, we have

$$M: \mathbb{R}^n_+ \setminus \{0\} \to \mathbb{R}^n_+ \setminus \{0\},$$

and since M is linear, it follows that

$$M(\lambda v) = \lambda M(v) \quad \forall\, v \in \mathbb{R}^n.$$

Thus, the quotient mapping $\widetilde{M}: K \to K$ is well defined and takes any ray $\{\lambda v,\, \lambda > 0\}$ to the ray $\{\lambda M(v),\, \lambda > 0\}$. By Brouwer's theorem, $\widetilde{M}$ has a fixed point, which means that there exists a $\lambda > 0$ and a $v \in \mathbb{R}^n_+ \setminus \{0\}$ such that $Mv = \lambda v$. Now let all entries of M be positive. Then, obviously,

$$M: \mathbb{R}^n_+ \setminus \{0\} \to \operatorname{int} \mathbb{R}^n_+,$$

whence it follows that $v \in \operatorname{int} \mathbb{R}^n_+$. To prove the uniqueness, it is convenient to consider the adjoint of M, which is specified by the transpose matrix M'. Let $\mu > 0$ be a positive eigenvalue of M' with eigenvector $y \in \operatorname{int} \mathbb{R}^n_+$. Obviously, the inner product of any two vectors in $\operatorname{int} \mathbb{R}^n_+$ is positive. Hence, the identities

$$\lambda(v, w) = (\lambda v, w) = (Mv, w) = (v, M'w) = (v, \mu w) = \mu(v, w)$$

imply that $\lambda = \mu$, whence the uniqueness follows.

Remark 2.2 Clearly, to prove that there exist $\lambda > 0$ and $v \in \mathbb{R}^n_+ \setminus \{0\}$ with $Mv = \lambda v$, it suffices to require M to be positively homogeneous in the cone $\mathbb{R}^n_+$, that is,

$$M(\lambda v) = \lambda M(v) \quad \forall\, \lambda > 0,\ v \in \mathbb{R}^n_+.$$

The additivity of M is nowhere used in the proof.

Let us return to idempotent analysis and begin with stating a theorem on the spectra of A-linear operators on A^n; this theorem is actually a direct corollary of the preceding assertion.

Theorem 2.6 *Let $B: A^n \to A^n$ be a continuous A-linear mapping satisfying the nondegeneracy condition $B^{-1}(\mathbb{0}) = \mathbb{0}$. (Here $\mathbb{0} \in A^n$ is the vector all of whose coordinates are equal to $\mathbb{0}$. In the matrix form, this condition implies* *that each column of B contains at least one entry different from $\mathbb{0}$.) Then B has an eigenvalue $\alpha \in A$, $\alpha \neq 0$, and an eigenvector $v \in A^n \setminus \{0\}$ such that*

$$B(v) = \alpha \odot v = \alpha + v.$$

If, moreover, B takes $A^n \setminus \{\mathbb{0}\}$ to $(A \setminus \{\mathbb{0}\})^n$, that is, all entries of B are different from $\mathbb{0}$, then $v \in (A \setminus \{\mathbb{0}\})^n$ and the eigenvalue $\alpha \neq 0$ is unique.

Remark 2.3 If B is degenerate, then B obviously has an eigenvector with eigenvalue $\mathbb{0}$.
Proof. The mapping

$$E: v = (v_1, \ldots, v_n) \mapsto Ev = (e^{-v_1}, \ldots, e^{-v_n})$$

is an isometry of A^n onto $\mathbb{R}^n_+$ (the norm on $\mathbb{R}^n_+$ is assumed to be given by the maximum of absolute values of the coordinates). This mapping transforms B into the operator

$$M = E \circ B \circ E^{-1}: \mathbb{R}^n_+ \to \mathbb{R}^n_+.$$

It is clear that the A-homogeneity condition for B,

$$B(\lambda \odot v) = \lambda \odot B(v),$$

is equivalent to the usual positive homogeneity of M. Hence, the existence of an eigenvalue of B follows from the Frobenius–Perron theorem (with regard to Remark 2.2). The proof of uniqueness reproduces the corresponding argument in the Frobenius–Perron theorem word for word; the only difference is that instead of the usual $\mathbb{R}^n$ inner product, the A-bilinear product (I.1.4) is used on A^n. The theorem is proved.

Let us now extend Theorem 2.6 to operators on general spaces $C_{\mathbb{0}}^\infty(X)$. The proof scheme remains the same, but the choice of a representation for the quotient space is somewhat more complicated and, instead of Brouwer's theorem, we use the more general Schauder theorem, or, more precisely, the following obvious corollary of this theorem: if T is a continuous self-mapping of a closed convex subset in a Banach space such that the image of T is precompact, then T has a fixed point.

Let us recall (see §2.2) that an operator $B: C_{\mathbb{0}}^\infty \to C_{\mathbb{0}}^\infty(X)$ is compact if its integral kernel $b: X \times X \to A$ is lower semicontinuous, bounded below ($b(x, y) \geq D > -\infty$), equicontinuous in x with respect to y, and tends to $\mathbb{0}$ uniformly with respect to y as $x \to \infty$. In addition, we shall assume that X is a locally compact metric space, d is a metric on X, and μ is a (usual) regular Borel measure on X such that $\mu(X) = 1$ and the measure of each open subset in X is positive (obviously, such a measure always exists). Set $D = \inf b(x, y)$.

We shall now impose some nondegeneracy conditions on B. These conditions are slightly stronger that the property $B^{-1}(\mathbb{0}) = \mathbb{0}$ (here $\mathbb{0}$ is the function identically equal to $\mathbb{0}$ on X), which is equivalent to the requirement that the interior of the set

$$\{y \in X : b(x,y) = \mathbb{0} \text{ for all } x \in X\}$$

is empty.

Condition (A) $\forall\, y\, \exists\, x$: $b(x,y) \neq \mathbb{0}$.

Condition (B) $\exists\, c \neq \mathbb{0}$: $\inf_x b(x,y) \leq c\ \forall\, y$.

Condition (C) The image $B(C_{\mathbb{0}}^{\infty}(X) \setminus \{\mathbb{0}\})$ consists of functions nowhere equal to $\mathbb{0}$ $\iff$ the interior of the set $\{y \in X \mid \exists\, x : b(x,y) = 0\}$ is empty.

Condition (B) is the main condition in the proof of Theorem 2.7.

Note that if X is a compact set and the integral kernel $b(x,y)$ is continuous, then condition (B) directly follows from Condition (A).

Theorem 2.7 *Let B be a compact A-linear operator on $C_{\mathbb{0}}^{\infty}(X)$ satisfying condition* (B)*. Then B has an eigenvalue $\alpha \neq \mathbb{0}$ and an eigenfunction $h \in C_{\mathbb{0}}^{\infty}(X)$, $h \not\equiv \mathbb{0}$, such that $Bh = \alpha \odot h = \alpha + h$. If, moreover, B satisfies condition* (C)*, then the eigenvalue is unique and the eigenfunction is nowhere equal to $\mathbb{0}$.*

Remark 2.4 The eigenfunction need not be unique.

First proof. The refinement related to condition (C) is obvious, and uniqueness can be proved just in the same way as in the Frobenius–Perron theorem and Theorem 2.6. Thus, we only have to prove the existence of an eigenvalue under condition (B).

The mapping

$$E\colon h(x) \mapsto (Eh)(x) = \exp(-h(x))$$

is an isometry of $C_{\mathbb{0}}^{\infty}(X)$ onto the cone $C_+(X)$ of nonnegative functions in the Banach space $C(X)$ of real continuous functions on X. It is easy to see that the continuous mapping

$$M = E \circ B \circ E^{-1}\colon C_+(X) \to C_+(X)$$

satisfies the following conditions:

(a) M is positive homogeneous, that is,
$$M(\lambda g) = \lambda M(g) \quad \forall\, g \in C_+(X)\ \ \forall\, \lambda > 0;$$

(b) M satisfies the estimates
$$\exp(-C)\|g\| \leq \|Mg\| \leq \exp(-D)\|g\| \quad \forall\, g \in C_+(X);$$
in particular, it follows that $M^{-1}(0) = 0$;

(c) $\forall \varepsilon > 0\ \exists\, \delta > 0$: $d(x_1,x_2) < \delta \implies |Mg(x_1) - Mg(x_2)| \leq \varepsilon\|g\|\ \forall\, g \in C_+(X)$.

Let us introduce the following equivalence relation on $C_+(X)$: $v \sim g$ if $v = \lambda g$ for some $\lambda > 0$. By Φ we denote the quotient space

$$\Phi = (C_+(X) \setminus \{0\})/\sim$$

and by $\widetilde{M}$ the corresponding quotient mapping $\widetilde{M}\colon \Phi \to \Phi$, which takes each ray $\{\lambda v,\, \lambda > 0\}$ to the ray $\{\lambda M(v),\, \lambda > 0\}$. It is clear that Φ can be identified with the subset $\{v \in C_+(X) : \|v\| = 1\}$ of the unit sphere in $C(X)$ and that in this representation of Φ the operator $\widetilde{M}$ is given by the formula

$$\widetilde{M} = \operatorname{pr} \circ M\colon \Phi \to \Phi,$$

where pr is the projection

$$\operatorname{pr}\colon C_+(X) \setminus \{0\} \to \Phi, \qquad \operatorname{pr}(v) = v/\|v\|.$$

It readily follows from the two-sided estimates (b) and from property (c) that the image $\widetilde{M}(\Phi) \subset \Phi$ is a precompact set, since it is bounded and equicontinuous.

The existence of an eigenvector and an eigenvalue of B is equivalent to the existence of a fixed point of the mapping $\widetilde{M}$ (cf. Theorem 2.6). However, Φ is not convex, and Schauder's theorem does not apply directly to this case.

To conclude that $\widetilde{M}$ has a fixed point, it suffices to show that Φ is homeomorphic to a bounded closed convex subset in $C(X)$. To this end, we use the above-defined measure μ on X. Let us define a mapping $\Pi\colon \Phi \to L$, where $L \subset C(X)$ is the hyperplane determined by the equation $\int g(x)\, d\mu(x) = 0$, by setting

$$\Pi(g) = g - \lambda(g),$$

where $\lambda(g)$ is the (obviously unique) number such that $g - \lambda(g) \in L$. Since the μ-measure of any open set in X is positive, it follows that $\lambda(g) \in (0, 1)$ for each $g \in \Phi$ and that the mapping Π is continuous and injective. It is easy to see that

$$\Pi(\Phi) = \{g \in L : \max g \leq 1 + \min g\}.$$

Clearly, $\Pi(\Phi)$ is a closed subset of the unit ball in $C(X)$. Let us prove that $\Pi(\Phi)$ is convex. Let $v, w \in \Pi(\Phi)$; then

$$\max v \leq 1 + \min v \quad \text{and} \quad \max w \leq 1 + \min w.$$

Let us take the sum of these inequalities with some weights $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. We obtain

$$\alpha \max v + \beta \max w \leq 1 + \alpha \min v + \beta \min w.$$

Since

$$\max(\alpha v + \beta w) \leq \alpha \max v + \beta \max w$$

and

$$\alpha \min v + \beta \min w \leq \min(\alpha v + \beta w),$$

it follows that

$$\max(\alpha v + \beta w) \leq 1 + \min(\alpha v + \beta w),$$

and consequently, $\alpha v + \beta w \in \Pi(\Phi)$. Thus, $\Pi(\Phi)$ is convex. We now apply Schauder's theorem and conclude that the mapping

$$\Pi \circ \widetilde{M} \circ \Pi^{-1} \colon \Pi(\Phi) \to \Pi(\Phi)$$

has a fixed point, and hence so does the mapping $\widetilde{M} \colon \Phi \to \Phi$. It follows that the operator B has an eigenvalue and an eigenvector, and the proof is complete.

Second proof (this proof was communicated to the authors by M. Bronstein and is in fact a generalization of the argument from [**?**]). Suppose momentarily that X is compact. By virtue of the conditions imposed on the kernel $b(x, y)$, for each $\varepsilon > 0$ we can choose a finite ε-net $J = \{x_i\} \subset X$ so that

1) $\forall\, x \in X\ \exists\, x_j \in J: \quad \rho(b(x_j, y), b(x, y)) < \varepsilon, \quad \forall\, y \in X;$
2) $\rho\Big(\inf_{y_j \in J} b(x, y_j), \inf_{y \in X} b(x, y)\Big) \leq \varepsilon, \quad \forall\, x \in X;$
3) $\min_{x_i \in J} b(x_i, y) < c + 1 \quad \forall\, y \in X.$

Using Theorem 2.6, we can choose a function $g_J(x_i)$ ($x_i \in J$) such that

$$\inf_{y_j \in J} b(x_i, y_j) + g_J(y_j) = \alpha_J + g(x_i)$$

for each x_i. Without loss of generality it can be assumed that

$$g_J(\hat{x}) = \min_{x_j} g_J(x_j) = 0.$$

Hence,

$$\begin{aligned} c + 1 &\geq \min_{x_i \in J} b(x_i, \hat{x}) > \min_{x_i, y_j \in J} (b(x_i, y_j) + g_J(y_j)) = \min_{x_i \in J} (\alpha_J + g_J(x_i)) \\ &\geq \alpha_J = \inf_{y_j \in J} (b(\hat{x}, y_j) + g_J(y_j)) \geq \inf_{(x,y) \in X \times X} b(x, y) \geq d, \end{aligned}$$

that is, $\alpha_J \in [d, c + 1]$.

Let us introduce the function

$$h_J(x) = \min_{y_j \in J}(b(x, y_j) + g_J(y_j)).$$

The function family h_J is equicontinuous, since

$$h_J(u) - h_J(v) \leq \max_y (b(u, y) - b(v, y)),$$

and is bounded in A. Thus, the family (α_J, h_J) has a limit point $(\alpha, h) \in [d, c+1] \times C(X, A)$. This limit point satisfies the equation

$$h(x) + \alpha = \min_{y \in X}(b(x, y) + h(y)).$$

Indeed, it follows from properties 1) and 2) of the nets J that for each $\varepsilon > 0$ there exists a net J such that

$$\begin{aligned}
&\rho\Big(h + \alpha, \min_{y \in X}(b(\,\cdot\,, y) + g(y)\Big) \\
&\qquad \leq \varepsilon + \sup_{x_j \in J} \Big(h_J(x_j) + \alpha_j, \min_{y \in X}(b(x_j, y) + h(y))\Big) \\
&\qquad = \varepsilon + \sup_{x_j \in J} \Big(\min_{y_i \in J} b(x_j, y_i) + h_J(y_i), \min_{y \in X}(b(x, y) + h(y))\Big) \leq 2\varepsilon.
\end{aligned}$$

Now suppose that X is not compact, but $X = \bigcup_{n \in \mathbb{N}} X_n$, where all X_n are compact sets. Let us extend X to $\widetilde{X} = X \cup \{c_\infty\}$, where the neighborhoods of c_∞ are the sets $\widetilde{X} \setminus X_n$ (Alexandroff's compactification). We continue $b(x, y)$ from $X \times X$ to $\widetilde{X} \times \widetilde{X}$ by setting

$$\begin{aligned}
&\tilde{b}(x, c_\infty) = \sup_{X_n} \inf_{y \in X \setminus X_n} b(x, y), \\
&\tilde{b}(c_\infty, y) = \infty \quad \forall\, y \in \widetilde{X}.
\end{aligned}$$

Then $\tilde{b}$ thus constructed satisfies all desired continuity assumptions, and we can refer to the proof for the case of a compact base.

Corollary 2.6 *Let an operator B satisfy conditions* (B) *and* (C)*, so that its eigenvalue α is unique and the eigenvector h is nowhere equal to $\mathbb{0}$. Let $f \leq h + c$ with some constant c. Then*

$$\lim_{m \to \infty} \frac{B^m f(x)}{m} = \alpha. \tag{2.15}$$

Proof. Obviously,

$$\inf_x (f - h) \leq Bf(x) - Bh(x) \leq \sup_x (f - h)$$

for all x. By induction, we obtain

$$\inf_x (f-h) \le B^m f(x) - B^m h(x) \le \sup_x (f-h), \tag{2.16}$$

which implies (2.15) since $B^m(h) = m\alpha + h$.

Another important corollary of Theorem 2.6 is a sufficient condition for the Neumann series to be convergent and finite. This series specifies the Duhamel solution (see §1.2) of the equation $g = Bg \oplus f$ for an unknown function g.

Corollary 2.7 *Under the conditions of Corollary* 2.6*, if* $\alpha > 0$*, then the Neumann series*

$$f \oplus B(f) \oplus B^2(f) \oplus \cdots \tag{2.17}$$

is finite, that is, is equal to the finite sum

$$B \oplus B(f) \oplus \cdots \oplus B^k(f)$$

for some k*.*

Proof. It follows from Eq. (2.16) that

$$B^m f(x) - f(x) \ge \inf_x (f-h) + h(x) - f(x) + m\alpha,$$

whence

$$B^m f(x) \ge f(x) + \inf_x (f-h) + m\alpha - c.$$

Consequently, $B^m f(x) > f(x)$ for all x provided that m is sufficiently large. Hence, the series (2.17) is finite.

The calculations of the iterations B^m is needed in solving optimization problems of the form

$$\sum_{k=0}^{m-1} b(x_k, x_{k+1}) + g(x_m) \to \min, \quad x_0 \text{ is fixed},$$

by the dynamic programming technique.

Namely, the desired minimum is $(B^m g)(x_0)$. Thus, Corollary 2.6 describes the asymptotic behavior of solutions of this problem for large m.

Let us now present a result concerning the uniqueness of the eigenfunctions. This is a generalization of a theorem in [**?**], which pertains to the case of a convex function $b(x,y)$.

Theorem 2.8 *Let* X *be a compact set, and let the integral kernel* $b(x,y)$ *of an operator* B *be a continuous function on* $X \times X$ *such that* $b(x,y)$ *is nowhere equal to* $\mathbb{0} = +\infty$ *and attains its minimum at a unique point* (w,w)*, which* *lies on the diagonal in $X \times X$. Let $b(w,w) = 0$ (this assumption does not result in any loss in generality, since it can always be ensured by a shift by an appropriate constant). Then the eigenvalue of B is equal to $\mathbb{1} = 0$, and the iterations B^n with integral kernels $b^n(x,y)$ converge as $n \to \infty$ to the operator $\overline{B}$ with separated kernel*

$$\overline{b}(x,y) = \varphi(x) + \psi(y),$$

where $\varphi(x) = \lim_{n\to\infty} b^n(x,w)$ is the unique eigenfunction of B and $\psi(x) = \lim_{n\to\infty} b^n(w,x)$ is the unique eigenfunction of the adjoint operator.

Proof. Let $\overline{y}_1, \dots, \overline{y}_{n-1}$ be the points at which the minimum is attained in the expression

$$b^n(x,z) = \min_{y_1,\dots,y_{n-1}} (b(x,y_1) + b(y_1,y_2) + \cdots + b(y_{n-1},z))$$

for the kernel of B^n.

Since $0 \le b^n(x,z) < b(x,w) + b(w,z)$, it follows that $b^n(x,z)$ is uniformly bounded with respect to x, z, and n and moreover, for any $\varepsilon > 0$ and any sufficiently large n, all but finitely many $\overline{y}_j$ lie in the ε-neighborhood U_ε of w. Since $b(x,y)$ is continuous, we see that

$$\forall\, \delta > 0\ \exists\, \varepsilon > 0: \quad b(t,z) < \delta \ \text{ for } \ t, z \in U_\varepsilon.$$

Let $\overline{y}_j \in U_\varepsilon$. Then for $m \ge 1$ we have

$$\begin{aligned} b^{n+m}(x,z) &\le b(x,\overline{y}_1) + \cdots + b(y_{j-1},\overline{y}_j) \\ &\qquad + b(\overline{y}_j,w) + b(w,\overline{y}_{j+1}) + \cdots + b(\overline{y}_{n-1},z) \\ &\le b^n(x,z) + 2\delta. \end{aligned}$$

Consequently, the sequence $b^n(x,z)$ is "almost decreasing," that is,

$$\forall\, \delta > 0 \quad \exists\, N \quad \forall\, n > N: \quad b^n(x,z) \le b^N(x,z) + \delta.$$

In conjunction with boundedness, this property implies that the limit

$$\lim_{n\to\infty} b^n(x,z) = \beta(x,z)$$

exists. Since, obviously,

$$b^{2n}(x,z) = b^n(x,t(n)) + b^n(t(n),z)$$

for some $t(n) \to w$ as $n \to \infty$, we obtain, by passing to the limit,

$$\beta(x,z) = \beta(x,w) + \beta(w,z).$$

Thus, the kernel of the limit operator is separated, which, in particular, implies that the eigenfunction is unique. Let us prove that $\beta(x, w)$ is an eigenfunction of B with eigenvalue $\mathbb{1} = 0$. Indeed,

$$B(\beta(x,w)) = \inf_y \Big(b(x,y) + \lim_{n\to\infty} b^n(y,w) \Big)$$
$$= \lim_{n\to\infty} \inf_y (b(x,y) + b^n(y,w)) = \lim_{n\to\infty} b^{n+1}(x,w) = \beta(x,w).$$

Let us also point out that the uniform continuity of $b(x, y)$ implies the continuity of $\beta(x, z)$ and that the convergence $b^n(x, z) \to \beta(x, z)$ is uniform with respect to (x, z).

Theorem 2.8 can readily be generalized to the case in which the performance function $b(x, y)$ has several points of minimum. It is only essential that these points lie on the diagonal in $X \times X$. In particular, the following result is valid.

Theorem 2.9 *Let X be a compact set, and let the integral kernel $b(x, y)$ of an operator B be a continuous function on $X \times X$ that is nowhere equal to $\mathbb{0} = +\infty$ and that attains its minimum λ at some points (w_j, w_j), $j = 1, \ldots, k$, on the diagonal in $X \times X$. Then the eigenvalue of B is equal to λ, the functions $\varphi_j = \lim_{n\to\infty} b^n(x, w_j)$ (respectively, $\psi_j = \lim_{n\to\infty} b^n(w_j, x)$), $j = 1, \ldots, k$, form a basis of the eigenspace of B (respectively, of the adjoint B'), and the iterations $(B - \lambda)^n$ converge to a finite-dimensional operator with separated kernel*

$$\overline{b}(x,y) = \bigoplus_{j=1}^{n} \varphi_j(x) \odot \psi_j(y) = \min_j (\varphi_j(x) + \psi_j(y)).$$

It is also easy to state a more general result, in which X is locally compact and $b(x, y)$ attains its minimum on the diagonal. In the general case, the connected components of the set of minima of $b(x, y)$ are used instead of the points (w_j, w_j). Possible generalizations to problems with continuous time are given in §3.2.

2.4. Infinite Extremals and Turnpikes in Deterministic Dynamic Optimization Problems

In this section, we first discuss a construction of infinite extremals in deterministic dynamic optimization problems with infinite planning horizon. This construction is based on spectral analysis of idempotent operators and was proposed by S. Yu. Yakovenko [**?**] (see [**?**] for details). Then we discuss turnpike theory and, in particular, present a simple proof of the well-known turnpike theorem for the classical von Neumann–Gale model in mathematical economics.

Numerous attempts has been made in optimal control theory and mathematical economics to define infinite extremals in the formal optimization problem

$$\sum_{k=0}^{\infty} b(x_k, x_{k+1}) \to \min,$$

where $b: X \times X \to A$ is a continuous function, X is a metric compactum, $x_0 = a$ is fixed, and $x_k \in X$, $k = 0, 1, \dots$.

The traditional approach is to define an infinite extremal as a maximal element with respect to some partial order on the set of trajectories, the partial order being produced by comparing the sums corresponding to various initial segments of the trajectories. Consider the following definition.

Definition 2.2 A trajectory $\kappa' = \{x'_k\}_{k=0}^{\infty}$ *overtakes* (respectively, *supertakes*) a trajectory $\kappa = \{x_k\}_{k=0}^{\infty}$ if $x_0 = x'_0$ and

$$\delta(\kappa', \kappa) = \varlimsup_{n\to\infty} \sum_{k=0}^{n-1} (b(x'_k, x'_{k+1}) - b(x_k, x_{k+1})) \le 0$$

(respectively, $\delta(\kappa', \kappa) < 0$). A trajectory is said to be *weakly optimal* if it is not supertaken by any other trajectory. A trajectory is said to be *overtaking* if it overtakes any other trajectory with the same starting point.

Although these notions are frequently used (e.g., see [**?**], where a variety of other possible definitions of the same type are discussed), the set of, say, weakly optimal trajectories is empty in quite a few reasonable optimization problems. However, if such trajectories do exist, they are infinite extremals in the sense of the definition given below.

Let $\operatorname{extr}_n(b, f)$ be the set of solutions (extremals) to the finite-horizon optimization problem

$$\sum_{k=0}^{n-1} b(x_k, x_{k+1}) + f(x_n) \to \min. \tag{2.18}$$

Then it follows from Bellman's optimality principle that

$$x_{k+1} \in \arg\min_{y\in X} (b(x_k, y) + (B^{n-k}) f(y))$$

for each $\{x_k\} \in \operatorname{extr}_n(b, f)$, where B is the Bellman operator with kernel $b(x, y)$, i.e.,

$$(Bf)(x) = \min_y (b(x, y) + f(y))$$

for any continuous real function f.

Definition 2.3 ([?]) Let h be an eigenfunction of the operator B, that is, a solution of the equation $Bh = \lambda \odot h = \lambda + h$. An infinite trajectory $\kappa = \{x_k\}_{k=0}^{\infty}$ is called an *infinite extremal* (or an *h-extremal*) if

$$x_{k+1} \in \arg\min_{y \in X}(b(x_k, y) + h(y))$$

for each $k = 0, 1, \ldots$.

Let $\mathrm{extr}_\infty(B, h)$ denote the set of all (infinite) h-extremals, and let $\lambda = \mathrm{Spec}(B)$. It is easy to see that

$$\mathrm{extr}_\infty(B, \lambda \odot h) = \mathrm{extr}_\infty(B, h).$$

The following result, which shows that the notion introduced is meaningful, is a direct consequence of the definition, Bellman's optimality principle, and the spectral theorem in §2.3.

Theorem 2.10 ([?]) *Let B be a Bellman operator with continuous real kernel, and let $a \in X$ be an arbitrary initial state. Then the following assertions hold.*

(a) *There exists an infinite extremal $\kappa = \{x_k\}_{k=0}^{\infty}$ issuing from a.*

(b) *The relationship between B and the set of its extremals is conjugation-invariant: if $B = C^{-1} \circ B' \circ C$, where an invertible operator C is the composition of a diagonal operator with a "change of variables" $f(x) \mapsto f(\beta^{-1}(x))$ for some homomorphism β (see* §2.2*), then*

$$\kappa \in \mathrm{extr}(B, h) \iff \beta(\kappa) \in \mathrm{extr}_\infty(B', Ch),$$

where

$$\beta(\kappa) = \{\beta(x_k)\}_{k=0}^{\infty}.$$

(c) *If $\kappa \in \mathrm{extr}_\infty(B, h)$, then each segment $\{x_k\}_{k=k'}^{k=k''}$ is a finite extremal of the n-step optimization problem* (2.18) *with fixed initial point and with terminant $f(x_n) = h(x_n)$ $(n = k'' - k')$. In particular, this segment is a solution of the optimization problem*

$$\sum_{k=k'}^{k''-1} b(x_k, x_{k+1}) \to \min$$

with fixed endpoints.

One can introduce a weaker notion of an extremal, which is also invariantly related to B.

Definition 2.4 Let $\lambda = \mathrm{Spec}(B)$. Then $\kappa = \{x_k\}_{k=0}^{\infty}$ is a λ-*trajectory* if

$$\sum_{k=0}^{n-1} b(x_k, x_{k+1}) = n\lambda + O(1) \quad \text{as } n \to \infty.$$

It is easy to see that each infinite extremal is a λ-trajectory. However, unlike in the case of extremals, a trajectory differing from a λ-trajectory by a finite number of states is itself a λ-trajectory. Thus, the notion of λ-trajectories reflects limit properties of infinite extremals. In what follows we assume that $\mathrm{Spec}(B) = 0$. This can always be achieved by adding an appropriate constant.

Generally speaking, the eigenfunction of an operator is not unique, so there exist several various types of infinite extremals issuing from a given point. However, one can always single out an infinite extremal that is (in a sense) the limit as $n \to \infty$ of finite extremals of problem (2.18) with fixed terminant. More precisely, the following theorem is valid.

Theorem 2.11 **([?])** *Let B be a Bellman operator with continuous kernel and with* $\mathrm{Spec}(B) = 0$. *Then there exists a unique "projection" operator Ω in $C(X)$ such that*

(a) Ω *is a linear operator in the semimodule* $C(X, A)$ (*here* $A = \mathbb{R} \cup \{+\infty\}$, $\oplus = \min$, *and* $\odot = +$);

(b) *the relation between Ω and B is conjugation-invariant, that is, if $B = C^{-1} \circ B' \circ C$ for some invertible operator C and Ω' is the projection operator corresponding to B', then $\Omega' = C^{-1} \circ \Omega \circ C$*;

(c) $\Omega f = f \iff f$ *is an eigenfunction of* B;

(d) Ωf *is an eigenfunction of B for any f*;

(e) $\Omega B = \Omega$.

Note that properties (c) and (d) are equivalent to the operator identities $B\Omega = \Omega$ and $\Omega^2 = \Omega$.

Proof. The existence and the properties of Ω readily follow from the explicit formula

$$\Omega f = \lim_{n \to \infty} \bigoplus_{n=N}^{\infty} B^n f = \lim_{n \to \infty} \inf_{n \geq N} B^n f.$$

Since $\mathrm{Spec}(B) = 0$, it follows that all $B^n f$ are bounded and the infimum exists. Let us prove the uniqueness. Suppose that $\widetilde{\Omega}$ satisfies the same conditions. Then

$$\begin{aligned} \Omega f &= \widetilde{\Omega} \Omega f = \widetilde{\Omega} \left(\lim_{N \to \infty} \bigoplus_{n=N}^{\infty} B^n f \right) \\ &= \lim_{N \to \infty} \bigoplus_{n=N}^{\infty} \widetilde{\Omega} B^n f = \lim_{N \to \infty} \widetilde{\Omega} f = \widetilde{\Omega} f. \end{aligned}$$

Let us now discuss the case, most important in mathematical economics and best studied, in which the kernel is convex. More precisely, suppose that $X \subset \mathbb{R}^n$ is a convex compact set and $b \colon X \times X \to \mathbb{R}$ is a continuous strictly convex function whose restriction $b(x, x)$ to the diagonal attains its minimum at some interior point $w \in \operatorname{int} X$. Then there exists a $p \in \mathbb{R}^n$ such that

$$b(x, y) \geq b(w, w) + \langle p, y - x \rangle \tag{2.19}$$

for any $x, y \in X$. Indeed, to find p, it suffices to extend the support plane $\{x = z,\ y = z,\ b = b(w, w)\}$ of the epigraph of the restriction of b to the diagonal to a support hyperplane of the epigraph of $b(x, y)$ through the point $x = y = w$ (the *epigraph* of a function $g(x)$ is the set of pairs (a, x) such that $a \geq g(x)$). It follows that by a conjugation followed by a translation, B can be reduced to an operator B' with strictly convex kernel $b'(x, y)$ such that

$$(b'(x, y) = 0 \iff x = y = w) \quad \text{and} \quad b'(x, y) \geq 0. \tag{2.20}$$

Indeed, $B' = C^{-1} \circ B \circ C - b(w, w)$, where C is the invertible operator with kernel $\langle p, y \rangle + \delta(x - y)$.

Thus, the above-considered case of a convex utility function is a special case of the situation considered in Theorem 2.8. The λ-trajectories for such operators B possess the turnpike property, which generalizes a related result in [**?**] pertaining to a convex translation function.

Theorem 2.12 *Let the assumptions of Theorem* 2.8 *be satisfied. Then*
(a) *If* $\kappa = \{x_k\}_{k=0}^{\infty}$ *is a* λ*-trajectory for* B*, then*
$$\lim x_k = w. \tag{2.21}$$
(b) *Each infinite extremal is a weakly optimal trajectory.*

Proof. (a) Note that

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 : \quad \rho(x, w) \geq \delta \implies \forall\, y \quad b(x, w) > \varepsilon, \tag{2.22}$$

where ρ is the distance function on X. If Eq. (2.21) is violated, then, according to Eq. (2.22), the sum $\sum b(x_k, x_{k+1})$ along κ tends to $+\infty = \mathbb{0}$, which contradicts the fact that $\operatorname{Spec}(B) = 0 = \mathbb{1}$ and Eq. (2.15).

(b) Obviously, a λ-trajectory can only be overtaken by another λ-trajectory. Now assume that some λ-trajectory $\kappa' = \{x_k\}_{k=0}^{\infty}$ supertakes an infinite extremal $\kappa = \{x_k\}_{k=0}^{\infty}$, $x_0' = x_0 = a$. Then, by definition, there exists a sequence N_j such that

$$\sum_{k=0}^{N_j} (b(x_k', x_{k+1}') - b(x_k, x_{k+1})) \leq -\varepsilon < 0. \tag{2.23}$$

But according to Theorem 2.10, $\sum_{k=0}^{N-1} b(x_k, x_{k+1})$ is the minimum in the N-step optimization problem with fixed endpoints $x_0 = a$ and x_N. Since $x_N \to w$, it follows that we can choose a neighborhood U of the point w so that the minima in the N-step problems with fixed endpoints $x_0 = a$ and $x \in U$ are uniformly close to one another for all $x \in U$ and $N \geq N_0$. This contradicts Eq. (2.23), and so the proof is complete.

Needless to say, the point w is a turnpike in problems with fixed (but large) planning horizon. Let us state the related result in a more general form.

Theorem 2.13 *Let X be a locally compact metric space with metric ρ, and let continuous functions*

$$f\colon X \to \mathbb{R} \cup \{+\infty\}, \qquad b\colon X \times X \to \mathbb{R} \cup \{+\infty\}$$

bounded below be given (thus, f and b are continuous bounded A-valued functions, where A is the standard idempotent semiring). Let F and λ be the greatest lower bounds of $f(x)$ and $b(x,y)$, respectively. Suppose that the set

$$W = \{w \in X : b(w,w) = \lambda\}$$

is not empty. Let $\kappa = \{x_k\}_{k=0}^n$ be an optimal trajectory for problem (2.18) with fixed starting point $x_0 = a \in X$, and suppose that there exists a $w \in W$ such that $b(a,w) + f(w) \neq +\infty$. Then for any $\varepsilon > 0$ there exists a positive integer K such that for all positive integers n, however large, the inequality $\rho(x_k, W) < \varepsilon$ is violated at most at K points of the trajectory κ.

Proof. Without loss of generality we can assume that $\lambda = 0$. Let $w \in W$ be an arbitrary point. Then the functional

$$\sum_{k=0}^{n-1} b(x_k, x_{k+1}) + f(x_n) \tag{2.24}$$

to be minimized attains the value $b(a,w) + f(w) \stackrel{\text{def}}{=} C$, independent of n, on the trajectory $\kappa_w = \{x_k\}_{k=0}^n$, where $x_0 = a$ and $x_j = w$, $j = 1, \dots, n$. Consequently, the value of problem (2.18) does not exceed c for all n. Furthermore, it follows from the continuity of $b(x,y)$ that for each $\delta > 0$ there exists an $\varepsilon > 0$ such that $b(x,y) \geq \delta > 0 = b(w,w)$ whenever $\rho(x,W) > \varepsilon$ or $\rho(y,w) > \varepsilon$. It follows that if the inequality $\rho(x_k,w) < \varepsilon$ is violated more than K times on some trajectory, then the value of the functional (2.24) on that trajectory exceeds $K\delta + F$, which is greater than C for $K > (C-F)/\delta$. Consequently, for these K the trajectory cannot be optimal.

For the sake of completeness, let us show how the well-known turnpike theorem for the classical von Neumann–Gale (NG) model can be derived from this theorem. Let us recall the definitions. An NG model is specified by a closed convex cone $\mathbb{Z} \subset \mathbb{R}_+^n \times \mathbb{R}_+^n$ such that $(0,y) \notin \mathbb{Z}$ for any $y \neq 0$ and the projection of $\mathbb{Z}$ on the second factor has a nonempty intersection with the interior of $\mathbb{R}_+^n$. The cone $\mathbb{Z}$ uniquely determines a set-valued mapping $a\colon \mathbb{R}_+^n \to 2^{\mathbb{R}_+^n}$ by the rule

$$y \in a(x) \iff (x,y) \in \mathbb{Z}.$$

In the economical interpretation, the mapping a describes possible transitions from one set of goods to another in one step of the production process under a prescribed technology. A triple (α, y, p), where $\alpha > 0$, $z = (y, \alpha y) \in \mathbb{Z}$, and $p \in \mathbb{R}^n_+ \setminus \{0\}$ is called an *equilibrium state* of the NG model if

$$\alpha(p, x) \geq (p, v) \quad \forall (x, v) \in \mathbb{Z}.$$

If, moreover, $(p, y) > 0$, then the equilibrium is said to be nondegenerate, the coefficient $\alpha > 0$ is referred to as the *growth rate*, and p is known as the vector of equilibrium prices. A *trajectory* in the NG model is a sequence $\{x_k\}_{k=1}^T$, $T \in \mathbb{N}$, such that $(x_k, x_{k+1}) \in \mathbb{Z}$ for all k. For a given utility function $u: \mathbb{R}^n_+ \to \mathbb{R}$, the planning problem on a time horizon $[0, T]$ is to find a trajectory $\{x_k\}_{k=1}^T$ on which the terminal performance functional $u(x_T)$ attains its maximal value. Such a trajectory is said to be optimal.

It will be convenient to use the angular metric $\rho(x, y) = |x/\|x\| - y/\|y\||$ on the set of rays in $\mathbb{R}^n_+$.

A ray $\{\alpha y : \alpha \in \mathbb{R}_+\}$, $y \in \mathbb{R}^n_+$, is called a *strong* (respectively, *weak*) *turnpike* if for each $\varepsilon > 0$ there exists a positive integer $K = K(\varepsilon)$ such that for each optimal trajectory $\{x_k\}_{k=1}^T$, regardless of the planning horizon $T \in \mathbb{N}$ and of the utility function from a given class U, the inequality $\rho(x_k, y) < \varepsilon$ can be violated only for the first K and the last K indices k (respectively, for at most K indices k). The optimization problem for the NG model is known [**?**, **?**] to be reducible to a multistep optimization problem on a compactum. Let us show how the well-known Radner turnpike theorem for the NG model can be derived from Theorem 2.13.

Radner's theorem about weak turnpikes is as follows.

Theorem 2.14 *Suppose that*

1) *An NG model is given, determined by a cone $\mathbb{Z}$ such that*

R1) *there exists an $\alpha > 0$ and a $y \in \mathbb{R}^n_+ \setminus \{0\}$ with $z = (y, \alpha y) \in \mathbb{Z}$;*

R2) *there exists a $p \in \mathbb{R}^n$ (a price vector) such that $\alpha(p, x) > (p, v)$ for any vector $(x, v) \in \mathbb{Z}$ that is not a multiple of $(y, \alpha y)$ (actually, this condition means that the cone $\mathbb{Z}$ is strictly convex in the vicinity of the point $(y, \alpha y)$);*

R3) *for each $x \in \mathbb{R}^n_+$ there exists an $L > 0$ such that $(x, Ly) \in \mathbb{Z}$ (this is a purely technical condition, which can be ensured by an arbitrarily small perturbation of the model and which means that the turnpike proportions can be achieved from an arbitrary initial state).*

2) *A class $U = \{u : \mathbb{R}^n_+ \to \mathbb{R}\}$ of utility functions is given such that each $u \in U$ satisfies the following conditions:*

R4) *$u(x)$ is continuous and nonnegative;*

R5) $u(\lambda x) = \lambda u(x)\ \forall x \in \mathbb{R}^n_+\ \forall \lambda > 0$;

R6) *$u(y) > 0$ (the consistency condition);*

R7) *there exists a $k > 0$ such that $u(y) \leq k(p, y)$.*

Then the ray $\{\alpha y : \alpha > 0\}$ is a weak turnpike.

To derive this theorem from Theorem 2.13 (more precisely, from its analog in which min is replaced by max), note that any optimal trajectory $\{x_k\}_{k=0}^T$

in an NG model satisfies the following *maximal expansion condition* at each step:

$$\max\{\mu : (x_{k-1}, \mu x_k) \in \mathbb{Z}\} = 1, \qquad k = 1, \dots, T.$$

Thus, in seeking optimal trajectories, only trajectories satisfying this condition will be considered feasible.

Let us now consider a multistep optimization problem on the set

$$\Pi = \{x \in \mathbb{R}^n_+ : (p, x) = 1\}$$

equipped with the metric induced by the angular metric on the set of rays. We introduce the transition function

$$b(x, v) = \ln \max\{\lambda > 0 : (x, \lambda v) \in \mathbb{Z}\},$$

where $b = -\infty$ is assumed if the set in the braces is empty. It follows from conditions R1)–R2) that

$$\alpha = b(y, y) = \max\{b(x, v) : x, v \in \Pi\}.$$

To each trajectory $\{x_k\}_{k=0}^T$ of the NG model there corresponds a unique sequence $\{v_k\}_{k=0}^T$ of the points in Π such that v_k and x_k lie on the same ray, $k = 0, \dots, T$. Moreover, by condition R5) we have

$$\ln u(x_T) = \sum_{k=0}^{T-1} b(v_k, v_{k+1}) + \ln u(v_T) \tag{2.25}$$

on the trajectories satisfying the maximal expansion condition, and so the problem of constructing optimal trajectory in the NG model is equivalent to the multistep optimization problem with the performance functional (2.25). Properties R1)–R7) of the model and of the utility function ensure the validity of all assumptions in Theorem 2.13. In particular, the set W is a singleton (its unique element lies on the turnpike ray $\{\alpha y : \alpha > 0\}$).

In the example of the proof of the weak turnpike theorem, we have shown how optimization problems arising in one of the most popular models in mathematical economics can be reduced to a general multistep optimization problem. This section was chiefly devoted to infinite extremals. On the basis of the described reduction, we obtain a natural definition of infinite extremals in the NG model, which coincides with the classical definition based on the Pareto order in $\mathbb{R}^n_+$ [**?**]. In closing, let us point out that various approaches to the construction of infinite extremals and discussion pertaining to specific situations can be found, e.g., in [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**].

In the next section we deal with stochastic multistep optimization models and with the related theory of homogeneous operators.

2.5. Homogeneous Operators in Idempotent Analysis and Turnpike Theorems for Stochastic Games and Controlled Markov Processes

Additive and homogeneous operators are important generalizations of linear operators. This section deals with operators homogeneous in the sense of the semiring $A = \mathbb{R} \cup \{+\infty\}$, i.e., operators B on function spaces such that

$$B(\lambda + h) = \lambda + B(h)$$

for any number λ and any function h. We shall show that the theory of such operators is closely related to game theory and obtain an analog of the eigenvalue theorem for such operators. We apply this analog to construct turnpikes in stochastic games. For simplicity, we only consider the case of a finite state space $X = \{1, \dots, n\}$ in detail.

First, let us show whence homogeneous operators appear. Let us define an antagonistic game on X. Let $p_{ij}(\alpha, \beta)$ denote the probability of transition from state i to state j if the two players choose strategies α and β, respectively (α and β belong to some fixed metric spaces with a metric ρ), and let $b_{ij}(\alpha, \beta)$ denote the income of the first player from this transition. The game is called a *game with value* if

$$\begin{aligned} &\min_\alpha \max_\beta \sum_{j=1}^n p_{ij}(\alpha,\beta)(h^j + b_{ij}(\alpha,\beta)) \\ &\qquad = \max_\beta \min_\beta \sum_{j=1}^n p_{ij}(\alpha,\beta)(h^j + b_{ij}(\alpha,\beta)) \end{aligned} \tag{2.26}$$

for all $y \in \mathbb{R}^n$. In that case, the operator $B\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $B_i(y)$ is equal to (2.26) is called the *Bellman operator* of the game. By the dynamic programming method [?], we can show that the value of the k-step game defined by the initial position i and the terminal income $h \in \mathbb{R}^n$ of the first player exists and is equal to $B_i^k(h)$.

It is clear that the operator B has the following two properties:

$$B(ae + h) = ae + B(h) \quad \forall a \in \mathbb{R},\ h \in \mathbb{R}^n,\ e = (1, \dots, 1) \in \mathbb{R}^n, \tag{2.27}$$

$$\|B(h) - B(g)\| \le \|h - g\| \quad \forall h, g \in \mathbb{R}^n, \tag{2.28}$$

where $\|h\| = \max |h^i|$.

Interestingly, these two properties are characteristic of the game Bellman operator [?]:

Theorem 2.15 *For each map $B\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying* (2.27) *and* (2.28)*, there exist functions $p_{ij}(\alpha, \beta)$ and $b_{ij}(\alpha, \beta)$* (*where α and β belong to some metric spaces*) *such that*

$$p_{ij}(\alpha,\beta) \ge 0, \qquad \sum_{j=1}^n p_{ij}(\alpha,\beta) = 1,$$

(2.26) *holds, and the value of* (2.26) *is equal to* $B_i(h)$.

Proof. It follows from (2.28) that B_i almost everywhere has partial derivatives such that $\sum_{j=1}^n |\partial B_i/\partial h^j| \leq 1$. Then property (2.27) implies that $\sum_{j=1}^n (\partial B_i/\partial h^j) = 1$. Hence, the gradients of B_i belong to the unit simplex

$$\Pi = \left\{ h = (h^1, \dots, h^n) \in \mathbb{R}^n : h^i \geq 0, \sum_{j=1}^n h^j = 1 \right\}.$$

Now let us represent the function B_i in the standard game form [**?**]

$$\begin{aligned} B_i(h) &= \min_{\alpha \in \mathbb{R}^n} \max_{\beta \in \mathbb{R}^n} (B_i(\alpha) + (F_i(\alpha, \beta), h - \alpha)) \\ &= \max_{\beta \in \mathbb{R}^n} \min_{\alpha \in \mathbb{R}^n} (B_i(\alpha) + (F_i(\alpha, \beta), h - \alpha)), \end{aligned}$$

where

$$F_i(\alpha, \beta) = \int_0^1 \operatorname{grad} B_i(\alpha + t(\beta - \alpha))\, dt.$$

By virtue of the cited properties of the gradient of B, the vector $F_i(\alpha, \beta)$ belongs to Π.

Now let us point out that the most important examples of games with value represent games with finitely many (pure) strategies α, β, when the value is attained at so-called mixed strategies. The sets of mixed strategies in that case coincide with some unit simplices in $\mathbb{R}^n$. It is clear that the set of the corresponding Bellman operators is a dense subset in the set of all operators with properties (2.27) and (2.28).

Now we define the quotient space Φ of the space $\mathbb{R}^n$ by the one-dimensional subspace generated by the vector $e = (1, \dots, 1)$. Let $\Pi \colon \mathbb{R}^n \mapsto \Phi$ be the natural projection. The quotient norm on Φ is obviously defined by the formula

$$\|\Pi(h)\| = \inf_{a \in \mathbb{R}} \|h + ae\| = \frac{1}{2} \Big(\max_j h^j - \min_j h^j \Big).$$

It is clear that Π has a unique isometric (but not linear) section $S \colon \Phi \mapsto \mathbb{R}^n$. The image $S(\Phi)$ consists of all $h \in \mathbb{R}^n$ such that $\max_j h^j = -\min_j h^j$. By virtue of properties (2.27) and (2.28) of B, the continuous quotient map $\widetilde{B} \colon \Phi \mapsto \Phi$ is well defined.

To state the main result of this section, we need some additional properties of the transition probabilities:

$$\exists \delta > 0 : \forall i, j, \alpha \quad \exists \beta : p_{ij}(\alpha, \beta) \geq \delta, \tag{2.29}$$

$$\exists \delta > 0 : \forall i, j \quad \exists m : \forall \alpha, \beta : p_{im}(\alpha, \beta) > \delta,\ p_{jm}(\alpha, \beta) > \delta. \tag{2.30}$$

Let all $|b_{ij}(\alpha, \beta)|$ be bounded by some constant C.

Lemma 2.1 A) *If* (2.29) *holds and* $\delta < 1/n$, *then* $\widetilde{B}$ *maps each ball of radius* $R \geq C\delta^{-1}$ *centered at the origin into itself.*

B) *If* (2.30) *holds, then*

$$\|\widetilde{B}(H) - \widetilde{B}(G)\| \leq (1-\delta)\|H - G\|, \qquad \forall H, G \in \Phi.$$

Proof. We shall prove only A); the proof of B) is similar. It follows from the definition that

$$B_i(h) - B_m(h) \leq \sum_{j=1}^{n} p_{ij}(\alpha_1, \beta_1)(b_{ij}(\alpha_1, \beta_1) + h^j) - \sum_{j=1}^{n} p_{mj}(\alpha_0, \beta)(b_{ij}(\alpha_0, \beta) + h^j),$$

where α_1, β_1, and α_0 depend on i, whereas h and β are arbitrary. Hence,

$$B_i(h) - B_m(h) \leq 2C + \|h\| \sum_{j=1}^{n} |p_{ij}(\alpha_1, \beta_1) - p_{mj}(\alpha_0, \beta)|.$$

Let us choose j_0 so that $p_{ij_0}(\alpha_1, \beta_1) > \delta$. Using condition A), we can take β so that $p_{ij}(\alpha_0, \beta) > \delta$. Then

$$\sum_{j=1}^{n} |p_{ij}(\alpha_1, \beta_1) - p_{mj}(\alpha_0, \beta)| \leq |p_{ij_0} - p_{mj_0}| + (1 - p_{ij_0}) + (1 - p_{mj_0}) \leq 2(1-\delta).$$

Consequently,

$$B_i(h) - B_m(h) \leq 2C + 2(1-\delta).$$

Using the definition of the norm in Φ, we have

$$\|\Pi(B(h))\| \leq C + \|h\|(1-\delta).$$

Thus, for $h = S(H)$ we obtain

$$\|\widetilde{B}(H)\| \leq C + \|H\|(1-\delta).$$

It follows that the map $\widetilde{B}$ takes the ball of radius R into itself provided that $C + R(1-\delta) \leq R$, i.e., $R \geq C\delta^{-1}$.

Theorem 2.16 A) *If* (2.29) *holds, then there exists a unique* $\lambda \in \mathbb{R}$ *and a vector* $h \in \mathbb{R}^n$ *such that*

$$B(h) = \lambda + h \tag{2.31}$$

and for all $g \in \mathbb{R}^n$ *we have*

$$\|B^m g - m\lambda\| \le \|h\| + \|h - g\|, \tag{2.32}$$

$$\lim_{m\to\infty} \frac{B^m g}{m} = \lambda. \tag{2.33}$$

B) *If* (2.30) *holds, then* h *is unique* (*up to equivalence*), *and*

$$\lim_{m\to\infty} S \circ \Pi(B^m(g)) = S \circ \Pi(h) \quad \forall g \in \mathbb{R}^n. \tag{2.34}$$

Proof. This follows readily from the lemma and from the fixed point theorems.

As a consequence, we find that the equilibria $\widetilde{\alpha}_i$ and $\widetilde{\beta}_i$ in (2.26), where h is a solution of (2.31), define stationary strategies in the infinite-time game. Theorem 2.16 also implies turnpike theorems for the game in question.

Theorem 2.17 *Let* (2.30) *hold. Then for all* $\varepsilon > 0$ *and* $\Omega > 0$ *there exists an* $M \in \mathbb{N}$ *such that if* $\{\alpha(i,t)\}$ *and* $\{\beta(i,t)\}$ *are equilibrium strategies in the* T*-step game,* $T > M$*, with terminal income of the first player defined by a vector* g *with* $\|\Pi(g)\| \le \Omega$*, then*

$$\rho(\alpha(t,i), \widetilde{\alpha}_i) < \varepsilon, \quad \rho(\beta(t,i), \widetilde{\beta}_i) < \varepsilon$$

for all $t < T - M$*.*

Proof. This readily follows from (2.34).

Let $\widetilde{\alpha}_i$ and $\widetilde{\beta}_1$ be defined uniquely. Let $Q^* = (q_1^*, \dots, q_n^*)$ denote the stationary distribution for the stationary Markov chain defined on the state space X by these strategies. Just as for the case of the Markov decision process [**?**], we obtain the following turnpike theorem on the state space.

Theorem 2.18 *For all* $\alpha > 0$ *and* $\Omega > 0$ *there exists an* $M \in \mathbb{N}$ *such that for each* T*-step game,* $T > 2M$*, with terminal income* $g \in \mathbb{R}^n$*,* $\|\Pi(g)\| < \Omega$*, of the first player we have*

$$\|Q(t) - Q^*\| < \varepsilon,$$

where $Q(t) = (q_1(t), \dots, q_n(t))$ *and* $q_i(t)$ *is the probability that the process is in a state* $i \in X$ *at time* t *if the game is carried out with the equilibrium strategies.*

In other words, q_j^* is the mean amount of time that each sufficiently long game with equilibrium strategies spends in position j.

Proof. It follows from Theorem 2.17 that for each $\varepsilon_1 > 0$ there exists an $M_1 \in \mathbb{N}$ such that for any t-step equilibrium game, $t > M_1$, with the first player's terminal income g, $\|\Pi(g)\| \le \Omega$, the transition probabilities at the first $t - M_1$ steps are ε_1-close to the transition probabilities $p_{ij}(\tilde{\alpha}_i, \tilde{\beta}_j)$. Consequently,

$$Q(t) = Q^0(P + \delta_1) \cdots (P + \delta_t) = Q^0(P^t + \Delta(t)),$$

where the matrices δ_k (and, hence, $\Delta(t)$) are ε_1-close to zero. By a theorem on the convergence of probability distributions in homogeneous Markov chains to a stationary distribution, we have

$$\|Q^0 P^t - Q^*\| \le (1-\delta)^{t-1}.$$

Thus, we can successively choose M_2 and ε_1 so that

$$\|Q(M_2) - Q^*\| < \varepsilon \quad \text{for all } Q^0.$$

Then

$$\|Q(t) - Q^*\| < \varepsilon \quad \text{for all } t \in [M_2, T - M_1].$$

There is a natural generalization of conditions (2.29) and (2.30) under which the cited results can still be proved. Namely, we require these conditions to be valid for some iteration of the operator B. This is the case of cyclic games [**?**]. Some generalizations to n-person games were obtained in [**?**].

If, in addition to the assumptions of Theorem 2.15, we require each coordinate of the operator B to be convex, then B is the Bellman operator of some controlled Markov process. Thus, Theorems 2.16 and 2.17, in particular, contain the turnpike theorems for Markov processes [**?**, **?**]. An analog of Theorem 2.16 for connected Markov processes (cyclic one-person games) was originally proved in [**?**], and the turnpike theorem (Theorem 2.17) for Markov processes with discounting was obtained in [**?**].

We now give a natural generalization of the obtained results to the case of continuous time. For simplicity, we consider only Markov processes (one-person games). We first carry out the argument for a finite state space and then proceed to an arbitrary measurable space.

Let $X = \{1, \dots, n\}$ be a finite state space. We define a controlled Markov process with continuous time by specifying a continuous mapping $\Lambda\colon u \mapsto \Lambda(u) = \{\Lambda_{ij}(u)\}$ of a compact set of controls into the set of differential-stochastic matrices (that is, matrices with zero sum of entries in each row, nonnegative off-diagonal entries, and nonpositive diagonal entries). A *strategy* is a collection of continuous mappings $f_i(t)\colon \mathbb{R}_+ \to U$, $i \in X$. Once a strategy $\{f_i(t)\} = F$ is chosen, the probability distribution $Q(t)$ of the process in question at time t is determined as the solution of the nonautonomous system of ordinary differential equations (with Carathéodory-type discontinuous right-hand side)

$$\dot{Q}(t) = \Lambda_F(t) Q(t), \qquad Q(0) = Q^0,$$

with matrix $\Lambda_F(t) = \{\lambda_{ij}(f_j(t))\}$ and with some initial distribution Q^0. Thus, the choice of a strategy specifies a nonhomogeneous Markov process on X.

To define a controlled process with income, let us specify continuous functions $c_i(u)$ and $b_{ij}(u)$, $i \ne j$, on U, with the intended meaning of the income per unit time when staying in a state i and the income gained by the transition from a state i to a state j, respectively, under a control $u \in U$. Let $g_i(t)$ be the maximum income (more precisely, the mathematical expectation of maximum income) available in time t if the process starts from a state i and the terminal income at time t is given by a prescribed vector $g(0)$. The optimality principle implies the following Bellman differential equation for $g(t)$:

$$\dot{g}_i(t) = \max_u \left\{ c_i(u) + \sum_{j \neq i} \lambda_{ij}(u)(b_{ij}(u) + g_j(t) - g_i(t)) \right\}. \tag{2.35}$$

If all $\lambda_i = -\lambda_{ii}$ are strictly positive, then, by introducing the functions

$$b_{ii}(u) = -\frac{c_i(u)}{\lambda_i(u)} = -c_i(u) \left(\sum_{j \neq i} \lambda_{ij}(u) \right)^{-1},$$

we can rewrite Eq. (2.35) in the form

$$\dot{g}_i(t) = \max_u \sum_{j=1}^{n} \lambda_{ij}(u)(b_{ij}(u) + g_j(t)). \tag{2.36}$$

The resolving operator B^t of the Cauchy problem for Eq. (2.35) is a continuous-time analog of the Bellman operator B for discrete-time processes. It turns out that the properties of these operators are quite similar. First, it is obvious that

$$B^t(\lambda + g) = \lambda + B^t g \quad \text{for all } \lambda \in \mathbb{R} \text{ and } g \in \mathbb{R}^n,$$

since for each solution $g(t)$ of Eq. (2.35) the function $\lambda + g(t)$ is also a solution for any λ. Thus, on analogy with the discrete case, we can define the quotient operator $\widetilde{B}^t \colon \Phi \to \Phi$ on the quotient space of $\mathbb{R}^n$ by the subspace of constants. The following assertion is valid.

Lemma 2.2 *Suppose that the functions $\lambda_i(u) = -\lambda_{ii}(u)$ are bounded away from zero and that there exists a j_0 such that $\lambda_{ij_0}(u) \geq \delta$ for all $i \neq j_0$ and $u \in U$ and for some fixed $\delta > 0$. Then the $\widetilde{B}^t$ are contraction operators; more precisely,*

$$\|\widetilde{B}^t G - \widetilde{B}^t H\| \leq e^{-t\delta} \|G - H\| \tag{2.37}$$

for any $G, H \in \Phi$.

Proof. By using the Euler approximations, we obtain

$$B^t = \lim_{n \to \infty} (B_{t/n})^n, \tag{2.38}$$

where the linear operator B_τ is given by the formula

$$(B_\tau g)_i = g_i + \max_u \tau \sum_{j=1}^n \lambda_{ij}(u)(b_{ij}(u) + g_j)$$
$$= \max_u \left\{ (1-\tau\lambda_i)\left(\frac{c_i(u)}{1-\tau\lambda_i(u)} + g_i \right) + \sum_{j\neq i} \tau\lambda_{ij}(b_{ij}(u)+g_j) \right\}.$$

For small τ (such that $1-\tau\lambda_i(u) \geq \delta\tau$ for all u), the quotient operator $\widetilde{B}_\tau$ is a contraction operator by Lemma 2.1,

$$\|\widetilde{B}_\tau G - \widetilde{B}_\tau H\| \leq (1-\delta\tau)\|G-H\|, \tag{2.39}$$

whence Eq. (2.37) follows in view of (2.38).

Just as in the case of discrete time, Lemma 2.2 implies the following theorem.

Theorem 2.19 *Under the assumptions of Lemma* 2.2, *there exists a unique* $\lambda \in \mathbb{R}$ *and a unique* (*modulo constants*) *vector* $h \in \mathbb{R}^n$ *such that*

$$B^t h = t\lambda + h$$

for all $t \geq 0$. *Moreover, the following limit relations are valid for any* $g \in \mathbb{R}^n$:

$$\lim_{t\to\infty} \frac{1}{t} B^t g = \lambda, \qquad \lim_{t\to\infty} S \circ \Pi(B^t g) = S \circ \Pi(h). \tag{2.40}$$

Once these relations are proved, the turnpike theorems 2.17 and 2.18 automatically extend to the case in question, and we do not discuss the subject any more.

Let us now show how to extend these results to general controlled jump processes. First, let us recall some definitions (e.g., see [**?**]).

Let (X,σ) be a measurable space with σ-algebra σ of subsets. A function $P: X \times \sigma \to \mathbb{R}_+$ is called a *stochastic kernel* if $P(\,\cdot\,,\Omega)$ is measurable on X for each $\Omega \in \sigma$ and if $P(x,\,\cdot\,)$ is a probability measure on X for each $x \in X$. A family

$$P_{st}(x,\Omega) = P(s,x,t,\Omega), \qquad s \leq t,\ s,t \in \mathbb{R}_+$$

of stochastic kernels is called a *Markov family* if it satisfies the Kolmogorov–Chapman equation

$$P(s,x,t,\Omega) = \int P(\tau,y,t,\Omega)P(s,x,\tau,dy)$$

for any $\tau \in [s,t]$ and the boundary conditions

$$P_{tt}(x,\Omega) = \chi(x,\Omega) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases}$$

for each $t \in \mathbb{R}_+$. Any Markov family defines a wide-sense Markov process, that is, a family $\{\xi(t)\}$ of random variables with values in X such that $P(s,x,t,\Omega)$ is the probability of the event $\xi(t) \in \Omega$ under the condition $\xi(s) = x$.

The notion of a jump process is a model of the following type of system behavior. The system spends random positive time in some state, then instantaneously (by jump) passes into another state (also random), spends random time in that state, etc. The Markov process determined by a Markov family $P_{st}(x,\Omega)$ of stochastic kernels on (X,σ) is called a regular jump process if the σ-algebra σ on X contains the singletons, if for any $s \in \mathbb{R}_+$, $x \in X$, and $\Omega \in \sigma$ the limit

$$\overline{\mu}_s(x,\Omega) = \lim_{t \to s+} \frac{P(s,x,t,\Omega) - \chi(x,\Omega)}{t-s}$$

exists and is finite, and if, moreover, the convergence is uniform with respect to x, Ω, and $s \in [0,t]$ for any t and the limit $\overline{\mu}$ is continuous in $s \in [0,t]$ uniformly with respect to x and Ω. In other words, the family $P(s,x,t,\Omega)$ is continuously right differentiable with respect to t at $t = s$.

Let us introduce the functions

$$\mu_s(x,\Omega) = \overline{\mu}_s(x,\Omega \setminus \{x\}), \qquad \mu_s(x) = \mu_s(x,X).$$

It readily follows from the definition of μ and $\overline{\mu}$ that

(C1) $\overline{\mu}_s(x,\Omega)$ is a finite charge and $\mu_s(x,\Omega)$ is a measure (i.e., a positive charge) on (X,σ) for any s and x, and moreover, $\overline{\mu}_s(x,\Omega) \le K(t)$ for all $s \le t$, x, and Ω and for some function $K(t)$;

(C2) $\overline{\mu}_s(x,X) = 0$ for all s and x; $\overline{\mu}_s(x,\Omega) \ge 0$ if $x \notin \Omega$, and, moreover,

$$\overline{\mu}_s(x,\{x\}) = -\overline{\mu}_s(x,X \setminus \{x\}) \le 0;$$

(C3) $\overline{\mu}_s(x,\Omega) = -\mu_s(x)\chi(x,\Omega) + \mu_s(x,\Omega)$.

Kolmogorov's theorem says that

1. For a regular jump process, the function $P(s,x,t,\Omega)$ is differentiable with respect to t for $t > s$, satisfies the *backward Kolmogorov equation*

$$\frac{\partial P(s,x,t,\Omega)}{\partial t} = -\int_\Omega \mu_t(y)P(s,x,t,dy) + \int_X \mu_t(y,\Omega)P(s,x,t,dy), \quad (2.41)$$

and is right continuous at $t = s$:

$$\chi(x,\Omega) = P_{ss}(x,\Omega) = \lim_{t \to s+} P(s,x,t,\Omega).$$

2. For a regular jump process, the function $P(s,x,t,\Omega)$ is differentiable with respect to s for $s < t$, satisfies the *forward Kolmogorov equation*

$$\frac{\partial P}{\partial s}(s,x,t,\Omega) = \mu_s(x)P(s,x,t,\Omega) - \int_X P(s,x,t,\Omega)\mu_t(x,dy),$$

and is left continuous at $s = t$:

$$\chi(x, \Omega) = \lim_{s \to t-} P(s, x, t, \Omega).$$

3. Conversely, if some functions μ and $\overline{\mu}$ satisfying conditions (C1)–(C3) are given, then the backward Kolmogorov equation is uniquely solvable and the solution $P(s, x, t, \Omega)$ specifies a regular Markov jump process.

Furthermore, a Markov process is said to be *homogeneous* if the stochastic kernel $P(s, x, t, \Omega)$ depends only on the difference $t - s$. For a jump process, this implies that the charge $\overline{\mu}_t(x, \Omega)$ is independent of t, $\overline{\mu}_t(x, \Omega) = \overline{\mu}(x, \Omega)$.

We define a controlled process by specifying a compact control set U, functions $c(u, x)$ and $b(u, x, y)$ (the income per unit time and the income gained by the transition from x to $y \neq x$ under a control $u \in U$), and bounded functions $\overline{\mu}(u, x, \Omega)$ and $\mu(u, x) = \overline{\mu}(u, x, \Omega \setminus \{x\})$ that satisfy (C1)–(C3) for each $u \in U$ and hence determine a homogeneous jump process by Kolmogorov's theorem.

Let $g(t, x)$ be the maximum income (more precisely, mathematical expectation of income) available in time t if the process begins in a state x and the terminal income is given by a prescribed function $g_0(x)$.

Let $L_\infty = L_\infty(X)$ denote the Banach space of bounded measurable real functions on (X, σ) with the norm $\|g\| = \sup_x |g(x)|$.

Suppose that $g(x, t)$, as well as the derivative $\dot{g}(t, x) = \frac{\partial g}{\partial t}(t, x)$, lies in L_∞. Then

$$g(t, x) = \sup_u \left\{ \tau c(u, x) + g(t - \tau, x)(1 - \tau \mu(u, x)) + \int (b(u, x, y) + g(t - \tau, y)) \tau \mu(u, x, dy) \right\}$$

modulo higher-order infinitesimals.

We substitute

$$g(t - \tau, x) = g(t, x) - \tau \dot{g}(t, x) + o(\tau)$$

into the last equation, cancel out the factors $g(t, x)$ and τ, and let $\tau \to 0$, thus obtaining

$$\dot{g}(t, x) = \sup_u \left\{ c(u, x) + \int b(u, x, y) \mu(u, x, dy) + \int (g(t, y) - g(t, x)) \mu(u, x, dy) \right\}. \tag{2.42}$$

We have given the standard heuristic derivation of the Bellman equation (2.42). To justify this equation rigorously, one should use the existence and uniqueness theorem given in the following. It is obvious that Eq. (2.42) is an analog of Eq. (2.35) for an arbitrary infinite space X.

Equation (2.42) can be rewritten in the shorter form $\dot g = A(g)$, where $A\colon L_\infty \to L_\infty$ is the nonlinear mapping given by the formula

$$(Ah)(x) = \sup_u \left\{ c(u,x) + \int (b(u,x,y) + h(t,y) - h(t,x))\mu(u,x,dy) \right\}.$$

It is clear that A is Lipschitz continuous,

$$\|A(h) - A(g)\| \le L\|h-g\| \tag{2.43}$$

for some constant L, and that

$$A(\lambda + h) = A(h) \tag{2.44}$$

for all $\lambda \in \mathbb{R}$ and $h \in L_\infty$. For example, inequality (2.43) follows from the estimates

$$(A(h) - A(g))(x) \le \sup_u \int (|h(t,y) - g(t,y)| + |h(t,x) - g(t,x)|)\mu(u,x,dy)$$

and from the boundedness of $\mu(u,x,dy)$.

It follows from Eq. (2.43) that the Bellman equation $\dot g = Ag$ has a unique solution in L_∞ on any time interval and for any initial function $g_0 \in L_\infty$; this solution is the limit of Euler's approximations,

$$B^t = \lim_{n\to\infty} (B_\tau)^n, \qquad \tau = \frac{t}{n}, \tag{2.45}$$

where B^t is the resolving operator of the Cauchy problem for the equation $\dot g = Ag$ and $B_\tau\colon L_\infty \to L_\infty$ is the finite-difference approximation determined by the formula $B_\tau g = g + \tau A(g)$. The proof is word for word the same as that of the corresponding theorem for ordinary differential equations on a line.

It follows from (2.44) that the mappings B^t and B_τ are homogeneous, that is,

$$B^t(\lambda + g) = \lambda + B^t g, \quad B_\tau(\lambda + g) = \lambda + B_\tau g \qquad \forall\, \lambda \in \mathbb{R},\ g \in L_\infty.$$

We proceed just as in the case of finite X and define the quotient space $L\Phi_\infty$ of L_∞ by the subspace of constant functions. Then the operators B^t and B_τ factor through the natural projection $L_\infty \to L\Phi_\infty$, and thus the continuous quotient mappings $\widetilde{B}^t$, $\widetilde{B}_\tau\colon L\Phi_\infty \to L\Phi_\infty$ are well-defined.

To each sufficiently small τ there corresponds a family $\mu^\tau(u,x,\Omega)$ of probability measures on X. It is given by the formula

$$\mu^\tau(u,x,\Omega) = \tau\mu(u,x,\Omega) + (1 - \tau\mu(u,x))\chi(x,\Omega).$$

To each pair of controls (u_1,u_2) and each pair of points $x_1, x_2 \in X$ we assign a charge (signed measure) $\nu^\tau_{1,2} = \nu^\tau(u_1,x_1,u_2,x_2)$ by the formula

$$\nu^\tau_{1,2}(\Omega) = \mu^\tau(u_1,x_1,\Omega) - \mu^\tau(u_2,x_2,\Omega).$$

The total variation of this charge satisfies $\operatorname{Var}|\nu_{1,2}^\tau| \le 2$, since μ^τ are probability measures, and $\nu_{1,2}^\tau = 0$ at $\tau = 0$.

As is the case with discrete-time systems, for the existence of turnpike control regimes in this controlled jump process it is necessary that some connectedness conditions for the state space be satisfied. The following analog of Eq. (2.30) is a quite general condition of this sort to be imposed on the transition probabilities.

(P) For any $u_1, u_2 \in U$ and $(x_1, x_2) \in X$, the total variation of the charge $\nu_{1,2}^\tau$ does not exceed $2(1 - \delta\tau)$ for some fixed $\delta > 0$, or, equivalently,

$$\mu^\tau(u_1, x_1, \Omega^-) + \mu^\tau(u_2, x_2, \Omega^+) \ge \delta,$$

where $\Omega^\pm$ are the positive and the negative sets in the Hahn decomposition of $\nu_{1,2}^\tau$.

Remark 2.5 Various conditions sufficient for (P) to be satisfied can be written out in terms of the original measure μ. These conditions are especially descriptive if X is a compact set and $\mu(u, x, \Omega)$ are continuous functions.

If (P) is satisfied, then $\widetilde{B}_\tau$ and $\widetilde{B}^t$ are contraction operators and satisfy the estimates (2.37) and (2.39) for $G, H \in L\Phi_\infty$. The estimate (2.39) follows from the inequality

$$\begin{aligned}
&(B_\tau g - B_\tau h)(x_1) - (B_\tau g - B_\tau h)(x_2) \\
&\qquad \le \|g - h\| \sup_{u_1, u_2} \int |\mu^\tau(u_1, x_1, dy) - \mu^\tau(u_2, x_2, dy)| \\
&\qquad \le \|g - h\|(1 - \delta\tau),
\end{aligned}$$

and the estimate (2.37) follows from Eqs. (2.39) and (2.45).

As is the case with finite X, it follows that there exists a unique $\lambda \in \mathbb{R}$ and a unique (modulo an additive constant) $h \in L_\infty$ such that

$$B^t h = \lambda t + h, \tag{2.46}$$

and the limit equations (2.40) are satisfied, where S and Π are defined by complete analogy with the case of finite X.

If we now assume that the functions $b(u, x, y)$, $c(u, x)$, and $\mu(u, x, dy)$ are continuous in $u \in U$, then sup can be replaced by max in Eq. (2.42) and other formulas. In particular, there is a set of optimal stationary strategies associated with the solution of Eq. (2.35). Theorem 2.17 about turnpikes on the set of strategies can now be transferred directly to the situation in question; the only difference is that optimal strategies (which need not exist for nonstationary processes with continuous time) should be replaced by ε-optimal strategies (i.e., strategies in which the mathematical expectation of income differs from the optimal value at most by ε).