Nonlinear Markov processes and games via SDEs driven by nonlinear Lévy noise

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Abstract and Plan

1. We introduce a general construction of Markov processes and semigroups (both linear and nonlinear) via SDEs driven by nonlinear Levy noise.

2. The corresponding fractional dynamics is described in terms of subordination by the hitting times of Levy subordinators.

3. Including control, allows us to extend the analysis to nonlinear controlled Markov processes and games.

1. SDEs driven by nonlinear Lévy noise

Weak equations of the form

$$\frac{d}{dt}(f,\mu_t) = (L_{\mu_t} f, \mu_t), \quad \mu_t \in \mathcal{P}(\mathbf{R}^d), \quad \mu_0 = \mu, \tag{1}$$

(that should hold, say, for all $f \in C_c^2(\mathbf{R}^d)$), where

$$L_\mu f(x) = \frac{1}{2}(G(x,\mu)\nabla,\nabla)f(x) + (b(x,\mu),\nabla f(x))$$

$$+\int (f(x+y)-f(x)-(\nabla f(x),y)\mathbf{1}_{B_1}(y))\nu(x,\mu,dy), \tag{2}$$

play indispensable role in the theory of interacting particles (mean field approximation) and exhaust all positivity preserving evolutions on measures subject to certain mild regularity assumptions. I call them *general kinetic equations* as they include Vlasov, Boltzmann, Smoluchovski, Landau-Fokker-Planck equations, McKean diffusions and many other models. The strong form is of course

$$\dot{\mu} = L_\mu^\star \mu.$$

A resolving semigroup $U_t : \mu \mapsto \mu_t$ of the Cauchy problem for this equation specifies a so called *generalized or nonlinear Markov process* $X(t)$, whose distribution μ_t at time t can be determined by the formula $U_{t-s}\mu_s$ from its distribution μ_s at any previous moment s.

In the case of diffusions (when ν vanishes) the theory of the corresponding semigroups is well developed, as well as pure jump case.

We exploit the idea of nonlinear integrators combined with a certain coupling of Lévy processes in order to push forward the probabilistic construction in a way that allows the natural Lipschitz continuous dependence of the coefficients G, b, ν on x, μ. Thus obtained extension of the standard SDEs with Lévy noise represents a probabilistic counterpart of the celebrated extension of the Monge mass transformation problem to the generalized Kantorovich one.

Wasserstein-Kantorovich metrics W_p, $p \geq 1$, on the set of probability measures $\mathcal{P}(\mathbf{R}^d)$ on $\mathbf{R}^d$ are defined as

$$W_p(\nu_1, \nu_2) = \left(\inf_{\nu} \int |y_1 - y_2|^p \nu(dy_1 dy_2)\right)^{1/p}, \tag{3}$$

where inf is taken over the class of probability measures ν on $\mathbf{R}^{2d}$ that couple ν_1 and ν_2, i.e. that satisfy

$$\int\int(\phi_1(y_1)+\phi_2(y_2))\nu(dy_1 dy_2) = (\phi_1, \nu_1)+(\phi_2, \nu_2) \tag{4}$$

for all bounded measurable ϕ_1, ϕ_2.

The Wasserstein distances between the distributions in the Skorohod space $D([0,T], \mathbf{R}^d)$:

$$W_{p,T}(X_1, X_2) = \inf\left(\mathbf{E} \sup_{t \leq T} |X_1(t) - X_2(t)|^p\right)^{1/p}, \tag{5}$$

where inf is taken over all couplings of the distributions of the random paths X_1, X_2.

To compare the Lévy measures, we extend these distances to unbounded measures with a finite moment.

For simplicity, we present the arguments for L_μ having the form

$$L_\mu f(x) = \frac{1}{2}(G(x,\mu)\nabla,\nabla)f(x) + (b(x,\mu),\nabla f(x))$$

$$+\int (f(x+z)-f(x)-(\nabla f(x),z))\nu(x,\mu;dz) \tag{6}$$

with $\nu(x,\mu;.) \in \mathcal{M}_2(\mathbf{R}^d)$. Let $Y_\tau(z,\mu)$ be a family of Lévy processes depending measurably on the points z and probability measures μ in $\mathbf{R}^d$ and specified by their generators

$$L[z,\mu]f(x) = \frac{1}{2}(G(z,\mu)\nabla,\nabla)f(x) + (b(z,\mu),\nabla f(x))$$

$$+\int (f(x+y)-f(x)-(\nabla f(x),y))\nu(z,\mu;dy) \tag{7}$$

where $\nu(z,\mu) \in \mathcal{M}_2(\mathbf{R}^d)$.

Our approach to solving (1) is via the solution to the following nonlinear distribution dependent stochastic equation with nonlinear Lévy type integrators:

$$X(t) = X + \int_0^t dY_s(X(s), \mathcal{L}(X(s))), \ \mathcal{L}(X) = \mu, \tag{8}$$

with a given initial distribution μ and a random variable X independent of $Y_\tau(z, \mu)$.

We shall define the solution through the Euler type approximation scheme, i.e. by means of the approximations X_μ^τ:

$$X_\mu^\tau(t) = X_\mu^\tau(l\tau) + Y_{t-l\tau}^l(X_\mu^\tau(l\tau), \mathcal{L}(X_\mu^\tau(l\tau))), \tag{9}$$

$\mathcal{L}(X_\mu^\tau(0)) = \mu$, where $l\tau < t \leq (l+1)\tau$, $l = 0, 1, 2, ...$, and $Y_\tau^l(x, \mu)$ is a collection (depending on l) of independent families of the Lévy processes $Y_\tau(x, \mu)$ introduced above. Clearly these approximation processes are cadlag.

Theorem 1. *Let an operator* L_μ *have form* (6)*. Moreover assume that*

$$\|\sqrt{G(x,\mu)}-\sqrt{G(z,\eta)}\|+|b(x,\mu)-b(z,\eta)|$$

$$+W_2(\nu(x,\mu;.),\nu(z,\eta;.))\leq\kappa(|x-z|+W_2(\mu,\eta)), \tag{10}$$

holds true with a constant κ *and*

$$\sup_{x,\mu}\left(\sqrt{G(x,\mu)}+|b(x,\mu)|+\int|y|^2\nu(x,\mu,dy)\right)<\infty. \tag{11}$$

Then

(i) for any $\mu\in\mathcal{P}(\mathbf{R}^d)\cap\mathcal{M}_2(\mathbf{R}^d)$ *the approximations* $X_\mu^{\tau_k}$*,* $\tau_k=2^{-k}$*, converge to a process* $X_\mu(t)$ *in the sense that*

$$\sup_\mu W^2_{2,t_0}\left(X_\mu^{\tau_k},X_\mu\right)\leq c(t_0)\tau_k, \tag{12}$$

where $c(t_0)$ *depends only on the upper bounds in* (10), (11)*;*

(ii) the processes

$$M(t) = f(X_\mu(t)) - f(X_\mu(0))$$
$$- \int_0^t (L_{\mathcal{L}(X_\mu(s))} f(X_\mu(s))\, ds \qquad (13)$$

are martingales for any $f \in C^2(\mathbf{R}^d)$*, and the distributions* $\mu_t = \mathcal{L}(X_\mu(t))$ *satisfy the weak nonlinear equation* (1) *(that holds for all* $f \in C^2(\mathbf{R}^d)$*);*

(iii) the resolving operators $U_t : \mu \mapsto \mu_t$ *of the Cauchy problem* (1) *form a nonlinear Markov semigroup, i.e. they are continuous mappings from* $\mathcal{P}(\mathbf{R}^d) \cap \mathcal{M}_2(\mathbf{R}^d)$ *(equipped with the metric* W_2*) to itself such that* U_0 *is the identity mapping and* $U_{t+s} = U_t U_s$ *for all* $s, t \geq 0$*. If* $L[z, \mu]$ *do not depend explicitly on* μ *the operators* $T_t f(x) = \mathbf{E} f(X_x(t))$ *form a conservative Feller semigroup preserving the space of Lipschitz continuous functions.*

For example, assumption on ν is satisfied if one can decompose the Lévy measures $\nu(x;.)$ in the countable sums $\nu(x;.) = \sum_{n=1}^{\infty} \nu_n(x;.)$ of probability measures so that

$$W_2(\nu_i(x;.), \nu_i(z;.)) \leq a_i|x - z|$$

and the series $\sum a_i^2$ converges. It is well known that the optimal coupling of probability measures (Kantorovich problem) can not always be realized via a mass transportation (a solution to the Monge problem), thus leading to the examples when the construction of the process via standard stochastic calculus would not work.

Another important particular situation is that of a common star shape of the measures $\nu(x;.)$, i.e. if they can be represented as

$$\nu(x; dy) = \nu(x, s, dr)\, \omega(ds), \quad r = |y|, s = y/r, \tag{14}$$

with a certain measure ω on S^{d-1} and a family of measures $\nu(x, s, dr)$ on $\mathbf{R}_+$. This allows to reduce the general coupling problem to a much more easily handled one-dimensional one.

2. Fractional dynamics via subordination

Let $X(u)$, $u \geq 0$ be a Lévy subordinator, i.e. an increasing càdlàg Feller process (adapted to a filtration on a suitable probability space) with the generator

$$Af(x) = \int_0^\infty (f(x+y) - f(x))\nu(dy) + a\frac{\partial f}{\partial x}, \tag{15}$$

where $a \geq 0$ and ν is a Borel measure on $\{y > 0\}$ such that

$$\int_0^\infty \min(1, y)\nu(dy) < \infty.$$

The inverse function process or the first hitting time process $Z(t)$ defined as

$$Z(t) = \inf\{u : X(u) > t\} = \sup\{u : X(u) \leq t\}, \tag{16}$$

is also an increasing càdlàd process.

Assume that there exist $\epsilon > 0$ and $\beta \in (0, 1)$ such that $\nu(dy) \geq y^{1+\beta}\, dy$ for $0 < y < \epsilon$. Then

(i) The process $X(u)$ is a.s. increasing at each point; (ii) distribution of $X(u)$ for $u > 0$ has a density $G(u, y)$ vanishing for $y < 0$, which is infinitely differentiable in both variables and satisfies the equation

$$\frac{\partial G}{\partial u} = A^{\star} G, \tag{17}$$

where $A^{\star}$ is the dual operator to A given by

$$A^{\star} f(x) = \int_0^{\infty} (f(x-y) - f(x)) \nu(dy) - a \frac{\partial f}{\partial x},$$

(iii) if extended by zero to the half-space $\{u < 0\}$ the locally integrable function $G(u, y)$ on $\mathbf{R}^2$ specifies a generalized function satisfying the equation

$$\frac{\partial G}{\partial u} = A^{\star} G + \delta(u)\delta(y). \tag{18}$$

Consequently: (i) the process $Z(t)$ is a.s. continuous and $Z(0) = 0$; (ii) the distribution of $Z(t)$ has a continuously differentiable probability density function $Q(t, u)$ for $u > 0$ given by

$$Q(t, u) = -\frac{\partial}{\partial u} \int_{-\infty}^{t} G(u, y)\, dy. \tag{19}$$

Theorem 2. *Density Q satisfies the equation*

$$A^\star Q = \frac{\partial Q}{\partial u} \tag{20}$$

for $u > 0$, where $A^\star$ acts on the variable t, and the boundary condition

$$\lim_{u \to 0_+} Q(t,u) = -A^\star \theta(t) \tag{21}$$

where $\theta(t)$ is the indicator function equal one (respectively 0) for positive (respectively negative) t. If Q is extended by zero to the half-space $\{u < 0\}$, it satisfies the equation

$$A^\star Q = \frac{\partial Q}{\partial u} + \delta(u) A^\star \theta(t), \tag{22}$$

in the sense of distribution (generalized functions).

Moreover the (point-wise) derivative $\frac{\partial Q}{\partial t}$ also satisfies equation (20) for $u > 0$ and satisfies the equation

$$A^\star \frac{\partial Q}{\partial t} = \frac{\partial}{\partial u}\frac{\partial Q}{\partial t} + \delta(u) \frac{d}{dt} A^\star \theta(t) \tag{23}$$

in the sense of distributions.

Remark 1. *In the case of a β-stable subordinator $X(u)$ with the generator*

$$Af(x) = -\frac{1}{\Gamma(-\beta)} \int_0^\infty (f(x+y) - f(x)) y^{-1-\beta} dy, \tag{24}$$

one has

$$A = -\frac{d^\beta}{d(-x)^\beta}, \quad A^\star = -\frac{d^\beta}{dx^\beta}, \tag{25}$$

in which case equation (22) *takes the form*

$$\frac{d^\beta Q}{dt^\beta} + \frac{\partial Q}{\partial u} = \delta(u) \frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{26}$$

This remark gives rise to the idea to call the operator (15) *a generalized fractional derivative.*

Theorem 3. *Under the conditions of the previous Theorem let $Y(t)$ be a Feller process in $\mathbf{R}^d$, independent of $Z(t)$, and with the domain of the generator L containing $(C_\infty \cap C^2)(\mathbf{R}^d)$. Denote the transition probabilities of $Y(t)$ by $T(t, x, dy)$.*

Then the averages $f(t, x) = Ef(Y_x(Z(t)))$ of the (time changed or subordinated) process $Y_x(Z(t))$ for $t > 0$ and $f \in (C_\infty \cap C^2)(\mathbf{R}^d)$ satisfy the (generalized) fractional evolution equation

$$A_t^\star f(t, x) = -L_x f(t, x) + f(x) A^\star \theta(t) \qquad (27)$$

(where the subscripts indicate the variables, on which the operators act), and their time derivatives $h = \partial f / \partial t$ satisfy for $t > 0$ the equation

$$A_t^\star h = -L_x h + f(x) \frac{d}{dt} A^\star \theta(t). \qquad (28)$$

Proof (sketch). For a continuous bounded function f one has for $t > 0$ that

$$Ef(Y_x(Z(t)) = \int_0^\infty E(f(Y_x(Z(t))|Z(t) = u)Q(t,u)\,du$$

$$= \int_0^\infty \int f(y)T(u,x,dy)Q(t,u)\,du$$

by the independence of Z and Y. Hence for $t > 0$

$$A_t^\star f = \lim_{\epsilon \to 0} \int_\epsilon^\infty \int T(u,x,dy)f(y)A_t^\star Q(t,u)\,du$$

$$= \lim_{\epsilon \to 0} \int_\epsilon^\infty \int T(u,x,dy)f(y)\frac{\partial}{\partial u}Q(t,u)\,du$$

$$= -\int_0^\infty \frac{\partial}{\partial u} \int T(u,x,dy)f(y)Q(t,u)\,du + \delta(x-y)A^\star\theta(t),$$

implying (27).

Remark 2. *In the case of a β-stable Lévy subordinator $X(u)$ with the generator (24), where (25) hold, the left hand sides of the above equations become fractional derivatives per se. In particular, if $Y(t)$ is a symmetric α-stable Lévy motion, equation (27) takes the form*

$$\frac{\partial^\beta}{\partial t^\beta} g(t, y-x) = \frac{\partial^\alpha}{\partial |y|^\alpha} g(t, y-x) + \delta(y-x) \frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{29}$$

deduced earlier by Saichev-Zaslavski and Uchaikin.

One can generalize this theory to the case of Lévy type subordinators $X(u)$ specified by the generators of the form

$$Af(x) = \int_0^\infty (f(x+y) - f(x)) \nu(x, dy) + a(x) \frac{\partial f}{\partial x} \tag{30}$$

with position depending Lévy measure and drift.

Nonlinear extension.

Suppose nonlinearity sits only in the drift (potential interaction) so that

$$L_\mu f = Lf + b(x,\mu)\frac{\partial f}{\partial x}$$

with L being a Levy-type generator. Let $Y_{x,\mu}$ be the corresponding process. Then for $f(t,x) = Ef(Y_{x,\mu}(Z(t)))$ one gets the same, but with an ugly nonlinear correction:

$$A_t^\star f(t,x) = -L_x f(t,x) + f(x)A^\star \theta(t)$$

$$-\int_0^\infty \int b(z,\mu_u)\frac{\partial f}{\partial z} f(z)\mu_u(dz)Q(t,u)\,du. \quad (31)$$

However, for μ_u with a density, the last term turns to the linear integral operator on f by transferring the derivative of f to the derivation of μ via the integration by parts.

3. Controlled Markov processes and games

Suppose first that L does not depend on μ explicitly, but there is additional controllable drift $f(x,\alpha,\beta)$ and an integral payoff given by $g(x,\alpha,\beta)$. This leads to the HJB equation

$$\frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial x}) + LS = 0 \qquad (32)$$

with

$$H(x,p) = \max_\alpha \min_\beta \left(f(x,\alpha,\beta)\frac{\partial S}{\partial x} + g(x,\alpha,\beta)\right).$$

Theorem 4. *Suppose $H(x,p)$ is Lipshitz in p uniformly in x with a Lipshitz constant κ, and the process generated by L has a heat kernel (Green's function) $G(t,x,\xi)$, which is of class C^1 with respect to all variables for $t > 0$. Moreover*

$$\sup_x \int_0^t \int \left| \frac{\partial}{\partial x} G(s,x,\xi)\right| ds d\xi < \infty$$

for $t > 0$. Then for any $S_0 \in C^1(\mathbf{R}^d)$ there exists a unique classical solution for the Cauchy problem for equation (32) yielding also the solution to the corresponding optimal control problem.

Proof (sketch). It is based of course on the fixed point argument for the mapping

$$\Phi_t(S) = \int G(t,x,\xi)S_0(\xi)d\xi$$

$$+\int_0^t \int G(t-s,x,\xi)H(\xi,\frac{\partial S_s}{\partial x})dsd\xi,$$

which is applicable, because for S^1, S^2 with $S_0^1 = S_0^2$

$$\left\|\frac{\partial \Phi_t(S^1)}{\partial x} - \frac{\partial \Phi_t(S^2)}{\partial x}\right\|$$

$$\leq \kappa \int_0^t \frac{\partial G}{\partial x}(t-s,x,\xi)d\xi ds \sup_{s\leq t}\left\|\frac{\partial S^1}{\partial x} - \frac{\partial S^2}{\partial x}\right\|$$

implying the contraction property of Φ for small enough t.

Example: controlled stable-like processes with the generator $\Delta^{\alpha(x)}$ or more generally

$$\int_{S^{d-1}} |(\nabla, s)|^{\alpha(x)}\mu(ds).$$

Example of an application: extension of Nash Certainty Equivalence (NCE) principle of P. Caines et al (obtained for interacting diffusions) to stable-like processes.

4. Controlled nonlinear Markov processes and games (a starter)

Returning to L depending on μ consider a single control variable u. Assume that μ only is observable, so that the control is based on μ. This leads to the following infinite-dimensional HJB equation

$$\frac{\partial S}{\partial t} + \max_u \left(L_{\mu,u} \frac{\delta S}{\delta \mu} + g_u, \mu) \right) = 0. \qquad (33)$$

If the Cauchy problem for the corresponding kinetic equation $\dot{\mu} = L^{\star}_{\mu,u}\mu$ is well posed (see book [7]) uniformly for controls u from a compact set, with a solution denoted by $\mu^t(\mu, u)$ this can be resolved via discrete approximations

$$S_k(t-s) = B^k S(t), \quad k = (t-s)/\tau,$$

$$BS(\mu) = \max_u [S(\mu^\tau(\mu, u) + (g_u, \mu)].$$

Convergence proof (yielding a Lipshitz continuous function for a Lipshitz continuous initial one) is the same as in book [5], Section 3.2, yielding a resolving operator $R_s(S)$ for the inverse Cauchy problem (33) as a *linear operator in the* max-*plus algebra*, i.e. satisfying the condition

$$R_s(a_1 \otimes S_1 \oplus a_2 \otimes S_2) = a_1 \otimes R_s(S_1) \oplus a_2 \otimes R_s(S_2)$$

with $\oplus = \max$, $\otimes = +$. This linearity allows for effective numeric schemes.

Extensions to a competitive control case (games) is settled via the approach with generalized dynamic systems as presented in Section 11.4 of book [6].

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