Markov models of interacting particles and the dynamic law of large numbers

Vassili N. Kolokoltsov
Department of Statistics, University of Warwick, Coventry CV4 7AL UK,
Email: v.kolokoltsov@warwick.ac.uk

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X is a locally compact separable metric space;
$X^j = X \times \cdots \times X$ (j times)

$\mathcal{X} = \cup_{j=0}^{\infty} X^j$,

$C_{sym}(\mathcal{X}) = C(S\mathcal{X})$

Pairing

$$(f, \rho) = \int f(\mathbf{x})\rho(d\mathbf{x})$$

$$= f^0\rho_0 + \sum_{n=1}^{\infty} \int f(x_1, ..., x_n)\rho(dx_1 \cdots dx_n),$$

$$f \in C_{\text{sym}}(\mathcal{X}),\ \rho \in \mathcal{M}(\mathcal{X}).$$

Inclusion $S\mathcal{X}$ to $\mathcal{M}(X)$:

$$\mathbf{x} = (x_1, ..., x_l) \mapsto \delta_{x_1} + \cdots + \delta_{x_l} = \delta_{\mathbf{x}},$$

Decomposable measures (states):

$$(Y^{\otimes})_n(dx_1 \cdots dx_n)$$

$$= Y^{\otimes n}(dx_1 \cdots dx_n) = Y(dx_1) \cdots Y(dx_n)$$

Decomposable observables (multiplicative or additive)

$$(Q^{\otimes})^n(x_1,...,x_n) = Q^{\otimes n}(x_1,...,x_n) = Q(x_1)\cdots Q(x_n)$$

and

$$(Q^{\oplus})(x_1,...,x_n) = Q(x_1) + \cdots + Q(x_n)$$

Binary particle interaction of pure jump is specified by a transition kernel

$$P^2(x_1,x_2;d\mathbf{y}) = \{P^2_m(x_1,x_2;dy_1\cdots dy_m)\}$$

with the intensity

$$P^2(x_1,x_2) = \int_{\mathcal{X}} P^2(x_1,x_2;d\mathbf{y})$$

$$= \sum_{m=0}^{\infty} \int_{X^m} P^2_m(x_1,x_2;dy_1\cdots dy_m).$$

The generator of interacting system is

$$(G_2 f)(x_1,...,x_n) = \sum_{I\subset\{1,...,.n\},|I|=2}$$

$$\int (f(\mathbf{x}_{\bar{I}},\mathbf{y}) - f(x_1,...,x_n))P^2(\mathbf{x}_I,d\mathbf{y})$$

$$= \sum_{m=0}^{\infty} \sum_{I \subset \{1,...,.n\}, |I|=2}$$

$$\int (f(\mathbf{x}_{\bar{I}}, y_1, ..., y_m) - f(x_1, ..., x_n)) P_m^k(\mathbf{x}_I; dy_1 ... dy_m).$$

Probabilistic description.

Interaction of kth order is specified by a transition kernel

$P^k(x_1, ..., x_k; d\mathbf{y}) = \{P_m^k(x_1, ..., x_k; dy_1 \cdots dy_m)\}$

leading to the following *generator of k-ary interacting particles*:

$$(G_k f)(x_1, ..., x_n)$$

$$= \sum_{I \subset \{1,...,n\}, |I|=k} \int (f(\mathbf{x}_{\bar{I}}, \mathbf{y}) - f(x_1, ..., x_n)) P^k(\mathbf{x}_I, d\mathbf{y}).$$

Interactions of all orders up to k, are given by the generator

$$G_{\leq k} f = \sum_{l=1}^{k} G_l f.$$

Changing the state space according to mapping

$$\mathbf{x} = (x_1, ..., x_l) \mapsto \delta_{x_1} + \cdots + \delta_{x_l} = \delta_{\mathbf{x}},$$

yields the Markov process on $\mathcal{M}_\delta^+(X)$.

Scaling: empirical measures $\delta_{x_1} + \cdots + \delta_{x_n}$ by a factor h and the operator of k-ary interactions by a factor h^{k-1}.

$$\Lambda_k^h F(h\delta_{\mathbf{x}}) = h^{k-1} \sum_{I \subset \{1,...,n\}, |I|=k} \int_{\mathcal{X}}$$

$$[F(h\delta_{\mathbf{x}} - h\delta_{\mathbf{x}_I} + h\delta_{\mathbf{y}}) - F(h\nu)] P(\mathbf{x}_I; d\mathbf{y}),$$

$$\Lambda_{\leq k}^h F(h\delta_{\mathbf{x}}) = \sum_{l=1}^{k} \Lambda_l^h F(h\delta_{\mathbf{x}}).$$

Aim: limit $h \to 0$ with $h\delta_{\mathbf{x}}$ converging to a finite measure (h is the inverse of the number of particles).

Applying the obvious equation

$$\sum_{I \subset \{1,\ldots,n\}, |I|=2} f(\mathbf{x}_I) =$$

$$\frac{1}{2} \int \int f(z_1, z_2) \delta_{\mathbf{x}}(dz_1) \delta_{\mathbf{x}}(dz_2)$$

$$-\frac{1}{2} \int f(z, z) \delta_{\mathbf{x}}(dz),$$

which holds for any $f \in C_{\mathsf{sym}}(X^2)$ and $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$:

$$\Lambda_2^h F(h\delta_{\mathbf{x}}) = -\frac{1}{2} \int_{\mathcal{X}} \int_X$$

$$[F(h\delta_{\mathbf{x}} - 2h\delta_z + h\delta_{\mathbf{y}}) - F(h\delta_{\mathbf{x}})] P(z, z; d\mathbf{y})(h\delta_{\mathbf{x}})(dz)$$

$$+\frac{1}{2h} \int_{\mathcal{X}} \int_{X^2} [F(h\delta_{\mathbf{x}} - h\delta_{z_1} - h\delta_{z_2} + h\delta_{\mathbf{y}}) - F(h\delta_{\mathbf{x}})]$$

$$P(z_1, z_2; d\mathbf{y})(h\delta_{\mathbf{x}})(dz_1)(h\delta_{\mathbf{x}})(dz_2).$$

On the linear functions

$$F_g(\mu) = \int g(y)\mu(dy) = (g, \mu)$$

this operator acts as

$$\Lambda_2^h F_g(h\delta_{\mathrm{x}}) = \frac{1}{2}\int_{\mathcal{X}}\int_{X^2}[g^{\oplus}(\mathbf{y}) - g^{\oplus}(z_1, z_2)]$$

$$P(z_1, z_2; d\mathbf{y})(h\delta_{\mathrm{x}})(dz_1)(h\delta_{\mathrm{x}})(dz_2)$$

$$-\frac{1}{2}h\int_{\mathcal{X}}\int_{X}[g^{\oplus}(\mathbf{y}) - g^{\oplus}(z, z)]P(z, z; d\mathbf{y})(h\delta_{\mathrm{x}})(dz).$$

Hence, $h \to 0$ and $h\delta_{\mathrm{x}} \to \mu$ (in other words, that the number of particles tends to infinity, but the "whole mass" remains finite), the evolution equation $\dot{F} = \Lambda_2^h F$ on linear functionals $F = F_g$ tends to the equation

$$\frac{d}{dt}(g, \mu_t) = \Lambda_2 F_g(\mu_t) = \frac{1}{2}\int_{\mathcal{X}}\int_{X^2}$$

$$(g^{\oplus}(\mathbf{y}) - g^{\oplus}(\mathbf{z}))P^2(\mathbf{z}; d\mathbf{y})\mu_t^{\otimes 2}(d\mathbf{z}),$$

which is the *general kinetic equation for binary interactions of pure jump type* in weak form. "Weak" means that it must hold for all $g \in C_\infty(X)$ (or at least its dense subspace).

A similar procedure with k-ary interactions leads to the *general kinetic equation for k-ary interactions of pure jump type* in weak form:

$$\frac{d}{dt}(g, \mu_t) = \Lambda_k F_g(\mu_t)$$

$$= \frac{1}{k!} \int_{\mathcal{X}} \int_{X^k} (g^{\oplus}(\mathbf{y}) - g^{\oplus}(\mathbf{z}))$$

$$P^k(\mathbf{z}; d\mathbf{y}) \mu_t^{\otimes k}(d\mathbf{z}), \quad \mathbf{z} = (z_1, ..., z_k).$$

and more generally:

$$\frac{d}{dt}(g, \mu_t) = \Lambda_{l \le k} F_g(\mu_t)$$

$$= \sum_{l=1}^{k} \Lambda_l F_g(\mu_t).$$

Examples.

General Feller generator in $S\mathcal{X}$ has the form $B = (B^1, B^2, ...)$, where

$$B^k f(x_1, ..., x_k) = A^k f(x_1, ..., x_k) +$$

$$\int_{\mathcal{X}} (f(\mathbf{y}) - f(x_1, ..., x_k)) P^k(x_1, ..., x_k, d\mathbf{y}),$$

where P^k is a transition kernel from SX^k to $S\mathcal{X}$ and A^k generates a symmetric Feller process in X^k. However, with this generator, the interaction of a subsystem of particles depends on the whole system: for the operator $(0, B^2, 0, ...)$, say, two particles will interact only in the absence of any other particle). To put all subsystems on an equal footing one should mix the interaction between all subsystems. Consequently, instead of B^k one is led to the *generator of k-ary interaction* of the form

$$I_k[P^k, A^k] f(x_1, ..., x_n) = \sum_{I \subset \{1,...,n\}, |I|=k} B_I f(x_1, ..., x_n)$$

$$= \sum_{I \subset \{1,...,n\}, |I|=k} \Big[(A_I f)(x_1, ..., x_n)$$

$$+ \int (f(\mathbf{x}_{\bar{I}}, \mathbf{y}) - f(x_1, ..., x_n)) P^k(\mathbf{x}_I, d\mathbf{y}) \Big],$$

where A_I (resp. B_I) is the operator $A^{|I|}$ (resp. $B^{|I|}$) acting on the variables $\mathbf{x}_I$. In quantum mechanics, the transformation $B^1 \mapsto I_1$ is called the *second quantization* of the operator B^1. The transformation $B^k \mapsto I_k$ for $k > 1$ can be interpreted as the tensor power of the second quantization.

The same limiting procedure as above:

$$\frac{d}{dt}\int g(z)\mu_t(dz) = \sum_{l=1}^{k} \frac{1}{l!}\int_{X^l}\Big[(Ag^{\oplus})(\mathbf{z})$$

$$+\int_{\mathcal{X}}(g^{\oplus}(\mathbf{y}) - g^{\oplus}(\mathbf{z}))P(\mathbf{z};d\mathbf{y})\Big]\mu_t^{\otimes l}(d\mathbf{z}).$$

More compactly it can be written in terms of the operators B^k as

$$\frac{d}{dt}\int g(z)\mu_t(dz)$$

$$= \sum_{l=1}^{k} \frac{1}{l!}\int_{X^l}(B^l g^{\oplus})(\mathbf{z})\mu_t^{\otimes l}(d\mathbf{z})$$

or as

$$\frac{d}{dt}\int g(z)\mu_t(dz) = \int_{\mathcal{X}}(Bg^{\oplus})(\mathbf{z})\mu_t^{\tilde{\otimes}}(d\mathbf{z}),$$

where the convenient normalized tensor power of measures are defined by

$$(Y^{\tilde{\otimes}})_n(dx_1 \cdots dx_n) = Y^{\tilde{\otimes} n}(dx_1 \cdots dx_n)$$

$$= \frac{1}{n!} Y(dx_1) \cdots Y(dx_n).$$

Finally, one can allow additionally for *mean field interaction*:

$$\frac{d}{dt}(g, \mu_t) = \int_{\mathcal{X}} (B[\mu_t] g^{\oplus})(\mathbf{z}) \mu_t^{\tilde{\otimes} l}(d\mathbf{z}),$$

which is the weak form of the *general kinetic equation describing the dynamic LLN for Markov models of interacting particles* with mean field and kth-order interactions.

If the Cauchy problem for this equation is well posed, its solution μ_t with a given $\mu_0 = \mu$ can be considered as a deterministic measure-valued Markov process. The corresponding semigroup is defined as $T_t F(\mu) = F(\mu_t)$. Using variational derivatives evolution equation for this semigroup can be written as

$$\frac{d}{dt}F(\mu_t) = (\Lambda F)(\mu_t)$$

$$= \int_{\mathcal{X}} B[\mu_t] \left(\frac{\delta F}{\delta \mu_t(.)}\right)^{\oplus} (\mathbf{z}) \mu_t^{\tilde{\otimes}}(d\mathbf{z})$$

$$= \int_{\mathcal{X}} A[\mu_t] \left(\frac{\delta F}{\delta \mu_t(.)}\right)^{\oplus} (\mathbf{z}) \mu_t^{\tilde{\otimes}}(d\mathbf{z})$$

$$+ \int_{\mathcal{X}^2} \left[\left(\frac{\delta F}{\delta \mu_t(.)}\right)^{\oplus} (\mathbf{y}) - \left(\frac{\delta F}{\delta \mu_t(.)}\right)^{\oplus} (\mathbf{z})\right] P(\mu_t, \mathbf{z}; d\mathbf{y}) \mu_t^{\tilde{\otimes}}(d\mathbf{z})$$

Kinetic equation above is nothing but a particular case of this equation for the linear functionals $F(\mu) = F_g(\mu) = (g, \mu)$.

One can naturally identify the nonlinear analogs of the main notions from the theory of Markov processes and observe how the fundamental connection between Markov processes, semigroups and martingale problems is carried forward into the nonlinear setting.

Let $\tilde{\mathcal{M}}(X)$ be a dense subset of the space $\mathcal{M}(X)$ of finite (positive Borel) measures on a metric space X (considered in its weak topology). By a nonlinear *sub-Markov* (resp. *Markov*) *propagator* in $\tilde{\mathcal{M}}(X)$ we shall mean any propagator $V^{t,r}$ of possibly nonlinear transformations of $\tilde{\mathcal{M}}(X)$ that do not increase (resp. preserve) the norm. If $V^{t,r}$ depends only on the difference $t-r$ and hence specifies a semigroup, this semigroup is called nonlinear or generalized *sub-Markov* or *Markov* respectively.

The usual, linear Markov propagators or semigroups correspond to the case when all the transformations are linear contractions in the whole space $\mathcal{M}(X)$. In probability theory these propagators describe the evolution of averages of Markov processes, i.e. processes whose evolution after any given time t depends on the past $X_{\leq t}$ only via the present position X_t. Loosely speaking, to any nonlinear Markov propagator there corresponds a process whose behavior after any time t depends on the past $X_{\leq t}$ via the position X_t of the process and its distribution at t. To be more precise, consider the nonlinear kinetic equation

$$\frac{d}{dt}(g, \mu_t) = (B[\mu_t]g, \mu_t)$$

with a certain family of operators $B[\mu]$ in $C(X)$ depending on μ as on a parameter and such that each $B[\mu]$ generates a Feller semigroup. (It was shown above that equations of this kind appear naturally as LLN for interacting particles, they also arise from the mere assumption of positivity preservation.)

Suppose also that for any weakly continuous curve $\mu_t \in \mathcal{P}(X)$ the solutions to the Cauchy problem of the equation

$$\frac{d}{dt}(g, \nu_t) = (B[\mu_t]g, \nu_t)$$

define a weakly continuous propagator $V^{t,r}[\mu_.]$, $r \leq t$, of linear transformations in $\mathcal{M}(X)$ and hence a Markov process in X. Then to any $\mu \in \mathcal{P}(X)$ there corresponds a Markov process X_t^μ in X with distributions $\mu_t = T_t(\mu)$ for all times t and with transition probabilities $p_{r,t}^\mu(x, dy)$ satisfying the condition

$$\int_{X^2} f(y) p_{r,t}^\mu(x, dy) \mu_r(dx) = (f, V^{t,r}\mu_r) = (f, \mu_t).$$

We shall call the family of processes X_t^μ a *nonlinear Markov process*.

Thus a nonlinear Markov process is a semigroup of the transformations of distributions such that to each trajectory is attached a 'tangent' Markov process with the same marginal distributions. The structure of these tangent processes is not intrinsic to the semigroup, but can be specified by choosing a stochastic representation for the generator.

As in the linear case the process X_t with càdlàg paths (or the corresponding probability distribution on the Skorohod space) solves the $(B[\mu], D)$-*nonlinear martingale problem* with initial distribution μ, meaning that X_0 is distributed according to μ and the process

$$M_t^f = f(X_t) - f(X_0) - \int_0^t B[\mathcal{L}(X_s)] f(X_s)\, ds, \quad t \geq 0$$

is a martingale for any $f \in D$, with respect to the natural filtration of X_t.

LLN and Conditional positivity.

One can show that bounded generators of measure-valued positivity preserving evolutions have necessarily a stochastic representation, leading directly to a probabilistic interpretation of the corresponding evolution. For a Borel space X we shall say that a mapping $\Omega : \mathcal{M}(X) \to \mathcal{M}^{\text{signed}}(X)$ is *conditionally positive* if the negative part $\Omega^-(\mu)$ of the Hahn decomposition of the measure $\Omega(\mu)$ is absolutely continuous with respect to μ for all μ. One easily deduces that continuous generators of positivity preserving evolutions should be conditionally positive in this sense.

Theorem.

Let X be a Borel space and $\Omega : \mathcal{M}(X) \to \mathcal{M}^{\text{signed}}(X)$ be a conditionally positive mapping. Then there exists a nonnegative function $a(x,\mu)$ and a family of kernels $\nu(x,\mu,.)$ in X such that

$$\Omega(\mu) = \int_X \mu(dz)\nu(z,\mu,.) - a(.,\mu)\mu. \qquad (1)$$

If moreover $\int \Omega(\mu)(dx) = 0$ for all μ (condition of conservativity), then this representation can be chosen in such a way that $a(x,\mu) = \|\nu(x,\mu,.)\|$, in which case

$$(g, \Omega(\mu)) = \int_X (g(y) - g(x))\nu(x,\mu,dy).$$

Proof. One can take $a(x,\mu)$ to be the Radon-Nicodyme derivative of $\Omega^-(\mu)$ with respect to μ and

$$\nu(x,\mu,dy) = \left(\int \Omega^-(\mu)(dz)\right)^{-1} a(x,\mu)\Omega^+(\mu)(dy).$$

CLT Binary interaction

if B^2 preserves the number of particles and hence can be written as

$$B^2 f(x,y) = \Big[\frac{1}{2}(G(x,y)\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) + \frac{1}{2}(G(y,x)\frac{\partial}{\partial y}, \frac{\partial}{\partial y})$$

$$+(\gamma(x,y)\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\Big] f(x,y)$$

$$+\left[(b(x,y), \frac{\partial}{\partial x}) + (b(y,x), \frac{\partial}{\partial y}) f(x,y)\right]$$

$$+\int_{X^2} \nu(x,y,dv_1 dv_2)\Big[f(x+v_1, y+v_2) - f(x,y)$$

$$-(\frac{\partial f}{\partial x}(x,y), v_1)\mathbf{1}_{B_1}(v_1) - (\frac{\partial f}{\partial y}(x,y), v_2)\mathbf{1}_{B_1}(v_2)\Big],$$

where $G(x,y), \gamma(x,y)$ are symmetric matrices such that $\gamma(x,y) = \gamma(y,x)$ and $\nu(x,y,dv_1 dv_2) =$

$\nu(y, x, dv_2 dv_1)$, then

$$O_t F(Y) = \left(B^2 \left(\frac{\delta F}{\delta Y} \right)^{\oplus}, Y \otimes \mu_t \right)$$
$$+ \int \frac{1}{2} \left(G(x,y) \frac{\partial}{\partial x}, \frac{\partial}{\partial z} \right) \frac{\delta^2 F}{\delta Y(z) \delta Y(x)} |_{z=x} \, \mu_t(dx) \mu_t(dy)$$
$$+ \int \left(\gamma(x,y) \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \frac{\delta^2 F}{\delta Y(y) \delta Y(x)} \mu_t(dx) \mu_t(dy)$$
$$+ \frac{1}{4} \int_{X^4} \left(\frac{\delta^2 F}{\delta Y(.) \delta Y(.)}, (\delta_{z_1+v_1} + \delta_{z_2+v_2} - \delta_{z_1} - \delta_{z_2})^{\otimes 2} \right)$$
$$\nu(z_1, z_2, dv_1 dv_2) \mu_t(dz_1) \mu_t(dz_2). \qquad (2)$$

CLT k-ary.

Let B^k be a conditionally positive operator $C(S\mathcal{X}) \mapsto C(SX^k)$ given by (**??**), i.e.

$$B^k f(x_1, ..., x_k) = A^k f(x_1, ..., x_k) + \int_{X^k}$$

$$(f(y_1, ..., y_k) - f(x_1, ..., x_k)) P^l(x_1, ..., x_k, dy_1 \cdots dy_k),$$

and specifying the scaled generator of k-ary interaction:

$$\Lambda_h^k F(h\delta_{\mathrm{x}}) = h^{k-1} \sum_{I \subset \{1,...,n\}, |I|=k} B_I^k F(h\delta_{\mathrm{x}}).$$

Then the limiting generator of the process of fluctuation writes down as

$$O_t F(Y) = \left(B^k \left(\frac{\delta F}{\delta Y} \right)^{\oplus}, Y \otimes \mu_t^{\tilde{\otimes}(k-1)} \right)$$

$$+ \Big(\frac{1}{2} B^k \sum_{i,j=1}^{k} \frac{\delta^2 F}{\delta Y(y_i) \delta Y(y_j)}$$

$$- \left(B^k_{y_1,\dots,y_k} \sum_{i,j=1}^{k} \frac{\delta^2 F}{\delta Y(z_i) \delta Y(y_j)} \right) |_{\forall i \ z_i = y_i}, \mu_t^{\tilde{\otimes} k} \Big),$$

where $B^k_{y_1,\dots,y_k}$ denotes of course the action of B^k on the variables $y_1, \dots, y_k$.

About the method. Remark: How to prove the usual functional CLT.

Recall: main result on the convergence of semigroups.

Using this fact, a functional central limit theorem can easily be proved as follows. Let Z_t be a continuous-time random walk on $\mathbf{Z}$ moving in each direction with equal probability. It is the process specified by the generator

$$Lf(x) = a[f(x+1) + f(x-1) - 2f(x)]$$

with coefficient $a > 0$. By scaling we can define a new random walk Z_t^h on $h\mathbf{Z}$ by the generator

$$L_h f(x) = \frac{a}{h^2}(f(x+h) + f(x-h) - 2f(x)).$$

If f is thrice continuously differentiable, then

$$L_h f(x) = af''(x) + O(1) \sup_y |f^{(3)}(y)|.$$

Hence for $f \in C^3_\infty(\mathbf{R}^d)$ (invariant cire) the generators $L_h f$ converge to the generator $Lf = af''$ of a Brownian motion. Done.

Remark. The process $Z_t^h = h(S_1 + \cdots + S_{N_t})$ is a random walk with $\mathbf{E}(N_t) = 2at/h^2$, so that $h \sim 1/\sqrt{N_t}$.

Extension: position dependent, stable-like, etc.

For interacting particles this method is used with three NO: (1) no compactness, (2) time non-homogeneous, (3) infinite dimensions.

One needs to show: (i) the approximating particle systems converge to the deterministic limit described by the kinetic equations, (ii) the fluctuation process converge to a limiting infinite-dimensional Gaussian process. In both cases we need a core for the limiting semigroup, and to get the rates of convergence we need an invariant core.

The generators of the LLN limit ΛF is expressed in variational derivative. But the limiting evolution is deterministic; i.e. it has the form $F_t(\mu) = F(\mu_t)$ with μ_t being a solution to kinetic equation. We therefore have

$$\frac{\delta F_t}{\delta \mu} = \frac{\delta F}{\delta \mu_t}\frac{\delta \mu_t}{\delta \mu},$$

which shows that in order to apply the general method outlined above, we need first-order variational derivatives of the solutions to the kinetic equation with respect to the initial data. Similarly, to study for fluctuation processes the second-order variational derivatives.

In other words: (1) smoothness of nonlinear Markov semigroups $T_t(\mu) = \mu_t$ given by

$$\frac{d}{dt}(g, \mu_t) = (A_{\mu_t} g, \mu_t), \quad \mu_0 = \mu \in \mathcal{M}(S),$$

with respect to initial data, where A_μ is a Lévy-Khintchine type operator depending on μ as on a parameter, and (ii) to identify an invariant core for the usual (linear) Markov semigroup $\Phi_t F(\mu) = F(\mu_t)$ of the deterministic measure-valued Markov process μ_t.

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