Rainbow options via the interval model of stock prices: risk-neutral selections, explicit formulas, algorithms. [1]

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Highlights

Game theoretic approach to option pricing (interval model): Different kind of generalizations of classical BS and CRR formulae with more rough assumptions on the underlying assets evolution.

A mysterious class of discrete-time dynamic games where the calculation of the competitive Belmann operator of a complicated form can be done explicitly on the final payoffs given by sub-modular functions (which includes the multiple strike rainbow options).

A natural unique selection among multiple risk neutral measures arising in incomplete markets for options specified by sub-modular functions.

Explicit formulae and new numeric schemes are developed.

Interval model for a market

Market with several securities in discrete time $k = 1, 2, \ldots$:
The risk-free bonds (bank account), priced B_k,
and J common stocks, $J = 1, 2\ldots$, priced S_k^i, $i \in \{1, 2, \ldots, J\}$.
$B_{k+1} = \rho B_k$, $\rho \geq 1$ is a constant interest rate,
$S_{k+1}^i = \xi_{k+1}^i S_k^i$, where $\xi_k^i, i \in \{1, 2, \ldots, J\}$, are unknown sequences taking values in some fixed intervals
$M_i = [d_i, u_i] \subset \mathbf{R}$ (*interval model*).
This model generalizes the colored version of the classical CRR model in a natural way.
In the latter a sequence ξ_k^i is confined to take values only among two boundary points d_i, u_i, and it is supposed to be random with some given distribution.

Rainbow (or colored) European Call Options

A premium function f of J variables specifies the type of an option.

Standard examples ($S^1, S^2, ..., S^J$ represent the expiration values of the underlying assets, and $K, K_1, ..., K_J$ represent the strike prices):

Option delivering the best of J risky assets and cash

$$f(S^1, S^2, ..., S^J) = \max(S^1, S^2, ..., S^J, K), \tag{1}$$

Calls on the maximum of J risky assets

$$f(S^1, S^2, ..., S^J) = \max(0, \max(S^1, S^2, ..., S^J) - K), \tag{2}$$

Multiple-strike options

$$f(S^1, S^2, ..., S^J) = \max(0, S^1 - K_1, S^2 - K_2, ..., S^J - K_J), \tag{3}$$

Portfolio options

$$f(S^1, S^2, ..., S^J) = \max(0, n_1 S^1 + n_2 S^2 + ... + n_J S^J - K), \tag{4}$$

Spread options: $f(S^1, S^2) = \max(0, (S^2 - S^1) - K)$.

Investor's (buyer of an option) control: one step

X_k the capital of the investor at the time $k = 1, 2, \ldots$. At each time $k-1$ the investor determines his portfolio by choosing the numbers γ_k^i of common stocks of each kind to be held so that the structure of the capital is represented by the formula

$$X_{k-1} = \sum_{i=1}^{J} \gamma_k^i S_{k-1}^i + (X_{k-1} - \sum_{i=1}^{J} \gamma_k^i S_{k-1}^i),$$

where the expression in bracket corresponds to the part of his capital laid on the bank account. The control parameters γ_k^i can take all real values, i.e. short selling and borrowing are allowed. The value ξ_k becomes known in the moment k and thus the capital at the moment k becomes

$$X_k = \sum_{i=1}^{J} \gamma_k^i \xi_k^i S_{k-1}^i + \rho(X_{k-1} - \sum_{i=1}^{J} \gamma_k^i S_{k-1}^i).$$

Investor's control: n step game

If n is the *maturity date*, this procedures repeats n times starting from some initial capital $X = X_0$ (selling price of an option) and at the end the investor is obliged to pay the premium f to the buyer.
Thus the (final) income of the investor equals

$$G(X_n, S_n^1, S_n^2, ..., S_n^J) = X_n - f(S_n^1, S_n^2, ..., S_n^J). \qquad (5)$$

The evolution of the capital can thus be described by the dynamic n-step game of the investor (strategies are sequences of real vectors $(\gamma_1, ..., \gamma_n)$ (with $\gamma_j = (\gamma_j^1, ..., \gamma_j^J)$)) with the Nature (characterized by unknown parameters ξ_k^i).
A position of the game at any time k is characterized by $J+1$ non-negative numbers $X_k, S_k^1, ..., S_k^J$ with the final income specified by the function

$$G(X, S^1, ..., S^J) = X - f(S^1, ..., S^J) \qquad (6)$$

Robust control (guaranteed payoffs, worst case scenario)

Minmax payoff (guaranteed income) with the final income G in a one step game with the initial conditions $X, S^1, ..., S^J$ is given by the *Bellman operator*

$$\mathbf{B} G(X, S^1, ..., S^J)$$

$$= \max_{\gamma} \min_{\xi} G(\rho X + \sum_{i=1}^{J} \gamma^i \xi^i S^i - \rho \sum_{i=1}^{J} \gamma^i S^i, \xi^1 S^1, ..., \xi^J S^J),$$

and the guaranteed income in the n step game with the initial conditions $X_0, S_0^1, ..., S_0^J$ is

$$\mathbf{B}^n G(X_0, S_0^1, ..., S_0^J).$$

Reduced Bellman operator

In our model G is given by (6).
As the class of function G of the form

$$\rho^k X - g(S^1, ..., S^J)$$

is clearly invariant under the action of **B**, it follows that in our model the guaranteed income in the n step game equals

$$\rho^n X_0 - (\mathcal{B}^n f)(S_0^1, ..., S_0^J), \tag{7}$$

where the *reduced Bellman operator* is defined as:

$$(\mathcal{B}f)(z^1, ..., z^J) = \min_{\gamma} \max_{\xi} [f(\xi^1 z^1, \xi^2 z^2, ..., \xi^J z^J) - \sum_{i=1}^{J} \gamma^i z^i (\xi^i - \rho)]. \tag{8}$$

Hedging

Main definition. A strategy $\gamma_1^i, ..., \gamma_n^i, i = 1, ..., J$, of the investor is called a *hedge*, if for any sequence $(\xi_1, ..., \xi_n)$ (with $\xi_j = (\xi_j^1, ..., \xi_j^J)$) the investor is able to meet his obligations, i.e.

$$G(X_n, S_n^1, ..., S_n^J) \geq 0.$$

The minimal value of the capital X_0 for which the hedge exists is called the *hedging price* H of an option.

Theorem (*Game theory for option pricing.*)

The minimal value of X_0 for which the income of the investor is not negative (and which by definition is the hedge price H) is given by

$$H^n = \frac{1}{\rho^n}(\mathcal{B}^n f)(S_0^1, ..., S_0^J). \tag{9}$$

Example. Standard CRR: J=1

$J = 1$ and f is convex non-decreasing (as for the standard European call with $f(S) = \max(S - K, 0)$).
Recall however that our assumptions are more general, we are working with the interval model.

$$(\mathcal{B}f)(z) = \min_{\gamma} \max_{\xi \in M} [f(\xi z) - \gamma z(\xi^i - \rho)],$$

where $M = [d, u] \subset \mathbf{R}$. By convexity,

$$(\mathcal{B}f)(z) = \min_{\gamma} \max[f(dz) - \gamma z(d - \rho), f(uz) - \gamma z(u - \rho)].$$

Clearly the minimum over γ is attained at

$$\gamma^h = \gamma^h(z, [f]) = \frac{f(uz) - f(dz)}{z(u - d)},$$

leading to

$$(\mathcal{B}f)(z) = \left[\frac{\rho - d}{u - d} f(uz) + \frac{u - \rho}{u - d} f(dz) \right].$$

Example J=1 completed

Clearly this operator is linear in the space of continuous functions on the positive half-line and preserves the set of convex non-decreasing function. Hence one can use this formula n times to find the hedge $H^n = \rho^{-n}(\mathcal{B}^n f)(S_0)$ leading to the following classical *CRR formula*

$$H^n = \rho^{-n} \sum_{k=0}^{n} C_n^k \left(\frac{\rho - d}{u - d} \right)^k \left(\frac{u - \rho}{u - d} \right)^{n-k} f(u^k d^{n-k} S_0),$$

where C_n^k are the standard binomial coefficients.

Example J=2 (two colors)

A nontrivial moment about this case is the possibility to calculate the Bellman operator again explicitly, but under certain additional assumption on f, which are remarkably satisfied for functions (1), (2) and (3) and with final formula depending on certain "coupling coefficient" reflecting the correlation between possible jumps of the first and second common stocks prices.

A function $f : R_+^2 \mapsto R^+$ is called *sub-modular* if it satisfies the inequality

$$f(z_1, \omega_2) + f(\omega_1, z_2) - f(z_1, z_2) - f(\omega_1, \omega_2) \geq 0$$

for every $z_1 < \omega_1$ and $z_2 < \omega_2$. A function $f : R_+^d \mapsto R^+$ is *sub-modular* whenever it is sub-modular with respect to every two variables.

Remark. If f is twice continuously differentiable, then it is sub-modular if and only if $\frac{\partial^2 f}{\partial z_i \partial z_j} \leq 0$ for all $i \neq j$.

Example J=2 (two colors) continued

Theorem

Let $J = 2$, f be convex sub-modular, and denote

$$\kappa = \frac{(u_1 u_2 - d_1 d_2) - \rho(u_1 - d_1 + u_2 - d_2)}{(u_1 - d_1)(u_2 - d_2)}. \tag{10}$$

If $\kappa \geq 0$, then $(\mathcal{B}f)(z_1, z_2)$ equals

$$\frac{\rho - d_1}{u_1 - d_1} f(u_1 z_1, d_2 z_2) + \frac{\rho - d_2}{u_2 - d_2} f(d_1 z_1, u_2 z_2) + \kappa f(d_1 z_1, d_2 z_2),$$

If $\kappa \leq 0$, the $(\mathcal{B}f)(z_1, z_2)$ equals

$$\frac{u_1 - \rho}{u_1 - d_1} f(d_1 z_1, u_2 z_2) + \frac{u_2 - \rho}{u_2 - d_2} f(u_1 z_1, d_2 z_2) + |\kappa| f(u_1 z_1, u_2 z_2),$$

Example J=2 (two colors) completed

The corresponding minimax strategies γ^{h1}, γ^{h2} can be also written explicitly.
Again by linearity, the powers of $\mathcal{B}$ can be found. Say, if $\kappa = 0$,

$$C_h = \rho^{-n} \sum_{k=0}^{n} C_n^k$$

$$\left(\frac{\rho - d_1}{u_1 - d_1}\right)^k \left(\frac{\rho - d_2}{u_2 - d_2}\right)^{n-k} f(u_1^k d_1^{n-k} S_0^1, d_2^k u_2^{n-k} S_0^2).$$

Example J=3 (three colors): setting I

In case $J > 2$ quite new qualitative effects can be observed. Namely the correspondent Bellman operator may turn out to be not linear, as above, but become a Bellman operator of a controlled Markov chain.
Denote vectors by bold letters , i.e. $\mathbf{z} = (z_1, z_2 ..., z_J)$.
For a set $I \subset \{1, 2, ..., J\}$ let us denote by $f_I(\mathbf{z})$ the value of $f(\xi^1 z_{1,} \xi^2 z_2, ..., \xi^J z_J)$ with $\xi^i = d_i$ for $i \in I$ and $\xi_i = u_i$ for $i \notin I$. For example, $f_{\{1,3\}}(\mathbf{z}) = f(d_1 z_1, u_2 z_2, d_3 z_3)$.
Suppose $0 < d_i < r < u_i$ for all $i \in \{1, 2, ..., J\}$ (otherwise trivial).

Example J=3 (three colors): setting II

Introduce the following coefficients:

$$\alpha_I = 1 - \sum_{j \in I} \frac{u_j - r}{u_j - d_j}, \text{where} \quad I \subset \{1, 2, ..., J\}.$$

In particular, in case $J = 3$

$$\begin{aligned}
\alpha_{123} &= \left(1 - \tfrac{u_1 - r}{u_1 - d_1} - \tfrac{u_2 - r}{u_2 - d_2} - \tfrac{u_3 - r}{u_3 - d_3}\right) \\
\alpha_{12} &= \left(1 - \tfrac{u_1 - r}{u_1 - d_1} - \tfrac{u_2 - r}{u_2 - d_2}\right) \\
\alpha_{13} &= \left(1 - \tfrac{u_1 - r}{u_1 - d_1} - \tfrac{u_3 - r}{u_3 - d_3}\right) \\
\alpha_{23} &= \left(1 - \tfrac{u_2 - r}{u_2 - d_2} - \tfrac{u_3 - r}{u_3 - d_3}\right).
\end{aligned} \tag{11}$$

Example J=3 (three colors): main result I

Theorem

Let $J = 3$ and f be convex and sub-modular.
(i) If $\alpha_{123} \geq 0$, then

$$(\mathcal{B}f)(\mathbf{z}) = \frac{1}{r}(\alpha_{123} f_{\emptyset}(\mathbf{z})$$

$$+\frac{u_1 - r}{u_1 - d_1} f_{\{1\}}(\mathbf{z}) + \frac{u_2 - r}{u_2 - d_2} f_{\{2\}}(\mathbf{z}) + \frac{u_3 - r}{u_3 - d_3} f_{\{3\}}(\mathbf{z})). \quad (12)$$

(ii) If $\alpha_{123} \leq -1$, then

$$(\mathcal{B}f)(\mathbf{z}) = \frac{1}{r}(-(\alpha_{123} + 1) f_{\{1,2,3\}}(\mathbf{z})$$

$$-\frac{d_1 - r}{u_{1-} d_1} f_{\{2,3\}}(\mathbf{z}) - \frac{d_2 - r}{u_{2-} d_2} f_{\{1,3\}}(\mathbf{z}) - \frac{d_3 - r}{u_{3-} d_3} f_{\{1,2\}}(\mathbf{z})). \quad (13)$$

Example J=3 (three colors): main result II

Theorem

As above, but now $0 \geq \alpha_{123} \geq -1$*.*
If $\alpha_{12} \geq 0$, $\alpha_{13} \geq 0$ *and* $\alpha_{23} \geq 0$, *then*

$$(\mathcal{B}f)(\mathbf{z}) = \frac{1}{r} \max(I, II, II),$$

$$I = -\alpha_{123} f_{\{1,2\}}(\mathbf{z}) + \alpha_{13} f_{\{2\}}(\mathbf{z}) + \alpha_{23} f_{\{1\}}(\mathbf{z}) + \frac{u_3 - r}{u_3 - d_3} f_{\{3\}}(\mathbf{z}),$$

$$II = -\alpha_{123} f_{\{1,3\}}(\mathbf{z}) + \alpha_{12} f_{\{3\}}(\mathbf{z}) + \alpha_{23} f_{\{1\}}(\mathbf{z}) + \frac{u_2 - r}{u_2 - d_2} f_{\{2\}}(\mathbf{z}),$$

$$III = -\alpha_{123} f_{\{2,3\}}(\mathbf{z}) + \alpha_{12} f_{\{3\}}(\mathbf{z}) + \alpha_{13} f_{\{2\}}(\mathbf{z}) + \frac{u_1 - r}{u_1 - d_1} f_{\{1\}}(\mathbf{z}).$$

For the cases (i) $\alpha_{ij} \leq 0$, $\alpha_{jk} \geq 0$, $\alpha_{ik} \geq 0$, *and (ii)* $\alpha_{ij} \geq 0$, $\alpha_{jk} \leq 0$, $\alpha_{ik} \leq 0$, *where* $\{i, j, k\}$ *is an arbitrary permutation of the set* $\{1, 2, 3\}$, *similar explicit formulae are available.*

Example J=3 (three colors): recursion

Denote by C_n^{ijk} the coefficient in the polynomial expansion

$$(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)^n = \sum_{i+j+k \le n} C_n^{ijk} \epsilon_1^{n-i-j-k} \epsilon_2^i \epsilon_3^j \epsilon_4^k.$$

Corollary

If $\alpha_{123} \ge 0$, the hedge price is equal to:

$$C_h = \frac{1}{\rho^n} \sum\nolimits_{i,j,k \in P_n} C_n^{ijk} (\alpha_{123})^{n-i-j-k} \left(\frac{u_1 - r}{u_1 - d_1}\right)^i \left(\frac{u_2 - r}{u_2 - d_2}\right)^j \left(\frac{u_3 - r}{u_3 - d_3}\right)^k$$
$$f(d_1^i u_1^{n-i} S_0^1, d_2^j u_2^{n-j} S_0^2, d_3^k u_3^{n-k} S_0^3), \tag{14}$$

and if $\alpha_{123} \le -1$,

$$C_h = \frac{1}{r^n} \sum\nolimits_{i,j,k \in P_n} C_n^{ijk} (-\alpha_{123} - 1)^{n-j-i-k} \left(\frac{r - d_1}{u_1 - d_1}\right)^i \left(\frac{r - d_2}{u_2 - d_2}\right)^j \left(\frac{r - d_3}{u_3 - d_3}\right)^k$$
$$f(d_1^{n-i} u_1^i S_0^1, d_2^{n-j} u_2^j S_0^2, d_3^{n-k} u_3^k S_0^3), \tag{15}$$

where $P_n = \{i, j, k \ge 0 : i + j + k \le n\}$.

Probabilistic interpretation I

Define a Markov process Z^t, $t = 0, 1, 2...$, on R_+^d: for $Z^t = \mathbf{z} \in R_+^d$ there are four possible positions of the process at the next time $t+1$, namely $(u_1 z_1, u_2 z_2, u_3 z_3)$, $(d_1 z_1, u_2 z_2, u_3 z_3)$, $(u_1 z_1, d_2 z_2, u_3 z_3)$, $(u_1 z_1, u_2 z_2, d_3 z_3)$, and they can occur with probabilities

$$
\begin{aligned}
P_{\mathbf{z}}^{u_1 z_1, u_2 z_2, u_3 z_3} &= \alpha_{123}, P_{\mathbf{z}}^{d_1 z_1, u_2 z_2, u_3 z_3} = \frac{u_1 - r}{u_1 - d_1}, \\
P_{\mathbf{z}}^{u_1 z_1, d_2 z_2, u_3 z_3} &= \frac{u_2 - r}{u_2 - d_2}, P_{\mathbf{z}}^{u_1 z_1, u_2 z_2, d_3 z_3} = \frac{u_3 - r}{u_3 - d_3},
\end{aligned}
$$

respectively. Since there are only finite number of possible jumps this Markov process is in fact a Markov chain.

Theorem

If $\alpha_{123} \geq 0$ *then*

$$
(\mathcal{B}^n f)(\mathbf{z}) = E_{\mathbf{z}} f(\mathbf{z}^n), \tag{16}
$$

where $E_{\mathbf{z}}$ *is the expectation of the process starting at* z.

Risk neutral probability selector

The probabilities

$$(\frac{u_1 - r}{u_1 - d_1}, \frac{u_2 - r}{u_2 - d_2}, \frac{u_3 - r}{u_3 - d_3}, \alpha_{123})$$

are called *risk-neutral probabilities*, as with these probabilities discounted stock prices become martingales.
Important:
Our method yields a unique selector among the $(J-1)$-parameter family of the risk -neutral probabilities for the incomplete market specified by our interval model.

Probabilistic interpretation II

In the case when $0 \geq \alpha_{123} \geq -1$, our Bellman operator (8) can be written in the form of the Bellman operator of a controlled Markov process, namely

$$(\mathcal{B}f)(\mathbf{z}) = \max_{i=1,2,3} \sum_{j=1}^{4} P_{\mathbf{z}}^{I_j^i(\mathbf{z})} f(I_j^i(\mathbf{z})). \tag{17}$$

For example, for $i = 1$, $I_j^1(\mathbf{z})$, $j = 1, 2, 3, 4$, could be the points

$$\begin{aligned} I_1^1(\mathbf{z}) &= (d_1 z_1, d_2 z_2, u_3 z_3), I_2^1(\mathbf{z}) = (d_1 z_1, u_2 z_2, u_3 z_3), \\ I_3^1(\mathbf{z}) &= (u_1 z_1, d_2 z_2, u_3 z_3), I_4^1(\mathbf{z}) = (u_1 z_1, u_2 z_2, d_3 z_3) \end{aligned}$$

and the corresponding probabilities of transitions from $\mathbf{z}$ to $I_j^1(\mathbf{z})$ are given by

$$P_{\mathbf{z}}^{I_1^1(\mathbf{z})} = -\alpha_{123}, P_{\mathbf{z}}^{I_2^1(\mathbf{z})} = \alpha_{23}, P_{\mathbf{z}}^{I_3^1(\mathbf{z})} = \alpha_{13}, P_{\mathbf{z}}^{I_4^1(\mathbf{z})} = \frac{u_3 - r}{u_3 - d_3}.$$

Concluding remarks

Representation (17) shows in particular that in this case the solution can not be written in form (16) and hence the obtained formula differs from what one can expect from the usual stochastic analysis approach to option pricing.

Problems. From a surprisingly simple linear form (12) and (13) of min-max Bellman operator (8) arises the question whether it can be generalized to other options, for example, those depending on $J > 3$ common stocks.

Another point to notice is an unexpectedly long and technical proof of the Theorems above resulting from a number of strange coincidence and cancelations. This leads to the following question for the theory of multistep dynamic games. What is the general justification for these cancelations and/or what is the class of game theoretic Bellman operators that can be reduced to a simpler Bellman operator of a controlled Markov chain?

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