$X_i \overset{iid}{\sim} \text{Beta}(\alpha, \alpha) \qquad f(x \mid \alpha) = \dfrac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} x^{\alpha-1}(1-x)^{\alpha-1}$

$x \in (0,1), \ \alpha > -1$

A. LIKELIHOOD FUNCTION

Want $\mathcal{L}(\alpha \mid \underline{x}) = f(\underline{x} \mid \alpha)$

$$X_i \text{ IID} \longrightarrow = \prod^{m} \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} x_i^{\alpha-1}(1-x_i)^{\alpha-1}$$

$$= \left(\frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}\right)^m \left(\prod_{i=1}^{m} x_i(1-x_i)\right)^{\alpha-1}$$

B. $T(\underline{X})$ SUFFICIENT

$$T(\underline{X}) = \frac{1}{m} \sum^{m} \log\left(x_i(1-x_i)\right) \quad < 0$$

$$\Rightarrow m\,T(\underline{X}) = \sum_{i=1}^{m} \log\left(x_i(1-x_i)\right) = \log \prod_{i=1}^{m}\left(x_i(1-x_i)\right)$$

$$\Rightarrow e^{m\,T(\underline{X})} = \prod_{i=1}^{m} x_i(1-x_i)$$

Likelihood above can be rewritten as

$$\mathcal{L}(\alpha \mid \underline{x}) = \left(\frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}\right)^m \left(e^{m\,T(\underline{X})}\right)^{\alpha-1}$$

Therefore Set $g\left(T(\underline{X}) \mid \alpha\right) = \underbrace{\left(\frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}\right)^m e^{m\,T(\underline{X})}}_{\text{Depends on } \underline{X} \text{ only through } T(\underline{X})}$

$\underbrace{h(\underline{x}) = 1}_{\text{independent of } \alpha}$

$$\mathcal{L}(\alpha \mid \underline{x}) = g\left(T(\underline{X}) \mid \alpha\right) h(\underline{x})$$

By factorisation theorem $T(\underline{X})$ is sufficient

C GENERALISED LIKELIHOOD RATIO

$$\mathbb{W}(\underline{X}) = \frac{\sup_{\alpha \in \Theta_1} \mathcal{L}(\alpha \mid \underline{x})}{\sup_{\alpha \in \Theta_0} \mathcal{L}(\alpha \mid \underline{x})} \qquad \Theta_1 = \{\tfrac{1}{2}, 2\} \quad \Theta_0 = \{1\}$$

Note: the notation for this question simplifies when we consider $\mathcal{L}(\alpha \mid \underline{x}) = g(T(\underline{x}) \mid \alpha)\, h(\underline{x})$ from part b.

$$\mathcal{L}(\alpha \mid \underline{x}) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} e^{m T(\underline{x})(\alpha - 1)}$$

$$\mathcal{L}(1 \mid \underline{x}) = 1 \qquad \mathcal{L}(\tfrac{1}{2} \mid \underline{x}) = \left(\frac{\Gamma(1)}{\Gamma(\frac{1}{2})^2}\right)^m e^{-\frac{m}{2} T(\underline{x})}$$

Note $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$, $\Gamma(n) = (n-1)!$

$$= \left(\frac{1}{\pi} e^{-\frac{T(\underline{x})}{2}}\right)^m$$

$$\mathcal{L}(2 \mid \underline{x}) = \left(\frac{\Gamma(4)}{\Gamma(2)^2}\right)^m e^{m T(\underline{x})}$$

$$= \left(6 e^{T(\underline{x})}\right)^m$$

$$\Rightarrow \quad \mathbb{W}(\underline{X}) = \max\left\{ \left(\frac{e^{-\frac{T(X)}{2}}}{\pi}\right)^m, \left(6 e^{T(X)}\right)^m \right\}$$

D. $\mathbb{E}_{\alpha=1} T(\underline{X})$ & $Var_{\alpha=1} T(\underline{X})$

Note: $f(x \mid \alpha = 1) = 1$

$$\Rightarrow \mathbb{E}_{\alpha=1}\left[T(\underline{X})\right] = \mathbb{E}\left[\frac{1}{m}\sum \log(X_i(1-X_i))\right]$$

$$\text{IID} \quad = \mathbb{E}\left[\log(X(1-X))\right]$$

$$= \mathbb{E}\left[\log(X) + \log(1-X)\right]$$

$$= \int_0^1 \left(\log(x) + \log(1-x)\right) \cdot 1 \, dx$$

$$= \text{Wolfram} = -2$$

$$Var_{\alpha=1}(T(\underline{X})) = \underbrace{\mathbb{E}\left(T(\underline{X})^2\right)}_{\mathbb{E}\left[\frac{1}{m}\sum^m \log^2(X_i(1-X_i))\right]^2} - \underbrace{\mathbb{E}\left(T(\underline{X})\right)^2}_{4}$$

$$= \frac{1}{m^2} \sum_i \sum_j \underbrace{\mathbb{E}\left[\log(X_i(1-X_i)) \log(X_j(1-X_j))\right]}$$

when $i = j \Rightarrow \mathbb{E} \log^2 X(1-X)$

$i \neq j \Rightarrow 4$

$$= \frac{1}{m^2}\left[m \underbrace{\mathbb{E}\left[\log(X(1-X))\right]}_{= \text{Wolfram} = 8 - \frac{\pi^2}{3}} + m(m-1)4\right]$$

$$Var_{\alpha=1}(T(\underline{X})) = \frac{1}{m}\left(8 - \frac{\pi^2}{3} + (m-1)4\right) - 4$$

$$= \frac{1}{m}\left(8 - \frac{\pi^2}{3} + (m-1)4 - m4\right)$$

$$= \frac{1}{m}\left(4 - \frac{\pi^2}{3}\right)$$

E P-value corresponding to W

Note: Because H_1 is not simply $\alpha \neq 1$ p-values based on $T(\underline{X})$ alone does not test our hypotheses. Therefore $W(\underline{X})$ must be used to derive the p-value

$$p_W(\underline{x}) = \mathbb{P}_{\alpha=1}\left(W(\underline{X}) \geq W(\underline{x})\right)$$

where $W(\underline{x}) = y$ is calculated from some realisation $\underline{x}$

$$\Rightarrow \mathbb{P}_{\alpha=1}\left(W(\underline{X}) \geq y\right)$$

Consider $W(\underline{X}) \geq y$

$y > 0$

Interested in $y > 1$

$$\Rightarrow \left(\frac{e^{-\frac{T(\underline{x})}{2}}}{\pi}\right)^m \geq y \quad \text{or} \quad \left(6\, e^{T(\underline{x})}\right)^m \geq y$$

$$m\left(-\frac{T(\underline{x})}{2} - \log \pi\right) \geq \log y \quad \text{or} \quad m\left(\log 6 + T(\underline{x})\right) \geq \log y$$

$$-\frac{T(\underline{X})}{2} - \log \pi \geq \frac{\log(y)}{m} \quad \text{or} \quad \log 6 + T(\underline{x}) \geq \frac{\log y}{m}$$

Set $w = \frac{\log(y)}{m}$

$$\Rightarrow \quad -T(\underline{x}) \geq 2(w + \log \pi)$$

$$T(\underline{x}) \leq -2(w + \log \pi) \quad \text{or} \quad T(\underline{x}) \geq w - \log 6$$

$$\Rightarrow \mathbb{P}\left(T(\underline{x}) \leq -2(w + \log \pi) \text{ or } T(\underline{x}) \geq w - \log 6\right)$$

Note $\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B)$

Note $-2(w+\log\pi) \leq w-\log 6$ when $y>1$

$$\therefore \quad \mathbb{P}\left(w-\log 6 \leq T(\underline{x}) \leq -2(w+\log\pi)\right) = 0$$

$$\Rightarrow pw(W(\underline{x})) = \mathbb{P}\left(T(\underline{x}) \leq -2(w+\log\pi)\right) + \mathbb{P}\left(T(\underline{x}) \geq w-\log 6\right)$$

Note
$$g(Y) = \frac{Y - \mathbb{E}_{\alpha=1}T(\underline{x})}{\sqrt{\mathrm{Var}_{\alpha=1}T(\underline{x})}}$$

$$g(T(\underline{x})) = Z \ \dot{\sim}\ \mathcal{N}(0,1)$$

$$\Rightarrow pw(W(\underline{x})) = \mathbb{P}\left(Z \leq g(-2(w+\log\pi))\right) + \mathbb{P}\left(Z > g(w-\log 6)\right)$$

$$F(z) = \mathbb{P}(Z \leq z)$$

$$pw(W(\underline{x})) = F\left(g(-2(w+\log\pi))\right) + 1 - F\left(g(w-\log 6)\right)$$

F. CONSTRUCT HYPOTHESIS TEST of POWER $\alpha^* = 0.05$

State decision Rule

$$\phi(\underline{x}) = \begin{cases} 1, & pw(\underline{x}) < \alpha^* \\ 0, & \text{otherwise} \end{cases}$$

with $pw(\underline{x})$ defined as above

Decision Rule describes Hypothesis test

with power α^* under $H_0: \alpha^* = 1$

Q PROB OF ACCEPTING H_0 WHEN NOT TRUE

$$*\mathbb{P}_{\alpha=\frac{1}{2}}\left(p_W(W(\underline{x})) > \alpha^*\right) \text{ and } \mathbb{P}_{\alpha=2}\left(p_W(W(\underline{x})) > \alpha^*\right)$$

Only going to answer for *

$$\mathbb{P}_{\alpha=\frac{1}{2}}\left(p_W(W(\underline{x})) \geq \alpha^*\right)$$

$$p_W(W(\underline{x})) \geq \alpha^*$$

$$\Rightarrow W(\underline{x}) \geq p_W^{-1}(\alpha^*)$$

$$\Rightarrow T(\underline{x}) \leq \overbrace{-2\left(\frac{\log(p_W^{-1}(\alpha^*))}{m} + \log\pi\right)}^{\lambda(\underline{x})}$$

$$\text{or } T(\underline{x}) \geq \underbrace{\frac{\log(p_W^{-1}(\alpha^*))}{m} - \log 6}_{\delta(\underline{x})}$$

Define $h(Y) = \dfrac{Y - \mathbb{E}_{\alpha=\frac{1}{2}} T(\underline{X})}{\sqrt{\mathrm{Var}_{\alpha=\frac{1}{2}} T(\underline{X})}} = \dfrac{Y + 2.77}{\sqrt{\frac{3.3042}{m}}}$

Note: $f(x \mid \alpha = \frac{1}{2}) = \begin{cases} \dfrac{(x(1-x))^{-\frac{1}{2}}}{\pi}, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases}$

Find $\mathbb{E}_{\alpha=\frac{1}{2}}(\cdot)$ & $\mathrm{Var}_{\alpha=\frac{1}{2}}(\cdot)$ as before

$$\Rightarrow \mathbb{P}\left(p_W(W(\underline{x})) \geq \alpha^*\right)$$

$$= \mathbb{P}\left(Z \leq h(\lambda(\underline{x}))\right) + \mathbb{P}\left(Z \geq h(\delta(\underline{x}))\right)$$