

Stochastic Perturbations of  
Deterministic Optimization  
Problems with Applications to  
a Spin Control Problem  
(Control of a Two-Level  
Atom)

Vassili N. Kolokoltsov\*

The main characteristics of a long-time optimal process are determined by the solutions  $(\lambda, h)$  (where  $\lambda$  is a number and  $h$  a function on the state space) of the equation  $Bh = \lambda \oplus h$ , where  $B$  is the Bellman operator of the optimization problem. Namely,  $\lambda$  is the mean income per step of the process, whereas  $h$  specifies stationary optimal strategies or even turnpike control modes. For deterministic control problems, in which  $B$  is linear in the sense of the operations  $\oplus = \min$  or  $\oplus = \max$  and  $\odot = +$ , this equation is the idempotent analog of an eigenvector equation in standard linear algebra. Lots of authors have studied this equation.

We develop results from V.N. Kolokoltsov. The stochastic Bellman equation as a nonlinear equation in Maslov spaces. Perturbation theory. Sov. Math. Dokl. **45:2** (1992), 294-300, (see also further publications, e.g. the book V.N.Kolokoltsov, V.P. Maslov. Idempotent Analysis and its Applications. Kluwer Academic, 1997) in order to show that:

if the influence of stochastic factors is small, it can be considered as nearly linear and perturbation theory can be developed. This kind of small stochasticity arises in some models of control over quantum systems that are presently subject to intensive research.

## Stochastic perturbations of discrete optimization

Let  $X$  be a finite or countable set, and  $b = (b_{ij})|_{i,j \in X}$  a matrix with values in  $\mathbf{R}$ . Let a subset  $J \subset X$  and a bounded set  $\{q_j^i\}_{i \in X, j \in J}$  of nonnegative numbers be given, and the family of operators  $\{B_\epsilon\}_{\epsilon \geq 0}$  in  $\mathbf{R}^X$  be defined by the formula

$$(B_\epsilon h)_i = \max_{k: (i,k) \in E} (b(i,k) + h_k) \left( 1 - \epsilon \sum_{j \in J} q_j^i \right) + \epsilon \sum_{j \in J} q_j^i (b(i,j) + h_j).$$

For  $\epsilon$  small enough, the operator  $B_\epsilon$  is the Bellman operator corresponding to a controlled Markov process on  $X$ . In this process, the transition from  $i \in X$  to  $j \in J$  has probability  $\epsilon q_j^i$ , and the transition to a chosen point  $k$  from the set  $\{k : (i,k) \in E\}$  occurs with probability  $1 - \epsilon \sum_{j \in J} q_j^i$ .

**Theorem 1.** *Let the maximum of  $b$  be attained at a unique point  $(V, V) \in E$ , where  $b(V, V) = 0$ ,  $V \notin J$  and there exists a  $j \in J$  such that the probability  $q_j^i$  is nonzero for all  $i$ . Then for each sufficiently small  $\epsilon > 0$ ,  $\exists$  unique  $\lambda_\epsilon \in \mathbf{R}$ ,  $h^\epsilon \in \mathbf{R}^X$  such that*

$$B_\epsilon h^\epsilon = \lambda_\epsilon + h^\epsilon, \quad h_V^\epsilon = 0, \quad (1)$$

*and  $\lambda_\epsilon$  and  $h^\epsilon$  are differentiable with respect to  $\epsilon$  at  $\epsilon = 0$ :*

$$\lambda_\epsilon = \epsilon \lambda' + o(\epsilon), \quad h^\epsilon = h^0 + \epsilon h' + o(\epsilon),$$

$$\lambda' = \sum_{j \in J} q_j^V (b(V, j) + h_j^0),$$

$$h^i = (I - B_h')^{-1} \left( \left\{ \sum_{j \in J} q_j^i (b(i, j) + h_j^0 - h_i^0) \right\} \right), \quad (2)$$

$$(B_h' g)_i = \max\{g_k : k \in \Gamma_h(i)\}; \quad (3)$$

*here  $\Gamma_h(i)$  is the set of vertices where  $b_{ij} + h_j^0$  attains the maximal value.*

## Extension to continuous state space

Let the state space  $X$  be locally compact and the control set  $U$  be compact, and let  $v$  and  $V$  be two distinct points of  $X$ . Suppose that the process dynamics is determined by a bounded continuous mapping  $y: X \times U \times [0, \epsilon_0] \rightarrow X$  and a bounded continuous function  $q: X \rightarrow \mathbf{R}_+$  as follows. If a control  $u \in U$  is chosen when the process is in a state  $x \in X$ , then at the current step the transition into the state  $y(x, u, \epsilon)$  takes place with probability  $1 - \epsilon q(x)$ , whereas with probability  $\epsilon q(x)$  the transition is into  $v$ . The income from residing in a state  $x \in X$  is specified by a Lipschitz continuous function  $b \in C_\infty(\mathbf{R})$ . The Bellman operator  $B_\epsilon$  acts in the space of continuous functions on  $X$  according to the formula

$$(B_\epsilon h)(x) = b(x) + \epsilon q(x)h(v) \\ + (1 - \epsilon q(x)) \max_{u \in U} h(y(x, u, \epsilon)).$$

**Theorem 2.** *Suppose that for each  $\epsilon$  the deterministic dynamics is controllable in the sense that by moving successively from  $x$  to  $y(x, u, \epsilon)$  one can reach any point from any other point in a fixed number of steps independent of the initial point. Suppose also that  $b$  attains its maximum at a unique point  $V$ , where  $b(V) = 0$  and moreover,*

$$V \in \{y(V, u, \epsilon) : u \in U\}.$$

*Then the equation*

$$Bh = \lambda + h \tag{4}$$

*is solvable, and the solution satisfies*

$$h^\epsilon - h^0 = O(\epsilon), \tag{5}$$

$$\lambda_\epsilon = q(V)h^0(v)\epsilon + o(\epsilon), \tag{6}$$

*where  $\lambda_0 = 0$  and  $h^0$  is the unique solution of (4) at  $\epsilon = 0$ .*

The proof is a generalization of that of Theorem 1.

## Generalized solutions of the HJB equation for jump controlled processes on a manifold

Let  $X$  be a smooth compact manifold, and let  $f(x, u, \epsilon)$  be a vector field on  $X$  depending on the parameters  $u \in U$  and  $\epsilon \in [0, \epsilon_0]$  and Lipschitz continuous with respect to all arguments. Consider a special case of the process described above, in which  $y(x, u, \epsilon)$  is the point reached at time  $\tau$  by the trajectory of the differential equation  $\dot{z} = f(z, u, \epsilon)$  issuing from  $x$  and the probability of the transition into  $v$  in one step of the process is equal to  $\tau \epsilon q(x)$ . As  $\tau \rightarrow 0$ , this process becomes a jump process in continuous time; this process is described by a stochastic differential equation with stochastic differential of Poisson type.



Let  $S_n^\epsilon(t, x)$  be the mathematical expectation of the maximal income per  $n$  steps of the cited discrete process with time increment  $\tau = (T - t)/n$  beginning at time  $t$  at a point  $x$  and with terminal income specified by a Lipschitz continuous function  $S_T(x)$ . Then  $S_n^\epsilon = (B_\epsilon^\tau)^n$ , where  $B_\epsilon^\tau$  is the Bellman operator corresponding to the discrete problem with step  $\tau$ .

**Theorem 3.** *The sequence of continuous functions  $S_n^\epsilon$  is uniformly convergent with respect to  $x$  and  $\epsilon$  to a Lipschitz continuous (and hence, almost everywhere smooth) function  $S^\epsilon(t, x)$ , which satisfies the functional-differential Bellman equation*

$$\begin{aligned} \frac{\partial S}{\partial t} + b(x) + \epsilon q(x)(S(v) - S(x)) \\ + \max_{u \in U} \left( \frac{\partial S}{\partial x}, f(x, u, \epsilon) \right) = 0 \end{aligned} \quad (7)$$

*at each point of differentiability.*

The limit function  $S^\epsilon$  may be called a generalized solution of the Cauchy problem for equation (7). This function specifies the mathematical expectation of the optimal income for the limit (as  $t \rightarrow 0$ ) jump process in continuous time. For  $\epsilon = 0$ , this solution coincides with that obtained in the framework of idempotent analysis. The proof of this theorem is quite lengthy and technical.

Theorems 2 and 3 imply the following result.

**Theorem 4.** *There exists a continuous function  $R^\epsilon$  and a unique  $\lambda_\epsilon$  such that the generalized solution of the Cauchy problem for equation (7) with terminal function  $S_T^\epsilon = h^\epsilon$  has the form*

$$S^\epsilon(t, x) = \lambda_\epsilon(T - t) + h^\epsilon(x), \quad (8)$$

$\chi_\epsilon$  satisfies the asymptotic formula (6), and the generalized solution  $S^\epsilon(t, x)$  of (7) with an arbitrary Lipschitz continuous terminal function  $S_T^\epsilon$  satisfies the limit equation

$$\lim_{t \rightarrow -\infty} \frac{1}{T - t} S^\epsilon(t, x) = \lambda_\epsilon. \quad (9)$$

## **Example from quantum control**

Dynamics will be given by a stochastic equation of Poisson type. A similar example of diffusive type can be found in V.N. Kolokoltsov. Long time behavior of continuously observed and controlled quantum systems (a study of the Belavkin quantum filtering equation). In: Quantum Probability Communications, QP-PQ, V. 10, Ed. R.L. Hudson, J.M. Lindsay, World Scientific, Singapore (1998), 229-243..

Consider a model of continuously observed quantum system interacting with an instrument (boson reservoir) by exchanging photons. The a posteriori dynamics (i.e., dynamics taking into account the measurement results) of this system can be described by Belavkin's quantum filtering equation

$$\begin{aligned} d\Phi + \left( i[E, \Phi] + \epsilon \left( \frac{1}{2} (R^* R \Phi + \Phi R^* R) - (\Phi, R^* R) \Phi \right) \right) dt \\ = \left( \frac{R \Phi R^*}{(\Phi, R^* R)} - \Phi \right) dN. \end{aligned} \quad (10)$$

Here  $N(t)$  is a counting Poisson process; its spectrum is the set of positive integers, and the result of measurement by time  $t$  is a random tuple  $\tau = \{t_1 < \dots < t_n\}$  of time moments at which the photon emission occurs. Furthermore,  $E$  and  $R$  are closed operators in the Hilbert state space  $H$  of the quantum system in question; the self-adjoint energy operator  $E$  specifies the free (nonobserved) dynamics, whereas  $R$  corresponds to the observed (measured) physical variable. The unknown density matrix  $\Phi$  specifies the a posteriori state in  $H$ , and the intensity of the jump process at the state  $\Phi$  is equal to  $\epsilon(\Phi, R^* R)$ .

Now suppose that the system is controllable; specifically, let the energy  $E$  be a function of a parameter  $u \in U$  whose value at each time can be chosen on the basis of the information  $\tau$  available by this time. The opportunity to evaluate a posteriori states from equation (10) permits one to construct control strategies as functions of state,  $u = u(t, \Phi(t))$ . Suppose that we intend to maximize some operator-valued criterion of the form

$$\int_t^T (\Phi(s), A) ds + (\Phi(T), G), \quad (11)$$

where  $A$  and  $G$  are self-adjoint operators in  $H$ . Let  $S(t, \Phi)$  denote the Bellman function, that is, the mathematical expectation (over all realizations of  $\tau$ ) of the maximum income of a process that starts at time  $t$  in a state  $\Phi$  and terminates at time  $T$ .

Bellman equation for the optimal income function  $S(t, \Phi)$ :

$$\begin{aligned} \frac{\partial S}{\partial t} + \epsilon \left( \Phi, R^* R, \left( S \left( \frac{R \Phi R^*}{(\Phi, R^* R)} \right) - S(\Phi) \right) + (\Phi, A) \right. \\ \left. + \max_{u \in U} \left( \text{grad}_{\Phi} S, i[E(u), \Phi] \right. \right. \\ \left. \left. + \frac{\epsilon}{2} (R^* R \Phi + \Phi R^* R) - \epsilon (\Phi, R^* R) \Phi \right) \right) = 0. \quad (12) \end{aligned}$$

However, the solution of this equation with the terminal condition  $S_T(\Phi) = (\Phi(T), G)$  is not uniquely determined in the class of functions smooth almost everywhere; hence, a well-grounded theory of generalized solutions should employ additional considerations so as to yield the Bellman function. For example, generalized solutions can be defined as limits of discrete approximations.

Consider the special case modeling the interaction of an atom with a Bose field by exchanging photons with simultaneous transitions of the atom from one level to another. In this model,  $R$  is the annihilation operator in the atom.

Let  $H = \mathbb{C}^2$  (two-level atom) and

$$E = \begin{pmatrix} \epsilon_1 & u_1 + iu_2 \\ u_1 - iu_2 & \epsilon_2 \end{pmatrix}, R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (13)$$

( $u_1 \in [-A_1, A_1]$  and  $u_2 \in [-A_2, A_2]$  specify external force (magnetic field)).

Let  $\epsilon_2 > \epsilon_1$ , and let  $v = (1, 0)$  and  $V = (0, 1)$  be, respectively, the lower (vacuum) and the excited eigenstates of the energy  $E(0, 0)$ .

Objective: to keep the atom maximally excited (as close as possible to the state  $V$ ) on a long observation interval. Then the operator  $A$  in equation (11) must be chosen as the operator  $R^*R$  of projection on  $V$ . This model can be regarded as the simplest model of laser pumping.

The density matrix of vector states of a two-dimensional atom is usually represented by the polarization vector  $P = (p_1, p_2, p_3) \in S$  according to the formulas

$$\Phi = \frac{1}{2}(I + P_\sigma) = \frac{1}{2}(I + p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3),$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices.



The filtering equation (10) in terms of  $P$  has the form

$$dP + (u_1 K_{u_1} + u_2 K_{u_2} + \Delta K_{\Delta} + \epsilon K_{\epsilon}) P dt = (v - P) dN,$$

where  $v = \{p_1 = p_2 = 0, p_3 = 1\}$ ,  $dN$  is the differential of the Poisson process with density  $\frac{\epsilon}{2}(1 - p_3)$  at the state  $P$ ,  $\Delta = \epsilon_2 - \epsilon_1$  is the difference between the energy levels of  $E(0, 0)$ , and the vector fields  $K$  on the sphere are determined by the formulas

$$K_{u_1}(P) = (0, -2p_3, 2p_2),$$

$$K_{u_2}(P) = (2p_3, 0, -2p_1),$$

$$K_{\Delta}(P) = (p_2, -p_1, 0),$$

$$K_{\epsilon}(P) = (\frac{1}{2}p_1p_3, \frac{1}{2}p_2p_3, -\frac{1}{2}(1 - p_3^2)).$$

The Bellman equation (12) acquires the form

$$\begin{aligned} & \frac{\partial S}{\partial t} + \frac{\epsilon}{2}(1 - p_3)(S(v) - S(p)) \\ & + \frac{1}{2}(1 - p_3) + \left( \frac{\partial S}{\partial p}, \Delta K_{\Delta} + \epsilon K_{\epsilon} \right) \\ & + \max_{u_1, u_2} \left( \frac{\partial S}{\partial p}, u_1 K_{u_1} + u_2 K_{u_2} \right) = 0. \end{aligned}$$

It is easy to see that we are just in the situation of Theorems 3 and 4. Hence, one can find the average income  $A_\epsilon$  per unit time in this process under a permanent optimal control in the first approximation. It is equal to  $1 + \epsilon h^0(v)$ , where  $h^0(v)$  is the income provided by the optimal transition from  $v$  to  $V$  neglecting the interaction; this income can be evaluated according to Pontryagin's maximum principle.