Contribution to the discussion of "Sequential Quasi-Monte-Carlo Sampling" – Gerber, Chopin

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We congratulate the authors on an excellent paper. It has inspired us to consider ways to incorporate QMC within SMC schemes in settings in which the transition density of the latent process is intractable and pseudo-marginal methods are deployed. In particular, consider filtering for partially observed (jump) diffusions (e.g. Fearnhead et al. (2008); Pollock (2013); Pollock et al. (2014)), in which (in the simplest setting) the latent process is a diffusion satisfying the SDE,

$$\mathrm{d}X_t = \alpha(X_t)\,\mathrm{d}t + \mathrm{d}B_t, \quad X_0 = x, \quad t \in [0,T]. \tag{1}$$

In this setting the data comprise partial observations (arising at a finite collection of time points) of the latent process; the extension to noisy observation is trivial. The transition density of the latent process (under certain regularity conditions) can be shown (Dachuna-Castelle and Florens-Zmirou, 1986) to have the following form between any two time points a and b, where $0 \le a < b \le T$, with $\mathbb{W}^{x_a,x_b}_{a,b}$ denoting the law of a Brownian bridge between x_a and x_b over $[a,b]$:

$$\begin{aligned} p_{b-a}(x_b|x_a) =& \underbrace{\mathcal{N}(x_b; x_a, b-a)\exp\left\{\int_{x_a}^{x_b}\alpha(u)\,\mathrm{d}u\right\}}_{\propto \tilde{p}_{b-a}(x_b|x_a)} \\ & \cdot \mathbb{E}_{\mathbb{W}^{x_a,x_b}_{a,b}}\underbrace{\left[\exp\left\{-\int_a^b \frac{\alpha^2(W_t)+\alpha'(W_t)}{2}\,\mathrm{d}t\right\}\right]}_{=:\psi(W)}. \end{aligned} \tag{2}$$

To propagate particles between consecutive observation times, a and b, one could simulate from the proposal $\tilde{p}_{b-a}(x_b|x_a)$, perhaps by rejection sampling, and modify the weight of each particle by a factor corresponding to an unbiased estimate of $\psi(W)$ (where $W \sim \mathbb{W}^{x_a,x_b}_{a,b}$). Supposing that

$$\forall t \in [0,T] \qquad \left(\alpha^2(W_t) + \alpha'(W_t)\right)/2 \in [L,U],$$

and letting $\kappa \sim \mathrm{Poi}\left[(U-L)\cdot(b-a)\right]$ and $(\xi_1,\dots) \overset{\text{iid}}{\sim} \mathrm{U}[a,b]$, we have the representation:

$$\psi(W) = e^{-L\cdot(b-a)} \cdot \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{\kappa}\frac{2U - \alpha^2(W_{\xi_i}) - \alpha'(W_{\xi_i})}{2(U-L)}\,\middle|\,\kappa, W\right]\,\middle|\,W\right]. \tag{3}$$

An unbiased estimate of $\psi(W)$, using a finite dimensional realisation of the sample path, is obtained by sampling κ and $\xi_1, \ldots, \xi_\kappa \overset{\text{iid}}{\sim} \mathrm{U}[a, b]$ and employing a simple Monte Carlo approximation (Beskos et al., 2006).

This scheme uses an unbiased estimator constructed by simulating κ and then using a κ-dimensional uniform random variable to approximate the inner expectation. Finding a lower variance unbiased estimator of $\psi(W)$ is desirable, and one would like to exploit RQMC. However, the dimension of the random variable being random, it is not straightforward to do this directly. One *could* instead sample κ in the usual manner, and approximate the inner expectation conditionally using an RQMC point set. There is clearly a computational cost associated with such a RQMC method, which will only be appropriate for problems in which the variance of the simple Monte Carlo estimator of $\psi(W)$ is large and κ is typically small.

References

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