Macroeconomic Determinants of Stock Market Volatility and Volatility Risk-Premiums

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Abstract

How does stock market volatility relate to the business cycle? We develop, and estimate, a no-arbitrage model to study the cyclical properties of stock volatility and the risk-premiums the market requires to bear the risk of fluctuations in this volatility. The level and fluctuations of stock market volatility is largely explained by business cycle factors, although some unobserved factor contributes to nearly 20% to the overall variation in volatility. At the same time, this unobservable factor cannot explain the ups and downs volatility experiences over time—the "volatility of volatility." Instead, the volatility of volatility relates to the business cycle. Finally, volatility risk-premiums are strongly countercyclical, even more so than stock volatility, and are partially responsible for the large swings the VIX index experienced during the 2007-2009 subprime crisis, which our model does capture in out-of-sample experiments.

Keywords: Aggregate stock market volatility; volatility risk-premiums; volatility of volatility; business cycle; no-arbitrage restrictions; simulation-based inference

JEL classification: E37, E44, G13, G17, C15, C32

1.   Introduction

Understanding the origins of stock market volatility has long been a topic of considerable interest to both policy makers and market practitioners. Policy makers are interested in the main determinants of volatility and in its spillover effects on real activity. Market practitioners are interested in the effects volatility exerts on the pricing and hedging of plain vanilla options and more exotic derivatives. In both cases, forecasting stock market volatility constitutes a formidable challenge but also a fundamental instrument to manage the risks faced by these institutions.

Many available models use latent factors to explain the dynamics of stock market volatility. For example, in the celebrated Heston's (1993) model, stock volatility is exogenously driven by some unobservable factor correlated with the asset returns. Yet such an unobservable factor does not bear an economic interpretation. Moreover, the model implies, by assumption, that volatility cannot be forecasted by macroeconomic factors such as industrial production or inflation. This circumstance is counterfactual. Indeed, there is strong evidence that stock market volatility has a very pronounced business cycle pattern, being higher during recessions than during expansions; see, e.g., Schwert (1989a,b), Hamilton and Lin (1996), or Brandt and Kang (2004).

In this paper, we develop a no-arbitrage model where stock market volatility is explicitly related to a number of macroeconomic and unobservable factors. The distinctive feature of this model is that stock volatility is linked to these factors by no-arbitrage restrictions. The model is also analytically convenient: under fairly standard conditions on the dynamics of the factors and risk-aversion corrections, our model is solved in closed-form, and is amenable to empirical work.

We use the model to quantitatively assess how market volatility and volatility-related risk-premiums change in response to business cycle conditions. Our model fully captures the procyclical nature of aggregate returns and the countercyclical behavior of stock volatility that we have been seeing in the data for a long time. It makes a fundamental prediction: macroeconomic factors can explain nearly 75% of the variation in the overall stock volatility. At the same time, our model, rigorously estimated through simulation-based inference methods, shows that the presence of some unobservable and persistent factor is needed to sustain the level of stock volatility that matches its empirical counterpart. Moreover, our model reveals that macroeconomic factors substantially help

explain the variability of stock volatility around its level—the volatility of volatility. That such a "vol-vol" might be related to the business cycle is indeed a plausible hypothesis, although clearly, the ups and downs stock volatility experiences over the business cycle are a prediction of the model in line with the data, not a restriction imposed while estimating the model. Such a new property we uncover, and model, brings practical implications. For example, business cycle forecasters might learn that not only does stock market volatility have predicting power, as discussed below; "vol-vol" is also a potential predictor of the business cycle.

Our second set of results relates to volatility-related risk-premiums. The volatility risk-premium is the difference between the expectation of future market volatility under the risk-neutral and the true probability. It quantifies how a representative agent is willing to pay to be ensured against the event that volatility will raise beyond his own expectations. Thus, it is a very intuitive and general measure of risk-aversion. We find that this volatility risk-premium is strongly countercyclical, even more so than stock volatility. Precisely, volatility risk-premiums are typically not very volatile, although in bad times, they may increase to extremely high levels, and quite quickly. We undertake a stress test of the model over a particularly uncertain period, which includes the 2007-2009 subprime turmoil. Ours is a stress test, as (i) we estimate the model using post-war data up to 2006, and (ii) feed the previously estimated model with macroeconomic data related to the subprime crisis. We compare the model's predictions for the crisis with the actual behavior of both stock volatility and the new VIX index, maintained by the Chicago Board Options Exchange (CBOE), which is, theoretically, the risk-adjusted expectation of future volatility within one month. The model tracks the dramatic movements in this index, and predicts that countercyclical volatility risk-premiums are largely responsible for the large swings in the VIX occurred during the crisis. In fact, we show that over this crisis, as well as in previous recessions, movements in the VIX index are determined by changes in such countercyclical risk-premiums, not by changes in the expected volatility.

Related literature

Stock volatility and volatility risk-premiums   The cyclical properties of aggregate stock market volatility have been the focus of recent empirical research, although early work relating stock volatility to macroeconomic variables dates back to King, Sentana and Wadhwani (1994), who rely

on a no-arbitrage model. In a comprehensive international study, Engle and Rangel (2008) find that high frequency aggregate stock volatility has both a short-run and long-run component, and suggest that the long-run component is related to the business cycle. Adrian and Rosenberg (2008) show that the short- and long- run components of aggregate volatility are both priced, cross-sectionally. They also relate the long-run component of aggregate volatility to the business cycle. Finally, Campbell, Lettau, Malkiel and Xu (2001), Bloom (2009), Bloom, Floetotto and Jaimovich (2009) and Fornari and Mele (2010) show that capital market uncertainty helps explain future fluctuations in real economic activity. Our focus on volatility risk-premiums relates, instead, to the seminal work of Dumas (1995), Bakshi and Madan (2000), Britten-Jones and Neuberger (2000), and Carr and Madan (2001), which has more recently stimulated an increasing interest in these premiums dynamics and determinants (see, for example, Bakshi and Madan (2006) and Carr and Wu (2009)). Notably, in seminal work, Bollerslev and Zhou (2006) and Bollerslev, Gibson and Zhou (2011) unveil a strong relation between volatility risk-premiums and a number of macroeconomic factors.

Our contribution hinges upon, and expands, over this growing literature, in that we formulate and estimate a fully-specified no-arbitrage model relating the dynamics of stock volatility and volatility risk-premiums to business cycle, and additional unobservable, factors. With the exception of King, Sentana and Wadhwani (1994) and Adrian and Rosenberg (2008), who still have a focus different from ours, the predicting relations in the previous papers, while certainly useful, are still part of reduced-form statistical models. In our out-of-sample experiments of the subprime crisis, we shall show that our no-arbitrage framework is considerably richer than that based on predictive linear regressions. We show, for example, that compared to our model's predictions about stock volatility and the VIX index, predictions from linear regressions are substantially flat over the subprime crisis.

The only antecedent to our paper is Bollerslev, Tauchen and Zhou (2009), who develop a consumption-based rationale for volatility risk-premiums, although then, the authors use this rationale only as a guidance to the estimation of reduced-form predictability regressions conditioned on the volatility risk-premium. In recent independent work discussed below, Drechsler and Yaron (2011) investigate the properties of the volatility risk-premium, implied by a calibrated consumption-based model with long-run risks. The authors, however, are not concerned with the cross-equation

restrictions relating the volatility risk-premium to state variables driving low frequency stock market fluctuations which, instead, constitute the central topic of our paper.

No-arbitrage regressions In recent years, there has been a significant surge of interest in consumption-based explanations of aggregate stock market volatility (see, for example, Campbell and Cochrane (1999), Bansal and Yaron (2004), Tauchen (2005), Mele (2007), or the two surveys in Campbell (2003) and Mehra and Prescott (2003)). These explanations are important because they highlight the main economic mechanisms through which markets and preferences affect equilibrium asset prices and, hence, stock volatility. In our framework, cross-equations restrictions arise through the weaker requirement of absence of arbitrage opportunities. In this respect, our approach is similar in spirit to the "no-arbitrage" vector autoregressions introduced in the term-structure literature by Ang and Piazzesi (2003) and Ang, Piazzesi and Wei (2006). Similarly as in those papers, we specify an analytically convenient pricing kernel affected by some macroeconomic factors, although we do not directly relate these to, say, markets, preferences or technology.

Our model works quite simply. We exogenously specify the joint dynamics of a number of macroeconomic and unobservable factors. We assume that the asset payoffs and the risk-premiums required by agents to be compensated for the fluctuations of the factors, are essentially affine functions of these factors, along the lines of Duffee (2002). We show that the resulting no-arbitrage stock price is affine in the factors. Our model does not allow for jumps or other market micro-structure effects, as our main focus is to model low frequency movements in the aggregate stock volatility and volatility risk-premiums, through the use of macroeconomic and unobservable factors. Our estimation results, obtained through data sampled at monthly frequency, are unlikely to be affected by measurement noise or jumps, say. In related work, Carr and Wu (2009), Todorov (2010), Drechsler and Yaron (2011), and Todorov and Tauchen (2011) do allow for the presence of jumps, although they do not analyze the relations between macroeconomic variables and aggregate volatility or volatility risk-premiums, which we do here.

Estimation strategy, and plan of the paper

In standard stochastic volatility models such as that in Heston (1993), volatility is driven by factors, which are not necessarily the same as those affecting the stock price—volatility is exogenous in these models. In our no-arbitrage model, volatility is endogenous, relating to a number of risks affecting (i) macroeconomic developments, (ii) unobserved factors and (iii) the very same asset returns—these risks affect both asset returns and volatility. To identify the premium required to bear the risk of volatility, we exploit derivative data, related to the new VIX index.

We implement a three-step estimation procedure that relies on simulation-based inference methods. In the first step, we estimate the parameters underlying the macroeconomic factors. In the second step, we use data on a broad stock market index, and the macroeconomic factors, and estimate reduced-form parameters linking the stock market index to the macroeconomic factors and the third unobservable factor, as well as the parameters underlying the dynamics of the unobservable factor. In the third step, we use data on the new VIX index, and the macroeconomic factors, and estimate the risk-premiums parameters. We implement these steps by matching model-based moments and impulse response functions to their empirical counterparts, relating to macroeconomic factors, realized returns, realized volatility and the VIX index. We develop, and utilize, a theory to consistently estimate the standard errors through block-bootstrap methods.

The remainder of the paper is organized as follows. In Section 2 we develop a no-arbitrage model for the stock price, stock volatility and volatility-related risk-premiums. Section 3 illustrates the estimation strategy. Section 4 presents our empirical results. Section 5 concludes, and the Supplemental material contains an appendix with technical details omitted from the main text.

2. The model

We develop a model where aggregate stock returns and volatility are tied up to macroeconomic developments and one unobservable factor. It is a three-factor model solved in closed form, a special case of a general multifactor model in Appendix A of the Supplemental material.

2.1. The macroeconomic environment

We consider a model with one unobservable factor, and two additional factors affecting the development of two aggregate macroeconomic variables, inflation and industrial production growth, and the stock market. Let $\boldsymbol{y}(t) = (y_1(t)\ y_2(t)\ y_3(t))$ be a vector-valued process, where $y_1(t)$ and $y_2(t)$ denote two observable factors, defined as $\ln(\mathrm{CPI}_t/\mathrm{CPI}_{t-12}) = \ln y_1(t)$ and $\ln(\mathrm{IP}_t/\mathrm{IP}_{t-12}) = \ln y_2(t)$, where CPI_t and IP_t are the consumer price index and industrial production as of month t, as further explained in Section 4.1. In Section 4.1, we also discuss the role these two macroeconomic factors have played in asset pricing. We also assume that a third, and unobservable, factor, $y_3(t)$, affects the stock price, but not the two macroeconomic aggregates, CPI_t and IP_t. Finally, we assume the two macroeconomic factors do not affect the unobservable factor y_3, although we allow for simultaneous feedback effects between inflation and industrial production growth, as explained below. The factors y_j are solution to,

$$\mathrm{d}y_j(t) = \left[\kappa_j \left(\mu_j - y_j(t) \right) + \bar{\kappa}_j \left(\bar{\mu}_j - \bar{y}_j(t) \right) \right] \mathrm{d}t + \sqrt{\alpha_j + \beta_j y_j(t)} \mathrm{d}W_j(t), \quad j = 1, 2, 3, \tag{1}$$

where $W_j(t)$ are standard Brownian motions, $\bar{\mu}_1 \equiv \mu_2$, $\bar{y}_1(t) \equiv y_2(t)$, $\bar{\mu}_2 \equiv \mu_1$, $\bar{y}_2(t) = y_1(t)$, $\bar{\kappa}_3 \equiv \bar{\mu}_3 \equiv \bar{y}_3(t) \equiv 0$ and, finally, Greek letters denote constant parameters. The two parameters, κ_1 and κ_2, are the speed of adjustment of inflation and industrial production growth towards their long run means, μ_1 and μ_2, and $\bar{\kappa}_1$ and $\bar{\kappa}_2$ are the feedback parameters. Appendix A of the Supplemental material reviews conditions guaranteeing Eq. (1) is well-defined, which we use as constraints whilst estimating the model.

We assume that asset prices, (i) respond to movements in the factors affecting macroeconomic conditions, and (ii) reflect a long-run trend in the asset payoffs. Precisely, we model the instantaneous dividends paid off by the asset at time t, $\mathrm{Div}(t)$ say, as the product of a stochastic trend, times a stationary component, as follows:

$$\mathrm{Div}(t) = G(t)\, \delta(\boldsymbol{y}(t)), \tag{2}$$

where $\delta(\boldsymbol{y})$ satisfies, for four constants δ_0 and $(\delta_j)_{j=1}^3$,

$$\delta(\boldsymbol{y}) = \delta_0 + \delta_1 y_1 + \delta_2 y_2 + \delta_3 y_3, \tag{3}$$

and $G(t)$ is a geometric Brownian motion with drift g and volatility σ_G,

$$\frac{dG(t)}{G(t)} = gdt + \sigma_G dW_G(t), \quad G(0) \equiv 1, \tag{4}$$

and $W_G(t)$ is a Brownian motion uncorrelated with the Brownian motions in Eq. (1). The rationale behind the assumption in Eq. (2) is to disentangle secular, yet stochastic, dividend growth, captured by $G(t)$, from short-run fluctuations of the dividend process, arising from business cycles, and captured by $\delta(\boldsymbol{y}(t))$. This assumption implies the asset price displays a similar property, being driven by a secular, growth component, and an additional, short-run component related to macroeconomic developments, as we now explain.

2.2. No-arbitrage

We model the pricing kernel, or the Arrow-Debreu price density, in the economy. Let $\mathbb{F}(T)$ be the sigma-algebra generated by the Brownian motion $[\boldsymbol{W}(t)^\top \ W_G(t)]^\top$, $t \leq T$, where $\boldsymbol{W}(t) = (y_1(t) \ y_2(t) \ y_3(t))$, and let P the associated physical probability. The Radon-Nikodym derivative of the risk-neutral probability Q with respect to P on $\mathbb{F}(T)$ is,

$$\xi(T) \equiv \frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left(-\int_0^T \boldsymbol{\Lambda}(t)^\top \mathrm{d}\boldsymbol{W}(t) - \frac{1}{2}\int_0^T \|\boldsymbol{\Lambda}(t)\|^2 \mathrm{d}t\right) \cdot \exp\left(-\lambda_G W_G(T) - \frac{1}{2}\lambda_G^2 T\right), \tag{5}$$

for some risk-premium process $\boldsymbol{\Lambda}(t)$ and constant λ_G. The interpretation of $\boldsymbol{\Lambda}(t)$ is that of a risk-premium required to compensate for the fluctuations of the factors $\boldsymbol{y}(t)$. The constant λ_G is, instead, the unit-risk premium for the stochastic fluctuations of secular growth, $G(t)$. While we model $\boldsymbol{\Lambda}(t)$ to be time-varying, we assume λ_G to be constant for analytical convenience.

We assume the risk-premium process satisfies an "essentially affine" specification, viz

$$\boldsymbol{\Lambda}(\boldsymbol{y}(t)) \equiv \boldsymbol{\Lambda}(t) = \boldsymbol{V}(\boldsymbol{y}(t))\boldsymbol{\lambda}_1 + \boldsymbol{V}^-(\boldsymbol{y}(t))\boldsymbol{\lambda}_2\boldsymbol{y}(t), \tag{6}$$

where $\boldsymbol{\lambda}_1 = (\lambda_{1(1)} \ \lambda_{1(3)} \ \lambda_{1(3)})$ is a parameter vector, $\boldsymbol{\lambda}_2$ is a diagonal matrix of parameters with diagonal elements equal to $\lambda_{2(j)}$, $j = 1, 2, 3$, $\boldsymbol{V}(\boldsymbol{y})$ is a diagonal matrix with $\sqrt{\alpha_j + \beta_j y_j}$ on its diagonal, and $\boldsymbol{V}^-(\boldsymbol{y}) : \boldsymbol{V}^-(\boldsymbol{y})\boldsymbol{V}(\boldsymbol{y}) = \boldsymbol{I}_{3\times3}$, for all $\boldsymbol{y}$, which it does under regularity conditions spelled out in Appendix A of the Supplemental material.

The functional form for $\boldsymbol{\Lambda}$ echoes that suggested by Duffee (2002) in the term-structure literature. If $\boldsymbol{\lambda}_2 = \boldsymbol{0}_{3\times3}$, $\boldsymbol{\Lambda}$ collapses to the "completely affine" specification introduced by Duffie and Kan

(1996), where the risk-premiums in $\boldsymbol{\Lambda}$ are tied up to the volatility of the fundamentals, $\boldsymbol{V}(\boldsymbol{y})$. While it is reasonable to assume that risk-premiums link to the *volatility* of fundamentals, the specification in Eq. (6) also allows risk-premiums to relate to the *level* of the fundamentals, through the additional term $\boldsymbol{\lambda}_2\boldsymbol{y}$. Including this term is, indeed, critical to our empirical results. Consider the *total* risk-premiums process, defined as,

$$
\boldsymbol{\pi}(\boldsymbol{y}) = \begin{pmatrix} \pi_1(y_1) \\ \pi_2(y_2) \\ \pi_3(y_3) \end{pmatrix} \equiv \boldsymbol{V}(\boldsymbol{y})\,\boldsymbol{\Lambda}(\boldsymbol{y}) = \begin{pmatrix} \alpha_1\lambda_{1(1)} + \left(\beta_1\lambda_{1(1)} + \lambda_{2(1)}\right)y_1 \\ \alpha_2\lambda_{1(2)} + \left(\beta_2\lambda_{1(2)} + \lambda_{2(2)}\right)y_2 \\ \alpha_3\lambda_{1(3)} + \left(\beta_3\lambda_{1(3)} + \lambda_{2(3)}\right)y_3 \end{pmatrix}. \tag{7}
$$

Each component of $\boldsymbol{\pi}(\boldsymbol{y})$, $\pi_j(y_j)$, depends on factor y_j due to the volatility of this factor (i.e. through β_j) and, also, due to the additional parameter $\lambda_{2(j)}$. Without $\lambda_{2(j)}$, we could not model the level of the risk-premiums separately from their sensitivities to changes in y_j—a sensible issue we have experienced whilst estimating our model. Consider, for example, the total risk premium for growth, $\pi_2(y_2)$. The coefficient $\lambda_{1(2)}$ affects both the intercept and the slope of π_2. The inclusion of $\lambda_{2(2)}$ allows to achieve flexibility in modeling the level of $\pi_2(y_2)$ and its sensitivity with respect to changes in y_2.

Finally, we assume that the safe asset is elastically supplied such that the short-term rate r (say) is constant. Whilst real rates are not as volatile as stock returns in the data, many existing models might likely predict rates to be too volatile. For example, models with habit formation predict the short-term rate is a function of the state, primarily due to intertemporal substitution effects. Campbell and Cochrane (1999) mitigate this issue with a well-known trick—they impose that intertemporal substitution effects are exactly offset by precautionary savings, thereby making the short-term rate constant. Additional models that cope with this challenge include those relying on non-expected utility, as in Bansal and Yaron (2004), or those with heterogeneous agents, as in Guvenen (2009), to cite a few. In this paper, we impose r to be constant for the purpose of keeping stock volatility tractable, as this facilitates the actual estimation of the model. How important is this assumption, quantitatively? Mele (2007) finds that in realistically calibrated models of habit formation, large countercyclical swings of stock volatility mainly arise due to risk-premiums effects, rather than interest rate volatility. It is an open question, however, whether such a result would

still hold in the economy we consider in the current paper.[1]

We are ready to determine the no-arbitrage stock price. As it turns out, the previous assumption on the pricing kernel and the assumption that $\delta(\cdot)$ in Eq. (3) is affine in $\boldsymbol{y}$ implies that the stock price is also affine in $\boldsymbol{y}$. Precisely, we have:

$$S(G, \boldsymbol{y}) = G \cdot \left(s_0 + \sum_{j=1}^{3} s_j y_j\right), \tag{8}$$

where

$$s_0 = \frac{1}{r - g + \sigma_G \lambda_G}\left[\delta_0 + \sum_{j=1}^{3} s_j\left(\kappa_j \mu_j + \bar{\kappa}_j \bar{\mu}_j - \alpha_j \lambda_{1(j)}\right)\right], \tag{9}$$

$$s_j = \frac{\delta_j\left(r - g + \sigma_G \lambda_G + \kappa_i + \lambda_{1(i)}\beta_i + \lambda_{2(i)}\right) - \delta_i \bar{\kappa}_i}{\prod_{h=1}^{2}\left(r - g + \sigma_G \lambda_G + \kappa_h + \lambda_{1(h)}\beta_h + \lambda_{2(h)}\right) - \bar{\kappa}_1 \bar{\kappa}_2}, \quad \text{for } j, i \in \{1, 2\} \text{ and } i \neq j, \tag{10}$$

$$s_3 = \frac{\delta_3}{r - g + \sigma_G \lambda_G + \kappa_3 + \lambda_{1(3)}\beta_3 + \lambda_{2(3)}}. \tag{11}$$

In the standard stochastic volatility literature, the asset price and, hence, its volatility, is taken as given, and volatility and volatility risk-premiums are modeled separately, as for example in the celebrated Heston's (1993) model, which many empirical studies take as a benchmark (e.g., Chernov and Ghysels (2000), Corradi and Distaso (2006), Garcia, Lewis, Pastorello and Renault (2011)). Moreover, a recent focus in this literature is to relate volatility risk-premiums the to business cycle (e.g., Bollerslev, Gibson and Zhou (2011)). Yet, while the empirical results in these papers are ground breaking, the Heston's model is not meant to capture, theoretically, the interplay between stochastic volatility, volatility risk-premiums and the business cycle.

Our model works differently, as it places restrictions on the asset price process directly, through our assumptions on the fundamentals of the economy, and absence of arbitrage. For our model, it is the asset price that determines, endogenously, volatility, which by Eq. (1) and Eq. (8) is:

$$\sigma(\boldsymbol{y}(t)) \equiv \sigma(t) = \sqrt{\sigma_G^2 + \frac{\sum_{j=1}^{3} s_j^2\left(\alpha_j + \beta_j y_j(t)\right)}{\left(s_0 + \sum_{j=1}^{3} s_j y_j(t)\right)^2}}. \tag{12}$$

Note that the model predicts that stock volatility embeds information about risk-corrections that agents require to invest in the stock market. We shall make use of this observation in the empirical

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[1]Our model has, however, implications for the nominal rate, which is $r - \ln \mathbb{E}\left(\left.\frac{\text{CPI}_t}{\text{CPI}_{t+12}}\right| \mathbb{F}_t\right)$ (for one year, say), where $\mathbb{E}$ is the expectation under Q. Evaluating this expression in steady state, through the estimates we obtain in Section 4, and assuming $r = 1\%$, yields 4.7%. In the data, the nominal rate for one year is, instead, 5.4%.

part of the paper. We now describe which measure of stock volatility we use to proceed with such a critical step of our analysis.

2.3. Arrow-Debreu adjusted volatility

In September 2003, the CBOE changed its volatility index VIX, to reflect recent advances in the option pricing literature. Given an asset price process $S(t)$ that is continuous in time (as that predicted by our model, in Eq. (8)), and all available information $\mathbb{F}(t)$ at time t, consider the economic value of the future integrated variance on a given interval $[t, t_0]$, IV_{t,t_0}, say, which is the sum of the future variances, weighted with the Arrow-Debreu state prices:

$$\mathbb{E}\left[IV_{t,t_0}\middle|\,\mathbb{F}(t)\right] \equiv \int_t^{t_0} \mathbb{E}\left[\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{var}\left[\ln S(\tau)\middle|\,\mathbb{F}(u)\right]\Bigg|_{\tau=u}\right)\middle|\,\mathbb{F}(t)\right]\mathrm{d}u, \tag{13}$$

where $\mathbb{E}$ is the expectation under Q. The new VIX index relies on the work of Dumas (1995), Bakshi and Madan (2000), Britten-Jones and Neuberger (2000), and Carr and Madan (2001), who showed that the risk-neutral expectation of the future integrated variance is a functional of put and call options written on the asset:

$$\mathbb{E}\left[IV_{t,t_0}\middle|\,\mathbb{F}(t)\right] = 2e^{r(t_0-t)}\left[\int_0^{F(t)} \frac{P_t(t_0,K)}{K^2}\mathrm{d}K + \int_{F(t)}^\infty \frac{C_t(t_0,K)}{K^2}\mathrm{d}K\right] \equiv (t_0-t)\cdot \mathrm{VIX}_t^2, \tag{14}$$

where $F(t) = e^{r(t_0-t)}S(t)$ is the forward price, $C_t(t_0,K)$ and $P_t(t_0,K)$ are the prices as of time t of call and put options expiring at t_0 and struck at K, and VIX_t is the new VIX index. In contrast, our model, which relies on the Arrow-Debreu state prices in Eq. (5), predicts that the risk-neutral expectation of the integrated variance is:

$$\mathbb{E}\left[IV_{t,t_0}\middle|\,\boldsymbol{y}(t)=\boldsymbol{y}\right] = \int_t^{t_0} \mathbb{E}\left[\sigma^2(\boldsymbol{y}(u))\middle|\,\boldsymbol{y}(t)=\boldsymbol{y}\right]\mathrm{d}u \equiv (t_0-t)\cdot \mathrm{VIX}^2(\boldsymbol{y}), \tag{15}$$

where $\sigma^2(\boldsymbol{y}(t))$ is given in Eq. (12). We shall estimate the risk-premium parameters in Eq. (7) so as to match the VIX index predicted by the model, $\mathrm{VIX}(\boldsymbol{y}(t))$ in Eq. (15), to its empirical counterpart, VIX_t in Eq. (14). Finally, our model makes predictions about how the volatility risk-premium, $\mathrm{VRP}(\boldsymbol{y}(t))$ say, changes with the factors $\boldsymbol{y}(t)$ in Eq. (1)

$$\mathrm{VRP}(\boldsymbol{y}(t)) \equiv \sqrt{\frac{1}{t_0-t}}\left(\sqrt{\mathbb{E}\left[IV_{t,t_0}\middle|\,\boldsymbol{y}(t)=\boldsymbol{y}\right]} - \sqrt{E\left[IV_{t,t_0}\middle|\,\boldsymbol{y}(t)=\boldsymbol{y}\right]}\right), \tag{16}$$

where E denotes the expectation taken under P.

3.   Statistical inference

We rely on a three-step procedure. In the first step, we estimate the parameters of the process underlying the dynamics of the two macroeconomic factors, $\boldsymbol{\phi}^\top = \left(\kappa_j, \mu_j, \alpha_j, \beta_j, \bar{\kappa}_j, j = 1, 2\right)$. In the second step, we estimate the parameters in Eq. (4), $\boldsymbol{\theta}_G^\top = (g, \sigma_G)$, the reduced-form parameters that link the asset price to the three factors in Eq. (8), and the parameters of the process for the unobserved factor, $\boldsymbol{\theta}^\top = \left(\kappa_3, \mu_3, \alpha_3, \beta_3, (s_j)_{j=0}^3\right)$, while imposing the identifiability condition that $\mu_3 = 1$, as explained below. In the third step, we estimate the risk-premiums parameters $\boldsymbol{\lambda}^\top = \left(\lambda_{1(1)}, \lambda_{2(1)}, \lambda_{1(2)}, \lambda_{2(2)}, \lambda_{1(3)}, \lambda_{2(3)}\right)$, relying on a simulation-based approximation of the model-implied VIX, which we match to the VIX index. At each of these steps, we do not have a closed form expression for the likelihood function, or for selected sets of moment conditions. For this reason, we need to rely on a simulation-based approach. Our estimation strategy relies on an hybrid of Indirect Inference (Gouriéroux, Monfort and Renault (1993)) and the Simulated Generalized Method of Moments (Duffie and Singleton (1993)).[2]

3.1.   Moment conditions for the macroeconomic factors

To simulate the factor dynamics in Eq. (1), we rely on a Milstein approximation scheme, with discrete interval Δ, say. We simulate H paths of length T of the two observable factors, and sample them at the same frequency as the available data, obtaining $y_{1,t,\Delta,h}^\phi$ and $y_{2,t,\Delta,h}^\phi$, where $y_{j,t,\Delta,h}^\phi$ is the value at time t taken by the j-th factor, at the h-th simulation performed with $\boldsymbol{\phi}$—the parameter vector relating to the process underlying the macroeconomic factors. Then, we estimate the following auxiliary models on both historical and simulated data,[3]

$$\boldsymbol{y}_t = \boldsymbol{w} + \boldsymbol{A}\boldsymbol{y}_{t-1} + \boldsymbol{\epsilon}_t^y, \tag{17}$$

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[2] The estimators we develop are not as efficient as Maximum Likelihood. Under some conditions, the methods put forward by Gallant and Tauchen (1996), Fermanian and Salanié (2004), Carrasco, Chernov, Florens and Ghysels (2007), Aït-Sahalia (2008), or Altissimo and Mele (2009), are asymptotic equivalent to Maximum Likelihood. In our context, they deliver asymptotically efficient estimators for the parameters in the first step. However, hinging upon these approaches in the remaining steps would make the two issues of unobservability of volatility and, especially, parameter estimation error considerably beyond the scope of this paper.

[3] The choice of lags for all the auxiliary models in the present section relies on the BIC criterion, and our additional concern to have non-overlapping regressors—with the exception of Eqs. (17), and a lag 6 in Eq. (23), which revealed to be empirically important. Appendix C.2 of the Supplemental material reports parameter estimates and R^2 for all these auxiliary models—both those referring to data and those implied by the model.

and

$$\boldsymbol{y}_{t,\Delta,h}^{\phi} = \boldsymbol{w}_h + \boldsymbol{A}_h \boldsymbol{y}_{t-1,\Delta,h}^{\phi} + \boldsymbol{\epsilon}_{t,h}^y, \tag{18}$$

where $\boldsymbol{y}_t = (y_{1,t} \; y_{2,t})^\top$, $\boldsymbol{w}$ and $\boldsymbol{A}$ denote a vector and a matrix of constants, $\boldsymbol{\epsilon}_t^y$ is a vector of normally distributed errors with zero mean and (diagonal) variance-covariance matrix $\boldsymbol{C}$, and the notation for Eq. (18) for simulated data follows the same rationale as that in Eq. (17).

Next, let $\tilde{\boldsymbol{\varphi}}_T = (\tilde{\boldsymbol{\varphi}}_{1,T}, \tilde{\boldsymbol{\varphi}}_{2,T}, \bar{y}_1, \bar{y}_2, \hat{\sigma}_1, \hat{\sigma}_2)^\top$ where $\tilde{\boldsymbol{\varphi}}_{1,T}$ and $\tilde{\boldsymbol{\varphi}}_{2,T}$ denote the ordinary least squares (OLS, henceforth) estimators of the parameters in Eq. (17), and $\bar{y}_i$ and $\hat{\sigma}_i$ are the sample mean and standard deviation of the macroeconomic factors. Let $\hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\phi)$ be the simulated counterpart to $\tilde{\boldsymbol{\varphi}}_T$ at simulation h, including the OLS estimator of the parameters in Eq. (18), and the sample means and standard deviations of the macroeconomic factors. The estimator of ϕ is:

$$\hat{\boldsymbol{\phi}}_T \equiv \arg\min_{\phi \in \boldsymbol{\Phi}_0} \left\| \frac{1}{H} \sum_{h=1}^{H} \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\phi) - \tilde{\boldsymbol{\varphi}}_T \right\|^2, \tag{19}$$

where $\boldsymbol{\Phi}_0$ is some compact set. Appendix B in the Supplemental material develops the asymptotic theory relating to this estimator.

3.2. Moment conditions for realized returns and volatility

Data on macroeconomic factors and stock returns do not allow us to identify the structural parameters of the model. In particular, there are many combinations of $\boldsymbol{\delta} = (\delta_j)_{j=0}^3$ and $\boldsymbol{\lambda} = (\lambda_{1(j)}, \lambda_{2(j)})_{j=1}^3$ in Eqs. (9)-(11), giving rise to the same asset price. In this second step, we estimate the parameters $\boldsymbol{\theta}_G = (g, \sigma_g)$ in Eq. (4), the reduced-form parameters, $(s_j)_{j=0}^3$ in Eqs. (9)-(11), and the parameters for the unobservable factor, $(\kappa_3, \mu_3, \alpha_3, \beta_3)$. The parameters $\boldsymbol{\lambda}$ shall be estimated in a third and final step, described in the next section. Note that, theoretically, it might be possible to collapse the second and third steps of our estimation procedure into a single one, where a combined use of data on dividends and volatility derivatives might help identify $\boldsymbol{\delta}$ and $\boldsymbol{\lambda}$. We do not pursue this approach because it revealed to be computationally prohibitive. Note, then, that our three-step methodology leads to identify the model's parameters through data relating to macroeconomic factors, and market data relating to stock returns and risk-neutral volatility. However, Appendix C.1 of the Supplemental material describes a calibration procedure relying on both the aggregate

asset price and dividends, which leads us to find our three-step estimation methodology has quite

reasonable implications for the dividends dynamics.

Even proceeding in this way, we cannot tell apart the loading on the unobservable factor, s_3, from the parameters underlying the dynamics of this factor, $(\kappa_3, \mu_3, \alpha_3, \beta_3)$, as this is independent of the observable ones. We impose the normalization $\mu_3 \equiv 1$. We estimate $\boldsymbol{\theta}_G$ using the time-series of the low-frequency component of the real stock price growth, extracted through the Hodrick-Prescott filter with smoothing parameter equal to 14400, given we are using monthly data (Hodrick and Prescott (1997)). We simulate H paths of length T of the unobservable factor $y_3(t)$, and the secular growth, $G(t)$, using a Milstein approximation with discrete interval Δ, and sample them at the same frequency as the data, obtaining for $\boldsymbol{\theta}_u = (\kappa_3, \alpha_3, \beta_3, s_3)$ and $\hat{\boldsymbol{\theta}}_{G,T} = (\hat{g}_T, \hat{\sigma}_{G,T})$, and simulation h, the series $y_{3,t,\Delta,h}^{\theta_u}$ and $G_{t,\Delta,h}^{\hat{\theta}_{G,T}}$. Likewise, let $S_{t,\Delta,h}^{\theta}(\hat{\boldsymbol{\theta}}_{G,T})$ be the simulated series of the stock price, when the parameters are fixed at $\boldsymbol{\theta} = (\boldsymbol{\theta}_u, (s_j)_{j=0}^3)$ and $\hat{\boldsymbol{\theta}}_{G,T}$:

$$\ln S_{t,\Delta,h}^{\theta}(\hat{\boldsymbol{\theta}}_{G,T}) = \ln G_{t,\Delta,h}^{\hat{\theta}_{G,T}} + \ln\left(s_0 + s_1 y_{1,t} + s_2 y_{2,t} + s_3 y_{3,t,\Delta,h}^{\theta_u}\right), \tag{20}$$

where $G_{0,\Delta,h}^{\hat{\theta}_{G,T}} \equiv 1$, as in Eq. (4). We fix the intercept, s_0, so as to make the model-implied average of the detrended stock price match its empirical counterpart: $s_0 = \bar{S}^d - s_1 \bar{y}_1 - s_2 \bar{y}_2 - s_3$, where $\bar{S}^d$ denotes the sample mean of the detrended stock price $S_t^d \equiv e^{-\hat{g}_T t} S_t$, S_t is the real stock price index observed at time t, and finally, $\bar{y}_1$ and $\bar{y}_2$ are the sample means of the two macroeconomic factors $y_{1,t}$ and $y_{2,t}$. Note, we simulate the stock price using the *observed* samples of $y_{1,t}$ and $y_{2,t}$, a feature of the estimation strategy that results in improved efficiency, as discussed below.

Following Mele (2007) and Fornari and Mele (2010), we measure the volatility of the monthly continuously compounded price changes, as:

$$\text{Vol}_t = \sqrt{6\pi} \cdot \frac{1}{12} \sum_{i=1}^{12} \left| \ln\left(\frac{S_{t+1-i}}{S_{t-i}}\right) \right|. \tag{21}$$

Next, define yearly returns as $R_t = \ln(S_t/S_{t-12})$, and let $R_{t,\Delta,h}^{\theta}(\hat{\boldsymbol{\theta}}_{G,T})$ and $\text{Vol}_{t,\Delta,h}^{\theta}(\hat{\boldsymbol{\theta}}_{G,T})$ be the simulated counterparts to R_t and Vol_t.[4]

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[4]Our asymptotics are not affected by the property that the returns $R_{t,\Delta,h}^{\theta}(\cdot)$ are driven by both a $I(0)$ and a $I(-1)$ components, as our laws of large numbers or central limit theorems still apply—similarly as in the realized volatility literature, for instance, where $I(-1)$ terms arise due to microstructure effects (see, e.g., Barndorff-Nielsen, Hansen, Lunde and Shephard (2008), or Aït-Sahalia, Mykland and Zhang (2011)). Campbell and Cochrane (1999) model of habit formation or Bansal and Yaron (2004) long-run risks model are instances of models predicting a similar property for asset returns.

Our estimator relies on two auxiliary models that capture the main statistical facts about stock returns and return volatility in our dataset. The auxiliary model for returns is:

$$R_t = a^{\mathrm{R}} + b^{\mathrm{R}}_{1,12} y_{1,t-12} + b^{\mathrm{R}}_{2,12} y_{2,t-12} + \epsilon^{\mathrm{R}}_t, \tag{22}$$

and that for return volatility is:

$$\mathrm{Vol}_t = a^{\mathrm{V}} + \sum_{i \in \{6,12,18,24,36,48\}} b^{\mathrm{V}}_i \mathrm{Vol}_{t-i} + \sum_{i \in \{12,24,36,48\}} b^{\mathrm{V}}_{1,i} y_{1,t-i} + \sum_{i \in \{12,24,36,48\}} b^{\mathrm{V}}_{2,i} y_{2,t-i} + \epsilon^{\mathrm{V}}_t. \tag{23}$$

Let $\tilde{\boldsymbol{\vartheta}}_T = \left(\tilde{\boldsymbol{\vartheta}}_{1,T}, \tilde{\boldsymbol{\vartheta}}_{2,T}, \bar{S}, \overline{\mathrm{Vol}}, \hat{\sigma}_{\mathrm{Vol}} \right)^{\top}$, where $\tilde{\boldsymbol{\vartheta}}_{1,T}$ is the OLS estimate of the parameters in Eq. (22), $\tilde{\boldsymbol{\vartheta}}_{2,T}$ is the OLS estimate of the parameters in Eq. (23), $\bar{S}$ is the sample mean of the real stock price, and, finally, $\overline{\mathrm{Vol}}$ and $\hat{\sigma}_{\mathrm{Vol}}$ are the sample mean and standard deviation of stock return volatility. Let $\hat{\boldsymbol{\vartheta}}_{T,\Delta,h}(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}_{G,T})$ be the simulated counterpart to $\tilde{\boldsymbol{\vartheta}}_T$ at simulation h, using $R^{\theta}_{t,\Delta,h}(\hat{\boldsymbol{\theta}}_{G,T})$ and $\mathrm{Vol}^{\theta}_{t,\Delta,h}(\hat{\boldsymbol{\theta}}_{G,T})$. The estimator of $\boldsymbol{\theta} = (\boldsymbol{\theta}_u, (s_j)^3_{j=0})$ is:

$$\hat{\boldsymbol{\theta}}_T = \arg\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} \left\| \frac{1}{H} \sum_{h=1}^{H} \hat{\boldsymbol{\vartheta}}_{T,\Delta,h}(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}_{G,T}) - \tilde{\boldsymbol{\vartheta}}_T \right\|^2, \tag{24}$$

where $\boldsymbol{\Theta}_0$ is a compact set. As shown in detail in Appendix B of the Supplemental material, the structure of the asymptotic covariance matrix of this estimator differs from that of $\hat{\boldsymbol{\phi}}_T$ in Eq. (19), due to two reasons. First, stock price paths are simulated through Eq. (20), with secular growth parameters fixed at their estimates, $\hat{\boldsymbol{\theta}}_{G,T}$, leading to parameter estimation error, which is asymptotically accounted for. Second, ours is, in fact, a conditional simulated inference estimator, in that the simulations in Eq. (20) occur conditionally upon the sample realizations of the observable factors, $y_{1,t}$ and $y_{2,t}$. This feature of the method results in a correlation among the auxiliary parameter estimates obtained over all the simulations, and leads to an efficiency improvement over unconditional (simulated) inference.

3.3. *Estimation of the risk-premium parameters*

We estimate the risk-premium parameters, $\boldsymbol{\lambda}$, by matching moments and impulse response functions of the model-based VIX, $\mathrm{VIX}(\boldsymbol{y}(t))$ in Eq. (15), to those of the model-free VIX index, VIX_t in Eq. (14), with $t_0 - t$ equal to one month. Since the new VIX index is available only since 1990, we use a sample of $\mathcal{T}$ observations in this step, with $\mathcal{T} < T$. Whilst $\mathrm{VIX}(\boldsymbol{y}(t))$ is not known

in closed-form, it can be accurately approximated through simulations, as explained in Appendix B of the Supplemental material. Note, also, that in the actual computation of Eq. (15), we replace the unknown parameters, s_0, $\left(s_j, \kappa_j, \alpha_j, \beta_j\right)_{j=1}^3$, $(\bar{\kappa}_i, \mu_i)_{i=1}^2$ and σ_G, with their estimated counterparts computed in the previous two steps: $\hat{\boldsymbol{\theta}}_T$, $\hat{\boldsymbol{\phi}}_T$ and $\hat{\sigma}_{G,T}$. As in the previous step, we use the observed samples of the macroeconomic factors $y_{1,t}, y_{2,t}$, and simulate samples for the latent factor only. We rely on the following auxiliary model:

$$\text{VIX}_t = a^{\text{VIX}} + b^{\text{VIX}}\text{VIX}_{t-1} + \sum_{i\in\{36,48\}} b_{1,i}^{\text{VIX}}y_{1,t-i} + \sum_{i\in\{36,48\}} b_{2,i}^{\text{VIX}}y_{2,t-i} + \epsilon_t^{\text{VIX}}. \tag{25}$$

Define, $\tilde{\boldsymbol{\psi}}_{\mathcal{T}} = \left(\tilde{\boldsymbol{\psi}}_{1,\mathcal{T}}, \overline{\text{VIX}}, \hat{\sigma}_{\text{VIX}}\right)^\top$, where $\tilde{\boldsymbol{\psi}}_{1,\mathcal{T}}$ is the OLS estimator of the parameters in Eq. (25), and $\overline{\text{VIX}}$ and $\hat{\sigma}_{\text{VIX}}$ are the sample mean and standard deviation of the VIX index. Likewise, define $\hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}(\hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\phi}}_T, \hat{\sigma}_{G,T}, \boldsymbol{\lambda})$, the simulated counterpart to $\tilde{\boldsymbol{\psi}}_{\mathcal{T}}$ at simulation h, obtained through simulations of the model-implied index, $\text{VIX}_{t,\Delta,h}(\hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\phi}}_T, \hat{\sigma}_{G,T}, \boldsymbol{\lambda})$ say, where the paths of the two macroeconomic factors, $y_{1,t}$ and $y_{2,t}$, are fixed at their sample values. The estimator of $\boldsymbol{\lambda}$ is:

$$\hat{\boldsymbol{\lambda}}_{\mathcal{T}} = \arg\min_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}_0} \left\| \frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}(\hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\phi}}_T, \hat{\sigma}_{G,T}, \boldsymbol{\lambda}) - \tilde{\boldsymbol{\psi}}_{\mathcal{T}} \right\|^2, \tag{26}$$

for some compact set $\boldsymbol{\Lambda}_0$. This estimator is, similarly as $\hat{\boldsymbol{\theta}}_T$ in Eq. (24), affected by parameter estimation error, arising because $\text{VIX}_{t,\Delta,h}(\hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\phi}}_T, \hat{\sigma}_{G,T}, \boldsymbol{\lambda})$, the model-implied VIX index, is simulated using parameters estimated in the previous two steps, $\hat{\boldsymbol{\phi}}_T$, $\hat{\boldsymbol{\theta}}_T$ and $\hat{\sigma}_{G,T}$. At the same time, the estimator $\hat{\boldsymbol{\lambda}}_{\mathcal{T}}$ in Eq. (24) is a conditionally simulated one, in that it relies on the observations of the macroeconomic factors $y_{1,t}$ and $y_{2,t}$, thereby resulting in efficiency gains.

3.4.   Bootstrap Standard Errors

The limiting variance-covariance matrices for $\hat{\boldsymbol{\phi}}_T$ in Eq. (19), $\hat{\boldsymbol{\theta}}_T$ in Eq. (24), and $\hat{\boldsymbol{\lambda}}_{\mathcal{T}}$ in Eq. (26) are characterized in Appendix B.1 of the Supplemental material. They are not known in closed form, and must be estimated through the computation of several numerical derivatives. Moreover, our sample sizes are relatively small, compared to those we usually have access to in empirical finance, and in particular such is that available for the estimation of the risk premium parameters. We rely on bootstrap standard errors consistent for those implied by the asymptotic variance-covariance matrices for $\hat{\boldsymbol{\phi}}_T$, $\hat{\boldsymbol{\theta}}_T$ and $\hat{\boldsymbol{\lambda}}_{\mathcal{T}}$. Bootstrap standard errors are not only easier

to compute, but also less prone to numerical errors, and likely to be more accurate than those based on asymptotic approximations, in finite samples. Finally, the auxiliary models we utilize are potentially misspecified, and they likely lead to a score that is not a martingale difference sequence. We appeal to the "block-bootstrap" to address this technical issue. Appendix B.2 of the Supplemental material develops results and algorithms that allow us to make use of this method within the simulation-based estimation procedure of this section.

4.   Empirical analysis

4.1.   Data

Our security data include the S&P 500 Compounded index, and the VIX index maintained by the CBOE. The VIX index is available daily, but only after January 1990. Our macroeconomic variables include the consumer price index (CPI), and the seasonally adjusted industrial production (IP) index for the US. Information related to the CPI and the IP indexes is made available to the market between the 19-th and the 23-th of every month. To possibly avoid overreaction to releases of information, we sample the S&P Compounded index and the VIX index every 25-th of the month. We compute the real stock price as the ratio between the S&P index and the CPI. Our dataset, then, includes (i) monthly observations of the VIX index, from January 1990 to December 2006, for a total of 204 observations; and (ii) monthly observations of the real stock price, the CPI and the IP indexes, from January 1950 to December 2006, for a total of 672 observations.

Our dataset also includes monthly observations of the University of Michigan Consumer Sentiment index, from January 1978 to December 2006 (for a total of 336 observations), Finally, we utilize additional data, from January 2007 to March 2009, to implement a stress test of how the previously estimated model would have performed over a particularly critical period. This out-of-sample period is critical for at least three reasons: first, the NBER determined that the US economy entered in a recession in December 2007, which is the third NBER-dated recession since the creation of the new VIX index; second, this period includes the quite unique events leading to the subprime crisis; third, both realized stock market volatility and the VIX index reached record highs, and possibly pose challenges to rational models of asset prices. Note that our out-of-sample

experiments are not intended to forecast the market, stock market volatility, and the level of the VIX index. Rather, we feed the model estimated up to December 2006, with macroeconomic data (the CPI and IP indexes) available from January 2007, and compare the predictions of the model with the actual movements of the market, stock market volatility and the VIX index.

Many theoretical explanations and, in fact, the empirical evidence, would lead us to expect that asset prices are, indeed, related to variables tracking business cycles (see, e.g., Cochrane (2005)), such as the CPI and the IP growth. For example, in their seminal article relating stock returns to the macroeconomy, Chen, Roll and Ross (1986) find that industrial production growth and inflation are among the most prominent priced factors. Theoretically, in standard theories of external habit formation, the pricing kernel volatility is driven by the surplus consumption ratio, defined as the percentage deviation of current consumption, C, from some habit level, H, i.e. $(C - H)/C$, which highly correlates with procyclical variables such as industrial production growth. Likewise, standard asset pricing models predict that compensation for inflation risk relates to variables that are highly correlated with inflation (e.g., Bakshi and Chen (1996), Buraschi and Jiltsov (2005)). Mainly for computational reasons, we refrain from considering additional factors to model the linkages of the pricing kernel to the business cycle.

Figure 1 depicts the two series $y_{1,t}$ (year-to-year gross inflation) and $y_{2,t}$ (year-to-year industrial production growth) along with NBER-dated recession events. Gross inflation is procyclical, although it peaked up during the 1975 and the 1980 recessions, as a result of the geopolitical driven oil crises that occurred in 1973 and 1979. Its volatility during the 1970s was large until the Monetary experiment of the early 1980s, although it dramatically dropped during the period following the experiment, usually referred to as the Great Moderation (e.g., Bernanke (2004)). At the same time, inflation is persistent: a Dickey-Fuller test rejects the null hypothesis of a unit root in $y_{1,t}$, although the rejection is at the marginal 95% level. The inclusion of inflation as a determinant of the pricing kernel displays one attractive feature. An old debate exists upon whether stocks provide a hedge against inflation (see, e.g., Danthine and Donaldson (1986)). While our no-arbitrage model is silent about the general equilibrium forces underlying inflation-hedge properties of asset prices, its data-driven structure allows us to assess quite directly the relations between inflation and the stock price, returns, volatility and volatility risk-premiums.

Figure 1 also shows that while the volatility of industrial production growth dropped during the Great Moderation, growth is still persistent, although less so than gross inflation: here, a Dickey-Fuller test rejects the null hypothesis of a unit root in $y_{2,t}$ at any conventional level. Finally, the properties of inflation and industrial production growth over our out-of-sample period, from January 2007 to March 2009, are discussed in Section 4.2.4.

4.2. Estimation results

4.2.1. Macroeconomic drivers

Table 1 reports parameter estimates and block-bootstrap standard errors for the joint process of the two macroeconomic variables, $y_{1,t}$ and $y_{2,t}$, as set forth in Section 3.1. The estimates are all largely significant, and confirm our discussion of Figure 1: inflation is more persistent than IP growth, as both its speed of adjustment in the absence of feedbacks, κ_1, and its feedback parameter, $\bar{\kappa}_1$, are much lower than the counterparts for IP growth, κ_2 and $\bar{\kappa}_2$. Finally, the estimates of β_1 and β_2 are both negative, implying that the volatility of these two macroeconomic variables are countercyclical, an interesting property, from an asset pricing perspective. However, we note that the estimate of β_1, albeit statistically significant, is also economically very small.[5]

4.2.2. Aggregate stock returns and volatility

Table 2 reports estimates and block-bootstrap standard errors for (i) the parameters affecting secular growth, (ii) the parameters linking the two macroeconomic factors and the unobservable factor to the asset price, and (iii) the parameters for the unobservable factor process, as explained in Section 3.1. The estimates are all largely significant and point to two conclusions. First, the stock price is positively related to IP growth and negatively related to inflation. Second, the unobservable factor is quite persistent, displaying high volatility, as the estimate of the speed of mean reversion, κ_3, is low. Note, the literature on long run risks started by Bansal and Yaron (2004) emphasizes

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[5]It is known since at least Friedman (1977) that high variability of inflation might link to high inflation. For example, Engle (1982) finds that inflation volatility increases during the middle 1970s. We have constructed measures of inflation volatility similar to that in Eq. (21), relating to the first difference of inflation, which confirm these findings. We also find that after the 1970s, inflation slowdowns tend to occur more rapidly than inflation increases although overall, a clear relation between inflation and inflation volatility is hard to establish. The estimate of β_1 for our continuous time model is likely to reflect these facts.

the asset pricing implications of long-run risks affecting the expected consumption growth rate. Interestingly, the presence of a persistent factor affecting stock returns and volatility emerges quite neatly from our estimation. Note, however, that in long-run risk models, expected consumption growth is unlikely to affect the dynamics of stock volatility which, instead, are inherited by those of the volatility of consumption growth. In our model, our unobservable factor does, instead, affect stock volatility, and substantially, as explained below.

Figure 2 shows the dynamics of stock returns and volatility predicted by the model, along with their sample counterparts, calculated as described in Section 3.2. These predictions are obtained by feeding the model with sample data for the two macroeconomic factors, $y_{1,t}$ and $y_{2,t}$, in conjunction with simulations of the third unobservable factor, using all the estimated parameters. For each point in time, we average over the cross-section of 1000 simulations, and report returns (in the top panel of Figure 2) and volatility (in the bottom panel). Returns are computed as we do with the data, and volatility is obtained through Eq. (12).

The model appears to capture the procyclical nature of stock returns and the countercyclical behavior of stock volatility. It generates *all* the market drops as well as *all* the volatility upward swings occurred during the NBER recessions, including the dramatic spike of the 1975 recession. In the data, average stock volatility is about 11.50%, with a standard deviation of about 4.0%. The model predicts an average volatility of about 13%, with a standard deviation of about 3.1%.

How much of the variation in volatility can be attributable to macroeconomic factors? It is a natural question, as the key innovation of our model is the introduction of these factors for the purpose of explaining volatility, on top of a standard unobservable factor. We address this issue and calculate: (i) the ratio of the instantaneous return variance due to factor y_j, $s_j^2 \left(\alpha_j + \beta_j y_j \left(t \right) \right)$, to the total instantaneous variance, $\sigma^2 \left(t \right)$ in Eq. (12), as well as (ii) the ratio of the instantaneous variance of secular growth, σ_G^2, to $\sigma^2 \left(t \right)$, as follows,

$$C_j \left(t \right) s^2 \left(\boldsymbol{y} \left(t \right) \right) \equiv \frac{s_j^2 \left(\alpha_j + \beta_j y_j \left(t \right) \right)}{\sigma^2 \left(t \right)}, \ j = 1, 2, 3, \quad \text{and} \quad C_G \left(t \right) \equiv \frac{\sigma_G^2}{\sigma^2 \left(t \right)}, \tag{27}$$

where $s \left(\boldsymbol{y} \right) \equiv s_0 + \sum_{j=1}^{3} s_j y_j$. Figure 3 depicts the time series of $C_j \left(t \right)$ and $C_G \left(t \right)$ implied by our estimated model, obtained, as usual, by feeding the model with the observed samples of $y_1 \left(t \right)$ and $y_2 \left(t \right)$, and averaging across 1000 simulations of $y_3 \left(t \right)$. The clear finding is that industrial production

growth makes the most important contribution to stock volatility: the time series average of $C_2(t)$ is above 73%, more than four times higher than $C_3(t)$, the contribution made by the unobserved factor. Panel A of Table 3 reports averages and standard deviations of the contributions made by all factors, and secular growth. Variations in industrial production growth and the unobserved factor are responsible, alone, for more than 90% of the variation in stock volatility. It is a striking result, as one challenge we face is to explain why we have observed a sustained stock market volatility, in spite of the Great Moderation. Our estimated model entails two clear conclusions.

First, as Figure 3 makes clear, the 73% average contribution of industrial production growth to stock volatility seems to be rather stable over time, at least once we exclude the 1950s—a period of sustained volatility for growth (see Figure 1). Accordingly, the Great Moderation does merely appear to have affected the variability of the linkages between industrial production growth and aggregate stock volatility, not the very same linkages. To illustrate, Panel A of Table 3 shows averages and standard deviations of the factors' contributions across different sampling periods. We take 1982 to be the year that marks the beginning of the Great Moderation, characterized by the inauguration of the Federal Reserve monetary policy turning point and a lower volatility of real macroeconomic variables (e.g., Blanchard and Simon (2001)). As is clear, whilst the average contributions are stable, the variability of these contributions has decreased over the Great Moderation. For example, the average of $C_2(t)$ is between 73% - 75%, across all sampling periods, whereas its standard deviation decreases to 3.47% during the 1982-2006 sample, from 9.65% (1950-1981) and 5.03% (1960-1981).

Second, the contribution of industrial production growth to volatility, albeit crucial, is not exhaustive. Our model predicts that stock volatility cannot be explained by macroeconomic variables only, as the unobserved factor accounts for about 17% of the fluctuations in $\sigma^2(t)$. Equally important is the observation that the contribution of industrial production to stock volatility is strongly countercyclical, exhibiting large upward swings starting at, and sometimes, anticipating, turning points, as in the case of the 1970s recessions and the most recent, 2001 recession. Instead, the contribution of the unobserved factor to stock volatility, $C_3(t)$, is procyclical, for the simple reason that the instantaneous volatility of $y_3(t)$ does not obviously link to the business cycle, thereby making the ratio $C_3(t)$ in Eq. (27) procyclical, due to the countercyclical nature of its denominator,

¹ $\sigma^2(t)$. All in all, our empirical results suggest that while unobserved factors are needed to explain

² the level of stock volatility, industrial production is needed to explain the countercyclical swings of

³ stock volatility that we have in the data—the volatility of volatility.

⁴   Finally, the contribution of secular growth to stock volatility is limited, being approximately 8%,

⁵ and that of gross inflation plays an even more marginal role, being less than 1%. Note, however,

⁶ that our model predicts that inflation links to asset returns and volatility in a manner comparable

⁷ to that in the data. For example, it is well-known since at least Fama (1981) that real stock returns

⁸ are negatively correlated with inflation, a property that hinders the ability of stocks to hedge against

⁹ inflation. In our sample, this correlation is -35%, while the correlation our model generates is -24%.

¹⁰ Finally, the correlation between stock volatility and inflation is about 20% in the data, while that

¹¹ implied by the model is about 25%.

¹²   The predictions of the model discussed so far rely on cross-sectional averages of simulations of

¹³ the unobserved factor, y_3. Yet what is the interpretation of this unobserved factor? Let us invert

¹⁴ the price function in Eq. (8), for y_3, and for each month, as follows:

¹⁵
$$-\hat{y}_{3,t} \equiv -\frac{1}{\hat{s}_3}\left(\frac{S_t}{\hat{G}_t} - \hat{s}_0 - \hat{s}_1 y_{1,t} - \hat{s}_2 y_{2,t}\right),\tag{28}$$

¹⁶ where S_t is the real stock price at time t, $(\hat{s}_j)_{j=0}^3$ are estimates of the pricing function coefficients,

¹⁷ as reported in Table 2, and $\hat{G}_t$ is the cross-sectional average of 1000 simulations of secular growth.

¹⁸   Figure 4 (top panel) depicts $-\hat{y}_{3,t}$ (in bold), along with 100 simulated trajectories of the un-

¹⁹ observed factor performed with the parameter estimates in Table 2. Reinsuringly, the range of

²⁰ variation of the model-implied factor roughly falls within that of the simulated trajectories of this

²¹ factor. Note that the estimate of s_3 is negative, such that $-\hat{y}_{3,t}$ positively affects the real stock

²² price—higher realizations of $-\hat{y}_{3,t}$ amount to good pieces of news to the stock market. There are

²³ episodes where $-\hat{y}_{3,t}$ comes close to the edges of the realized range of variation experienced by the

²⁴ unobserved factors during the simulations. These episodes are interesting, as they correspond to: (i)

²⁵ the lows of the late 1970s and the early 1980s, and (ii) the highs of the dotcom bubble that occurs in

²⁶ the late 1990s. The extracted factor oscillates between (about) its minimum and its maximum over

²⁷ those approximate twenty years. The rise and fall over this period have a clear economic interpreta-

²⁸ tion, with the late 1970s and early 1980s being particularly bad times, marked by the occurrence of a

1 double dip recession, and the extraordinary market boom over the dotcom bubble being notoriously

2 suspected to be one of exuberance (e.g., Shiller (2005)). These observations motivate us to explore

3 the extent to which our extracted factor links to indexes of "sentiment," following a recent strand

4 of the literature that attempts to link asset price movements to factors such as investors uncer-

5 tainty (as in David and Veronesi (2006)), confidence risk (as in Bansal and Shaliastovich (2010)), or

6 Knightian uncertainty (as in Drechsler (2010) or Mele and Sangiorgi (2011)). The bottom panel of

7 Figure 4 compares the time series behavior of the model-implied unobserved factor, $-\hat{y}_{3,t}$, with that

8 of an index of consumer confidence—the University of Michigan Consumer Sentiment (UMCSENT)

9 index, available from January 1978.

10   Note how the UMCSENT index tracks the lows and the highs of the market that have so slowly

11 occurred over the last thirty years: the bad times of the late 1970s and early 1980s, the rise occurring

12 over the late 1980s and culminating with the dotcom bubble of the late 1990s and, finally, the drop

13 of the late 2000s, corresponding to the subprime crisis—a period we study in detail in the next

14 section. Interestingly, our extracted factor, $-\hat{y}_{3,t}$, co-moves positively with the UMCSENT index,

15 correlating with it at about 50%. In contrast, its correlation with the macroeconomic factors is

16 modest (10% with inflation and 30% with industrial production growth). Interestingly, then, the

17 pattern our extracted factor exhibits is one that mostly tracks long-run movements of the market,

18 even more so than the short-term movements relating to business cycles. In Appendix C.3 of

19 the Supplemental material, we produce variance decompositions statistics, obtained by feeding the

20 model with both $\hat{y}_{3,t}$ in Eq. (28) and the UMCSENT index, instead of relying on simulations of y_3,

21 and document results similar to those summarized by Figure 3 and Table 3.

22 *4.2.3. Volatility risk-premiums and the dynamics of the VIX index*

23   Table 4 reports parameter estimates and block-bootstrap standard errors for the vector of the

24 risk-premiums coefficients $\boldsymbol{\lambda}$ in Eq. (7), as set forth in Section 3.3. The estimates, all significant,

25 imply that the risk-premiums processes are all positive, and quite large, especially those relating

26 to the two macroeconomic factors. Moreover, the risk compensation for inflation increases with

27 inflation and that for industrial production is countercyclical, given the sign of the estimated values

28 for the loadings of inflation, $\left(\beta_1\lambda_{1(1)} + \lambda_{2(1)}\right)$ (positive), and industrial production, $\left(\beta_2\lambda_{1(2)} + \lambda_{2(2)}\right)$

(negative), in the risk-premium process of Eq. (7). While gross inflation does receive compensation, the countercyclical behavior of the risk-premium for industrial production growth is even more critical, as we explain below. Our estimated model predicts that in bad times, the risk-premium for industrial production growth goes up, and future expected economic conditions even worsen, under the risk-neutral probability, which boosts future expected volatility, under the same risk-neutral probability. In part because of these effects, the VIX index predicted by the model is countercyclical. This reasoning is quantitatively sound. Figure 5 (top panel) depicts the VIX index, along with the VIX index predicted by the model and the (square root of the) model-implied expected integrated variance. The model appears to track the swings the VIX index has undergone over the 1991 and the 2001 recession episodes.

The top panel of Figure 5 also shows the dynamics of volatility expected under the physical probability. This expected volatility is certainly countercyclical, although it does not display the large variations the model predicts for its risk-neutral counterpart, the VIX index. The VIX index predicted by the model is countercyclical because, as explained, the risk-premiums required to bear the fluctuations of the macroeconomic factors are (i) positive and (ii) countercyclical, and, also, because (iii) current volatility is countercyclical. Under the physical probability, expected volatility is countercyclical only because of the third effect. However, quantitatively, movements of volatility risk-premiums account for variations in the VIX index sensibly more than those of the volatility expected under the physical probability, as clearly summarized by Figure 5.

Which factors mostly contribute to the dynamics of the VIX? Panel B of Table 3 reports averages and standard deviations of the contributions of each factor, as predicted by our estimated model. We calculate each of these contributions by evaluating C_j and C_G in Eq. (27) under the risk-neutral probability and, then, aggregating the average paths of C_j and C_G for every month, and, finally, taking cross-sectional averages over 1000 simulations of the unobserved factor. Similarly as for the results in Section 4.2.2 on realized volatility, we find, again, that our model predicts industrial production growth to account for the bulk of variation of the VIX index. The unobserved factor accounts for less than 10%, and inflation and secular growth play a quite marginal role, explaining no more than 5%, of the model-implied VIX. Interestingly, Stock and Watson (2003) find that the linkages of asset prices to growth are stronger than for inflation. Our results further qualify this

finding: inflation does not seem to affect too much the dynamics of neither realized volatility nor future expected volatility under the risk-neutral probability.

Finally, the bottom panel in Figure 5 plots the volatility risk-premium, defined as in Eq. (16). This risk-premium is countercyclical, and this property arises for exactly the same reasons we put forward to explain the swings the model predicts for the VIX index: positive compensation for risk, combined with countercyclical variation of the premiums required to compensate for the risk of fluctuations of the macroeconomic factors.

4.2.4. Out-of-sample predictions of the model, and the subprime crisis

We undertake out-of-sample experiments to investigate the model's predictions over a quite exceptional period, that from January 2007 to March 2009. This sample covers the subprime turmoil, and features unprecedented events, both for the severity of capital markets uncertainty and the performance of the US economy. The market witnessed to a spectacular drop accompanied by an extraordinary surge in volatility. In March 2009, yearly returns plummeted to -58.30%, a performance even worse than that experienced in October 1974 (-58.10%). Furthermore, according to our estimates, obtained through Eq. (21), aggregate stock volatility reached 28.20% in March 2009, the highest level ever experienced in our sample. Finally, the VIX index hit its highest value in our sample in November 2008 (72.67%), and remained stubbornly high for several months. The time series behavior of stock returns, stock volatility and the VIX index during our out-of-sample period are depicted over the shaded areas in Figures 2 and 5.

Macroeconomic developments over our out-of-sample period (the shaded area in Figure 1) were equally extreme, with yearly inflation rates achieving negative values in 2009, and yearly industrial production growth being as low as -13%, in March 2009. Under such macroeconomic conditions, we expect our model to produce the following predictions: (i) stock returns drop, (ii) stock volatility rises, (iii) the VIX index rises, and more than stock volatility. The mechanism is, by now, clear. Asset prices and, hence, returns, plummet, as they are positively related to growth, which crashed. (Note that inflation also decreased but the (negative) price sensitivity to it is much smaller than that of growth.) Moreover, volatility increases, with the VIX index increasing even more, due to our previous finding of (i) sizeable macroeconomic risk-premiums and (ii) strong countercyclical

1 variation in these premiums.

2   Figures 2 and 5 confirm our reasoning, and reveal that the model is able to trace out the dynamics

3 of stock returns and volatility (Figure 2), and the VIX index (Figure 5), over the out-of-sample

4 period. The market literally crashes, as in the data, although only less than a half as much as in the

5 data: the lowest value for yearly stock returns the model predicts, out-of-sample, is -21.77%, which

6 is the second lowest figure our model produces, since after the quite volatile periods occurring over

7 the 1950s and the early 1960s. (The lowest level the model predicts after those periods is -29.91%,

8 for March 1975, the last month of the second severe recession of the 1970s.) Instead, the model

9 predicts that stock volatility and the VIX index surge even more than in the data, reaching record

10 highs of 26.68% (volatility) and 61.27% (VIX).

11   Figure 6 provide additional details about the period from January 2000 to March 2009. It

12 compares stock volatility and the VIX index with the predictions of the model and those of a OLS

13 regression. The OLS for volatility is that in Eq. (23), excluding the lag for six months, related

14 to the autoregressive term. The OLS for the VIX index is that in Eq. (25). OLS predictions are

15 obtained by feeding the OLS predictive part with its regressors, using parameter estimates obtained

16 with data up to December 2006. The following table reports Root Mean Squared Errors (RMSE)

17 for both our model and OLS, calculated over the out-of-sample period.

<div align="center">

RMSE for the model and OLS

	Model	OLS
Volatility	0.0478	0.0700
VIX Index	0.1119	0.1319

</div>

18   Overall, OLS predictions do not seem to capture the countercyclical behavior of stock volatility.

19 As for the VIX index, the OLS model (in fact, by Eq. (25), an autoregressive, distributed lag model)

20 produces predictions that are not as accurate as the model, and generate overfit. The model, instead,

21 predicts the swings we see in the data, in both the last two recession episodes. The RMSEs clearly

22 favour the model against OLS, although it appears to do so more with realized volatility than with

23 the VIX index, as Figure 6 informally reveals. Appendix C.3 of the Supplemental material confirms

24 these findings in additional experiments, performed by feeding the model with both $\hat{y}_{3,t}$ in Eq. (28)

25 and the UMCSENT index, rather than by simulations of y_3.

5. Conclusion

How does aggregate stock market volatility relate to the business cycle? This old question has been formulated at least since Officer (1973) and Schwert (1989a,b). We learnt from recent theoretical explanations that the countercyclical behavior of stock volatility can be understood as the result of a rational valuation process. However, how much of this countercyclical behavior is responsible for the sustained level aggregate volatility has experienced for centuries? This paper develops a model where approximately 75% of the variations in stock volatility can be explained by macroeconomic factors, and where some unobserved component is also needed to make stock volatility consistent with rational asset valuation.

We show that risk-premiums arising from fluctuations in this volatility are strongly countercyclical, certainly more so than stock volatility alone. In fact, the risk-compensation for the fluctuation of the macroeconomic factors is large and countercyclical, and helps explain the swings in the VIX index that we observe during recessions. We undertake out-of-sample experiments that cover the 2007-2009 subprime crisis, when the VIX reached a record high of more than 70%, which our model can at least partially track, through a countercyclical variation in the volatility risk-premiums. Again, our model predicts that a business cycle factor such as industrial production growth can explain more than 85% of the variations of the VIX index.

The key aspect of our model is that the relations among the market, stock volatility, volatility risk-premiums and the macroeconomic factors, are consistent with no-arbitrage. In particular, volatility is endogenous in our framework: the same variables driving the payoff process and the volatility of the pricing kernel, and hence, the asset price, are those that drive stock volatility and volatility-related risk-premiums. A question for future research is to explore whether the no-arbitrage framework in this paper can be used to improve forecasts of real economic activity. In fact, stock volatility and volatility risk-premiums are driven by business cycle factors, as this paper clearly demonstrates. A challenging and fundamental question is to explore the extent to which business cycle, stock volatility and volatility risk-premiums do endogenously develop.

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1 Tables

<div align="center">

Table 2

</div>

Parameter estimates and block-bootstrap standard errors for the joint process of the two macroeconomic factors, gross inflation, $y_{1,t} \equiv \mathrm{CPI}_t / \mathrm{CPI}_{t-12} \equiv y_1(t)$ and gross industrial production growth, $y_{2,t} \equiv \mathrm{IP}_t / \mathrm{IP}_{t-12} \equiv y_2(t)$, where CPI_t is the Consumer price index as of month t, IP_t is the real, seasonally adjusted industrial production index as of month t, and:

$$\begin{bmatrix} \mathrm{d}y_1(t) \\ \mathrm{d}y_2(t) \end{bmatrix} = \begin{bmatrix} \kappa_1 & \bar{\kappa}_1 \\ \bar{\kappa}_2 & \kappa_2 \end{bmatrix} \begin{bmatrix} \mu_1 - y_1(t) \\ \mu_2 - y_2(t) \end{bmatrix} \mathrm{d}t + \begin{bmatrix} \sqrt{\alpha_1 + \beta_1 y_1(t)} & 0 \\ 0 & \sqrt{\alpha_2 + \beta_2 y_2(t)} \end{bmatrix} \begin{bmatrix} \mathrm{d}W_1(t) \\ \mathrm{d}W_2(t) \end{bmatrix},$$

where $W_j(t)$, $j = 1, 2$, are two independent Brownian motions, and the parameter vector is $\phi^\top = \left(\kappa_j, \mu_j, \alpha_j, \beta_j, \bar{\kappa}_j, j = 1, 2 \right)$. Parameter estimates are obtained through the first step of the estimation procedure set forth in Section 3.1, relying on Indirect Inference and Simulated Method of Moments. Matching conditions relate to (i) parameter estimates for the auxiliary Vector Autoregressive models in Eq. (17), and (ii) the sample mean and standard deviation of $y_{1,t}$ and $y_{2,t}$. The sample covers monthly data for the period from January 1950 to December 2006.

	Estimate	Std error
κ_1	0.0231	0.0095
μ_1	1.0375	0.3784
α_1	$2.4408 \cdot 10^{-4}$	$1.0918 \cdot 10^{-4}$
β_1	$-1.0005 \cdot 10^{-6}$	$0.3374 \cdot 10^{-6}$
κ_2	0.9025	0.4037
μ_2	1.0386	0.3962
α_2	0.0253	0.0126
β_2	-0.0198	0.0084
$\bar{\kappa}_1$	-0.2995	0.1293
$\bar{\kappa}_2$	1.3723	0.6423

Table 2

Parameter estimates and block-bootstrap standard errors for stochastic secular growth, the real stock price and the unobservable factor:

$$S\left(t\right) = G\left(t\right)\left(s_0 + \sum_{i=1}^{3} s_i y_i\left(t\right)\right), \quad \frac{dG\left(t\right)}{G\left(t\right)} = gdt + \sigma_G dW_G\left(t\right),$$

where $S\left(t\right)$ is the real stock price, $G\left(t\right)$ is stochastic secular growth, W_G is a standard Brownian motion, $y_1\left(t\right)$ and $y_2\left(t\right)$ are the observed gross inflation and gross industrial production growth, as defined in Table 1, $y_3\left(t\right)$ is an unobserved factor, with the following dynamics:

$$\mathrm{d}y_3\left(t\right) = \kappa_3\left(\mu_3 - y_3\left(t\right)\right)\mathrm{d}t + \sqrt{\alpha_3 + \beta_3 y_3\left(t\right)}\mathrm{d}W_3\left(t\right),$$

and $W_3\left(t\right)$ a standard Brownian motion. The parameter vector to be estimated is $\boldsymbol{\theta}^\top = (\boldsymbol{\theta}_u, (s_j)_{j=0}^3)$, where $\boldsymbol{\theta}_u = (\kappa_3, \mu_3, \alpha_3, \beta_3)$, and the long run mean of the unobservable factor, μ_3, is set equal to one for the purpose of model's identification. Parameter estimates are obtained through the second step of the estimation procedure set forth in Section 3.2, relying on Indirect Inference and Simulated Method of Moments, with parameters $\boldsymbol{\theta}_G^\top = (g, \sigma_G^2)$ estimated on the low frequency component of secular growth of the real stock price, extracted through the Hodrick-Prescott filter with smoothing parameter 1600. Matching conditions relate to (i) parameter estimates for the auxiliary model for stock returns, Eq. (22), and for the auxiliary model for stock volatility, Eq. (23), and (ii) the sample mean and standard deviation of the real stock price, the real and return volatility. The sample covers monthly data for the period from January 1950 to December 2006.

	Estimate	Std error
g	0.0264	0.0171
σ_G^2	0.0012	$5.2089 \cdot 10^{-4}$
s_0	0.2272	0.1154
s_1	-0.8956	0.4426
s_2	1.8925	0.8873
s_3	-0.0560	0.0292
κ_3	0.0101	0.0043
μ_3	1	restricted
α_3	1.2009	0.4699
β_3	0.0201	0.0106

Table 3

Variance decomposition statistics for (i) realized volatility as calculated in Section 4.2.2 (Panel A) and expected volatility under the risk-neutral probability as calculated in Section 4.2.2 (Panel B). Panel A reports averages and standard deviations of the contributions $C_j(t)$ and $C_G(t)$ to the total variance, $\sigma^2(t)$ in Eq. (12), made by: (i) the two macroeconomic factors, gross inflation, $y_1(t)$, and gross industrial production growth, $y_2(t)$, as defined in Table 1, (ii) the unobserved factor, $y_3(t)$, and (iii) secular growth, defined respectively, as:

$$C_j(t)\, s^2(\boldsymbol{y}(t)) \equiv \frac{s_j^2\left(\alpha_j + \beta_j y_j(t)\right)}{\sigma^2(t)},\ j=1,2,3, \quad \text{and} \quad C_G(t) \equiv \frac{\sigma_G^2}{\sigma^2(t)},$$

where $s(\boldsymbol{y}) \equiv s_0 + \sum_{j=1}^{3} s_j y_j$. Paths for the contributions $C_j(t)$ and $C_G(t)$ are generated by feeding the model with the two macroeconomic factors $y_1(t)$ and $y_2(t)$ and by averaging over the cross-section of 1000 simulations of the unobserved factor. The sample covers monthly data for the period from January 1950 to December 2006. Panel B reports statistics for the risk-neutral counterparts to the average paths of $C_j(t)$ and $C_G(t)$. The sample covers monthly data for the period from January 1990 to December 2006.

Panel A: Contributions of factors to stock volatility

Averages

	1950-2006	1950-1981	1960-1981	1982-2006
Gross inflation	0.87%	0.92%	0.88%	0.83%
Gross growth	73.47%	71.97%	73.57%	75.10%
Unobserved factor	17.23%	18.10%	17.41%	16.27%
Secular growth	8.43%	9.01%	8.13%	7.78%

Standard deviations

	1950-2006	1950-1981	1960-1981	1982-2006
Gross inflation	0.18%	0.23%	0.12%	0.08%
Gross growth	7.53%	9.65%	5.03%	3.47%
Unobserved factor	3.69%	4.67%	2.46%	1.71%
Secular growth	3.67%	4.75%	2.40%	1.68%

Panel B: Contributions of factors to the VIX Index

	Averages	Standard deviations
Gross inflation	1.55%	0.03%
Gross growth	86.91%	1.28%
Unobserved factor	8.79%	0.56%
Secular growth	2.75%	0.17%

Table 4

Parameter estimates and block-bootstrap standard errors for the risk-premium parameters of the total risk-premium process in Eq. (7):

$$\pi_1\left(y_1\left(t\right)\right) = \alpha_1\lambda_{1(1)} + \left(\beta_1\lambda_{1(1)} + \lambda_{2(1)}\right)y_1\left(t\right) \quad \text{(inflation)}$$
$$\pi_2\left(y_2\left(t\right)\right) = \alpha_2\lambda_{1(2)} + \left(\beta_2\lambda_{1(2)} + \lambda_{2(2)}\right)y_2\left(t\right) \quad \text{(industrial production)}$$
$$\pi_3\left(y_3\left(t\right)\right) = \alpha_3\lambda_{1(3)} + \left(\beta_3\lambda_{1(3)} + \lambda_{2(3)}\right)y_3\left(t\right) \quad \text{(unobservable factor)}$$

where $y_1\left(t\right)$ and $y_2\left(t\right)$ are gross inflation and gross industrial production growth, as defined in Table 1, and $y_3\left(t\right)$ is the unobserved factor. The parameter vector is $\boldsymbol{\lambda}^\top = \left(\lambda_{1(1)}, \lambda_{2(1)}, \lambda_{1(2)}, \lambda_{2(2)}, \lambda_{1(3)}, \lambda_{2(3)}\right)$. Parameter estimates are obtained through the third step of the estimation procedure set forth in Section 3.3, relying on Indirect Inference and Simulated Method of Moments. Matching conditions relate to (i) parameter estimates for the auxiliary model for the VIX index, Eq. (25), and (ii) the sample mean and standard deviation of the VIX index. The sample covers monthly data for the period from January 1990 to December 2006.

	Estimate	Std error
Inflation		
$\lambda_{1(1)}$	$-2.1533{\cdot}10^3$	$0.9683{\cdot}10^3$
$\lambda_{2(1)}$	32.0141	15.5655
Ind. Prod.		
$\lambda_{1(2)}$	$5.6760{\cdot}10^2$	$2.7643{\cdot}10^2$
$\lambda_{2(2)}$	5.5717	2.7952
Unobs.		
$\lambda_{1(3)}$	0.0019	0.0008
$\lambda_{2(3)}$	$5.9837{\cdot}10^{-4}$	$2.9526{\cdot}10^{-4}$

Figures
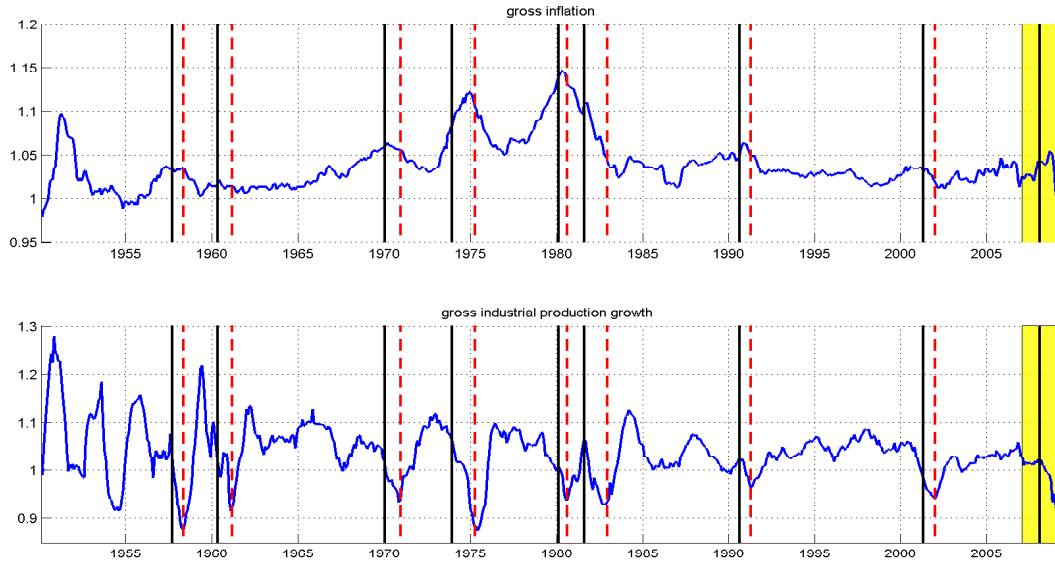
Figure 1 – Industrial production growth and inflation, with NBER dated recession periods. This figure plots the one-year, monthly gross inflation, defined as $y_{1,t} \equiv \mathrm{CPI}_t/\mathrm{CPI}_{t-12}$, and the one-year, monthly gross industrial production growth, defined as $y_{2,t} \equiv \mathrm{IP}_t/\mathrm{IP}_{t-12}$, where CPI_t is the Consumer price index as of month t, and IP_t is the real, seasonally adjusted industrial production index as of month t. The sample covers monthly data for the period from January 1950 to December 2006. Vertical solid lines (in black) track the beginning of NBER-dated recessions, and vertical dashed lines (in red) indicate the end of NBER-dated recessions. The shaded area (in yellow) covers the out-of-sample period, from January 2007 to March 2009.

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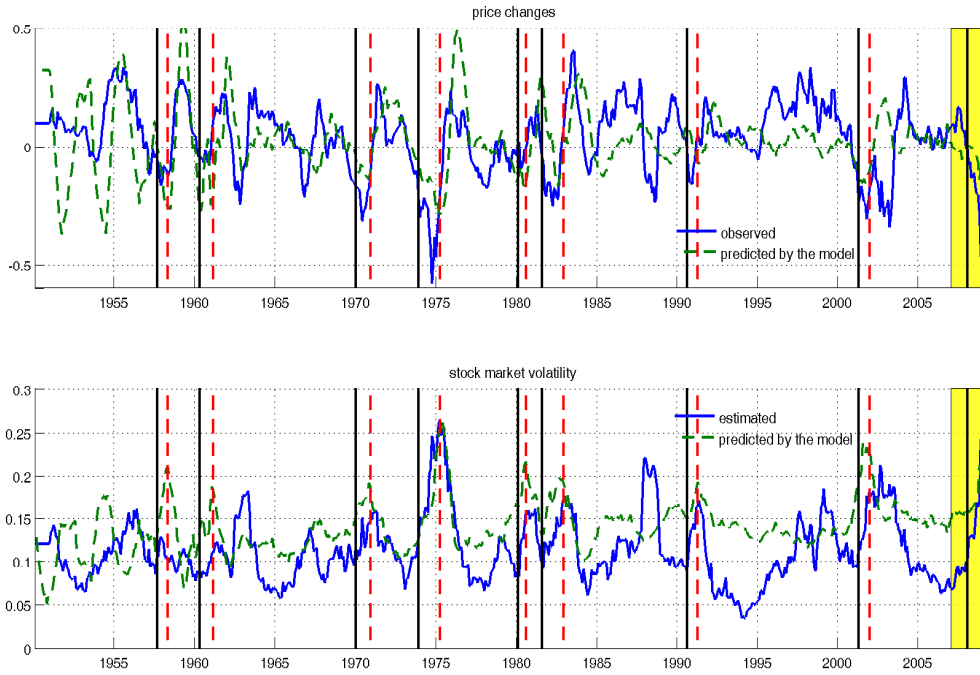
Figure 2 – Stock returns and volatility along with the model predictions, with NBER dated recession periods, and out-of-sample predictions. This figure plots one-year ex-post price changes and one-year return volatility, along with their counterparts predicted by the model. The top panel depicts continuously compounded price changes, defined as $R_t \equiv \ln(S_t/S_{t-12})$, where S_t is the real stock price as of month t. The bottom panel depicts smoothed return volatility, defined as $\text{Vol}_t \equiv \sqrt{6\pi} \cdot 12^{-1} \sum_{i=1}^{12} |\ln(S_{t+1-i}/S_{t-i})|$, along with the instantaneous standard deviation predicted by the model, obtained through Eq. (12). Each prediction at each point in time is obtained by feeding the model with the two macroeconomic factors depicted in Figure 1 (inflation and growth) and by averaging over the cross-section of 1000 simulations of the unobserved factor. The sample covers monthly data for the period from January 1950 to December 2006. Vertical solid lines (in black) track the beginning of NBER-dated recessions, and vertical dashed lines (in red) indicate the end of NBER-dated recessions. The shaded area (in yellow) covers the out-of-sample period, from January 2007 to March 2009.
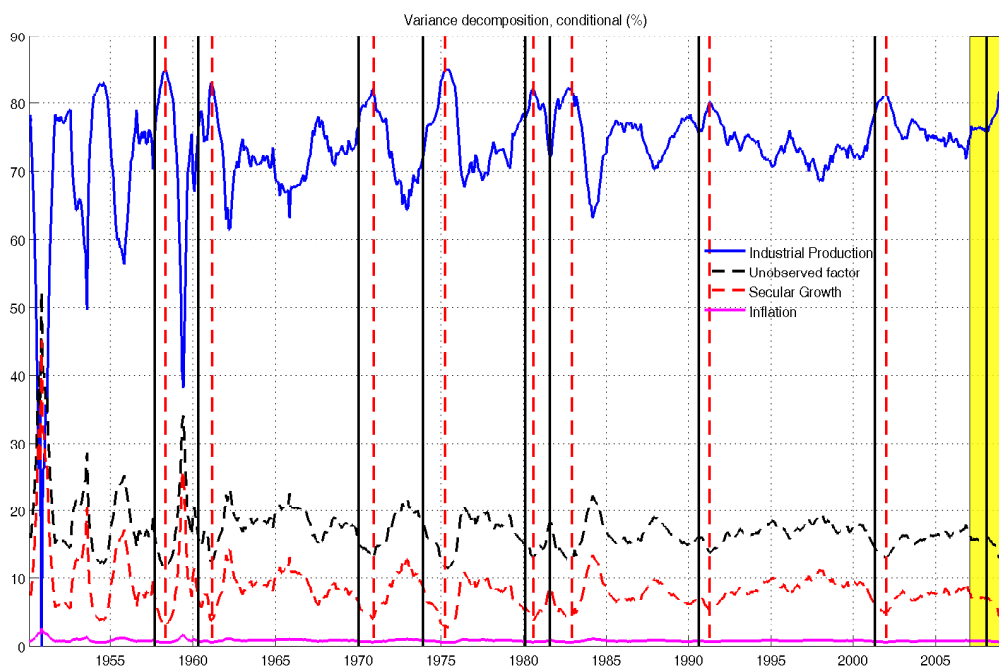
Figure 3 – Contributions to total stock volatility made by macroeconomic and unobservable factors, with NBER dated recession periods. This figure plots the contributions to stock volatility, $C_j(t)$ and $C_G(t)$ in Eq. (27), obtained as the ratios of the instantaneous stock return variance due to factor y_j to the total instantaneous variance, $\sigma^2(t)$, $C_j(t)$ ($j = 1, 2, 3$), as well as the ratio of the instantaneous variance of secular growth to $\sigma^2(t)$, $C_G(t)$. From top to bottom, "Industrial Production" is $C_2(t)$, "Unobservable factor" is $C_3(t)$, "Secular Growth" is $C_G(t)$, and "Inflation" is $C_1(t)$. Each prediction at each point in time is obtained by feeding the model with the two macroeconomic factors depicted in Figure 1 (inflation and growth) and by averaging over the cross-section of 1000 simulations of the unobserved factor. The sample covers monthly data for the period from January 1950 to December 2006. Vertical solid lines (in black) track the beginning of NBER-dated recessions, and vertical dashed lines (in red) indicate the end of NBER-dated recessions. The shaded area (in yellow) covers the out-of-sample period, from January 2007 to March 2009.

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Figure 4 − The dynamics of the unobserved factor implied by the estimated model, with NBER dated recession periods. This figure plots the dynamics of the unobserved factor as implied by the estimated model, and a comparison with an index of consumer confidence—the University of Michigan Consumer Sentiment (UMCSENT) index. The top panel plots: (i) 100 simulated trajectories of the unobserved factor, obtained using the parameter estimates in Table 2, and (ii) (minus) the model-implied unobserved factor (the bold line), estimated as $-\hat{y}_{3,t} \equiv -\hat{s}_3^{-1}\left(\frac{S_t}{\hat{G}_t} - \hat{s}_0 - \hat{s}_1 y_{1,t} - \hat{s}_2 y_{2,t}\right)$, where S_t is the real stock price as of month t, $y_{1,t}$ and $y_{2,t}$ are gross inflation and gross industrial production growth as of month t, as defined in Figure 1, S_t is the observed real stock price as of month t, $\hat{G}_t$ is the cross-sectional average of 1000 simulations of secular growth, and $(\hat{s}_j)_{j=0}^3$ are coefficient estimates, as reported in Table 2. The bottom panel depicts the time series of the UMCSENT index, de-meaned and standardized by its own standard deviation, together with the negative of the model-implied unobserved factor, $-\hat{y}_{3,t}$, de-meaned and standardized by its own standard deviation. The sample covers monthly data for the period from January 1950 to December 2006. The UMCSENT index is available for the sample period starting from January 1978. Vertical solid lines (in black) track the beginning of NBER-dated recessions, and vertical dashed lines (in red) indicate the end of NBER-dated recessions. The shaded area (in yellow) covers the out-of-sample period, from January 2007 to March 2009.

Figure 5 – The VIX Index and volatility risk-premia, with NBER dated recession periods, and out-of-sample predictions. This figure plots the VIX index along with model's predictions. The top panel depicts (i) the VIX index, (ii) the VIX index predicted by the model, and (iii) the VIX index predicted by the model in an economy without risk-aversion, i.e. the expected integrated volatility under the physical probability. The bottom panel depicts the volatility risk-premium predicted by the model, defined as the difference between the model-generated expected integrated volatility under the risk-neutral and the physical probability,

$$
\mathrm{VRP}\left(\boldsymbol{y}\left(t\right)\right) \equiv \sqrt{\frac{1}{T-t}} \left(\sqrt{\mathbb{E}\left(\left.\int_t^T \sigma^2\left(\boldsymbol{y}\left(u\right)\right)\mathrm{d}u\right| \boldsymbol{y}\left(t\right)\right)} - \sqrt{E\left(\left.\int_t^T \sigma^2\left(\boldsymbol{y}\left(u\right)\right)\mathrm{d}u\right| \boldsymbol{y}\left(t\right)\right)} \right),
$$

where $T - t = 12^{-1}$, $\mathbb{E}$ is the conditional expectation under the risk-neutral probability, E is the conditional expectation under the true probability, $\sigma^2\left(\boldsymbol{y}\right)$ is the instantaneous variance predicted by the model, obtained through Eq. (12), and $\boldsymbol{y}$ is the vector of three factors: the two macroeconomic factors depicted in Figure 1 (inflation and growth) and one unobservable factor. Each prediction at each point in time is obtained by feeding the model with the two macroeconomic factors depicted in Figure 1 (inflation and growth) and by averaging over the cross-section of 1000 simulations of the unobserved factor. The sample covers monthly data for the period from January 1990 to December 2006. Vertical solid lines (in black) track the beginning of NBER-dated recessions, and vertical dashed lines (in red) indicate the end of NBER-dated recessions. The shaded area (in yellow) covers the out-of-sample period, from January 2007 to March 2009.
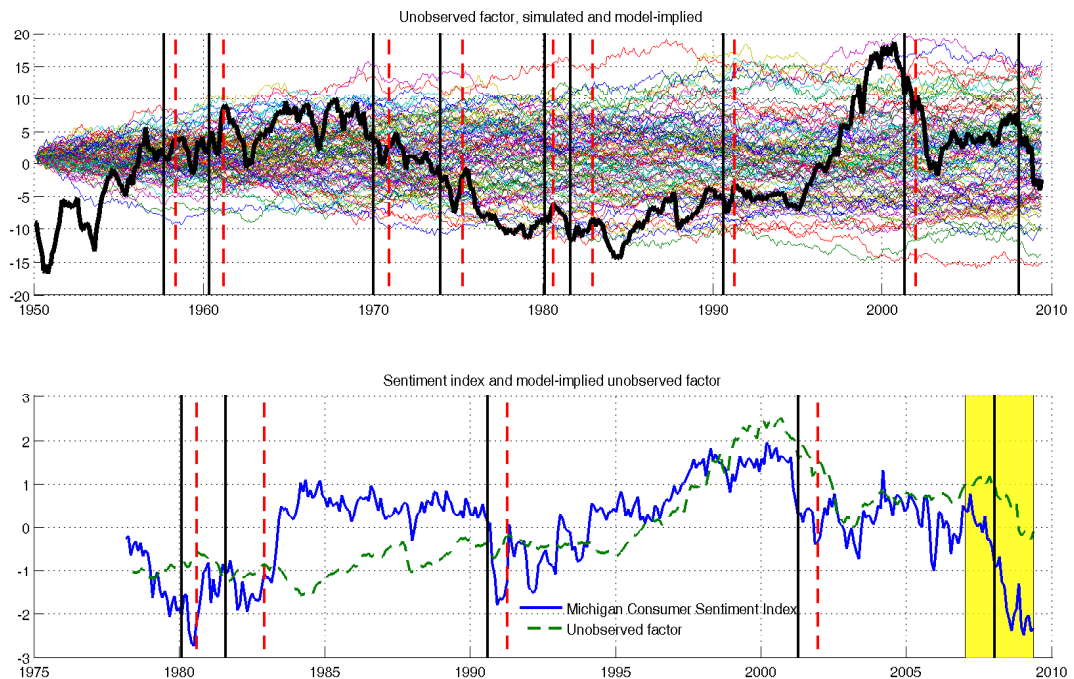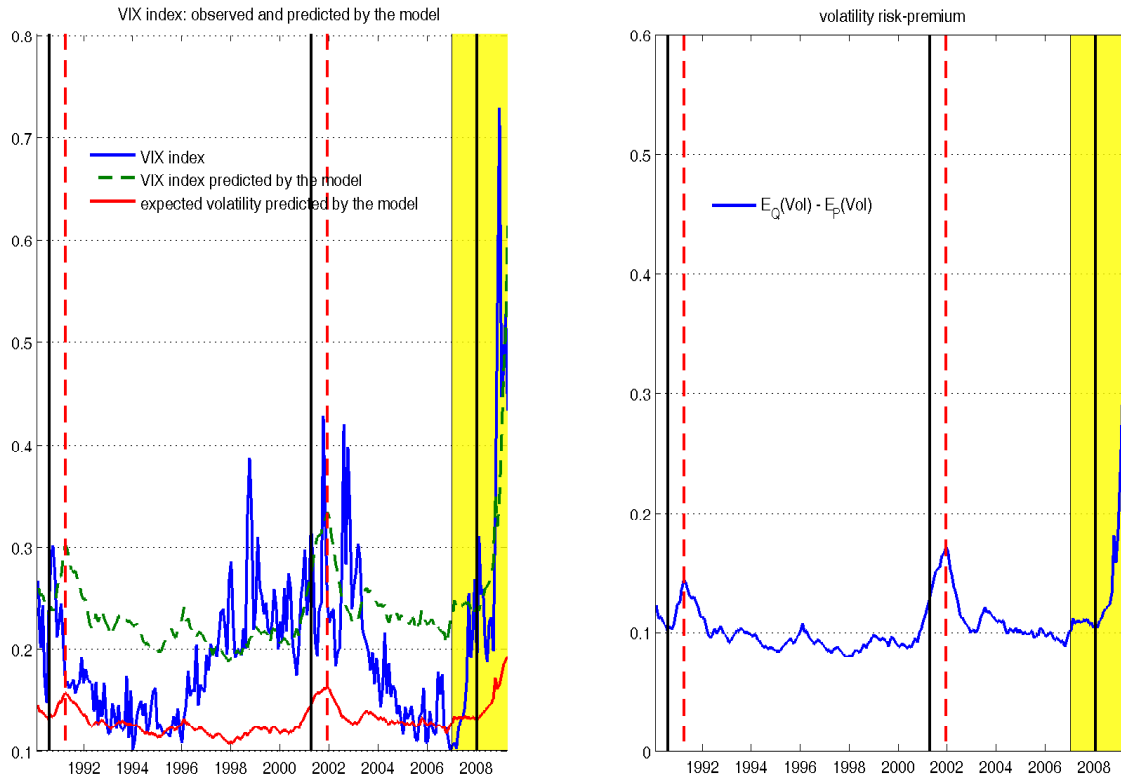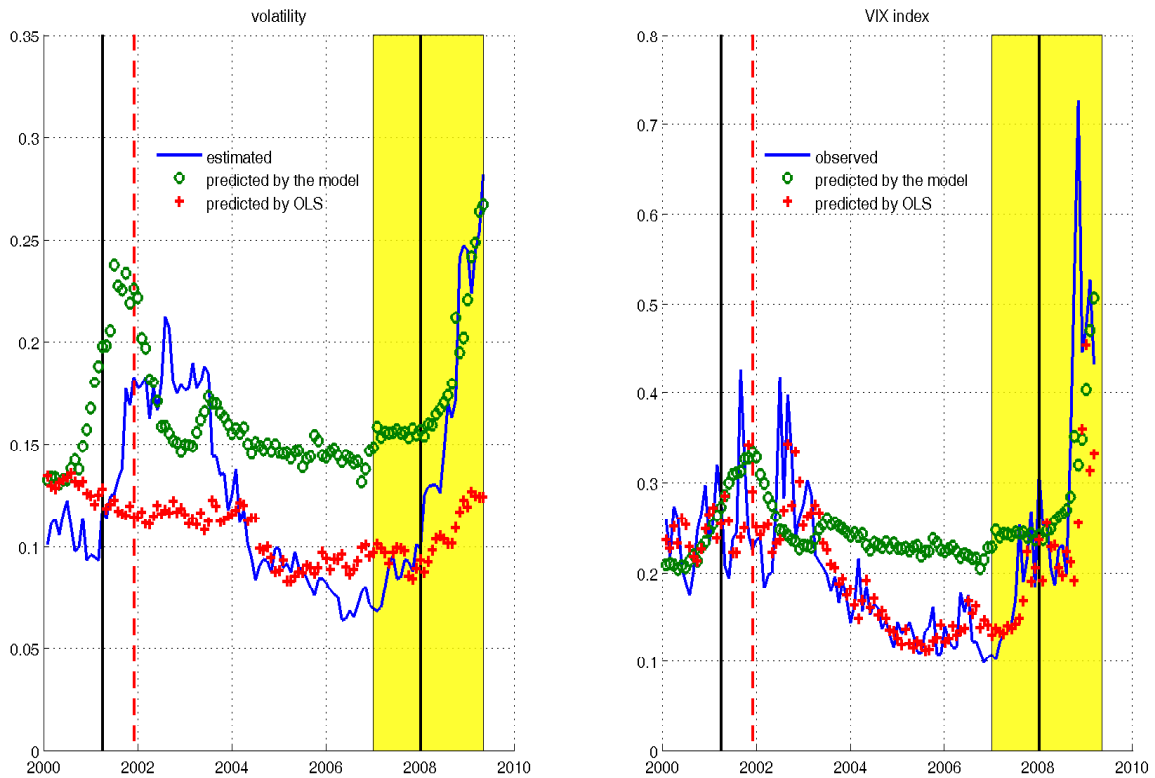
Figure 6 – Out of sample predictions and the subprime crisis. This figure plots one-year return volatility and the VIX index, along with its counterparts predicted by the model and by an OLS regression. The left panel depicts smoothed return volatility, defined as $\mathrm{Vol}_t \equiv \sqrt{6\pi} \cdot 12^{-1} \sum_{i=1}^{12} |\ln (S_{t+1-i}/S_{t-i})|$, where S_t is the real stock price as of month t, along with the instantaneous standard deviation predicted by (i) the model, through Eq. (12), and (ii) the predictive part of an OLS regression of Vol_t on to past values of Vol_t, inflation and industrial production growth. The right panel depicts the VIX index, along with the VIX index predicted by (i) the model; and (ii) the predictive part of an OLS regression of the VIX index on to past values of the VIX index, inflation and industrial production growth. Each prediction is obtained by feeding the model and the predictive part of the OLS regression with the two macroeconomic factors depicted in Figure 1 (inflation and growth) and, for the model, by averaging over the cross-section of 1000 simulations of the unobserved factor. The sample depicted in the figure spans the period from January 2000 to March 2009. The estimation of both the model and the OLS regressions relates to the period from January 1950 to December 2006. Vertical solid lines (in black) track the beginning of NBER-dated recessions, and the vertical dashed line (in red) indicates the end of the NBER-dated recession, occurred in November 2001. The shaded area (in yellow) covers the out-of-sample period, from January 2007 to March 2009, which includes the NBER recession announced to have occurred in December 2007, and the subprime crisis, which started in June 2007.

₁ Supplemental material: Appendix
₂ [Not for publication]

₃ A. Supplemental material for Section 2

₄ *A multifactor model*

₅   The model we consider differs from those in Bekaert and Grenadier (2001), Ang and Liu (2004) or Mamaysky
₆ (2002), for a number of reasons. First, we consider a continuous-time framework, which avoids theoretical challenges
₇ pointed out by Bekaert and Grenadier (2001). Furthermore, Ang and Liu (2004) consider a discrete-time setting
₈ in which expected returns are exogenous, while in our model, expected returns are endogenous. Finally, our model
₉ works differently from Mamaysky's because it endogenously determines the price-dividend ratio.

₁₀   We consider a multifactor model where a vector-valued process $\boldsymbol{y}(t)$ is solution to a n-dimensional affine diffusion,

$$
₁₁ \quad \mathrm{d}\boldsymbol{y}(t) = \boldsymbol{\kappa}\left(\boldsymbol{\mu} - \boldsymbol{y}(t)\right)\mathrm{d}t + \boldsymbol{\Sigma}\boldsymbol{V}\left(\boldsymbol{y}(t)\right)\mathrm{d}\boldsymbol{W}(t), \tag{29}
$$

where $\boldsymbol{W}(t)$ is a d-dimensional Brownian motion ($n \leq d$), $\boldsymbol{\Sigma}$ is a full rank $n \times d$ matrix, and $\boldsymbol{V}$ is a full rank $d \times d$ diagonal matrix with elements,

$$
V\left(\boldsymbol{y}\right)_{(ii)} = \sqrt{\alpha_i + \boldsymbol{\beta}_i^{\top}\boldsymbol{y}}, \quad i = 1, \cdots, d,
$$

₁₂ for some scalars α_i and vectors $\boldsymbol{\beta}_i$. We assume that the Brownian motion driving secular growth, $W_G(t)$ in Eq. (4),
₁₃ is uncorrelated with $\boldsymbol{W}(t)$ in Eq. (29). We shall review soon sufficient conditions known to ensure that Eq. (29)
₁₄ has a strong solution with $\boldsymbol{V}\left(\boldsymbol{y}(t)\right)_{(ii)} > 0$ almost surely for all t.

    The model we estimate, Eq. (1) in Section 2 of the main text, is a special case of Eq. (29), with $n = d = 3$, the matrix $\boldsymbol{\kappa}$ given by:

$$
\boldsymbol{\kappa} = \begin{bmatrix} \kappa_1 & \bar{\kappa}_1 & 0 \\ \bar{\kappa}_2 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{bmatrix},
$$

₁₅ and with $\boldsymbol{\Sigma} = \boldsymbol{I}_{3\times3}$ and the vectors $\boldsymbol{\beta}_i$ being such that $\beta_j \equiv \beta_{jj}$.

₁₆   While one does not necessarily observe every single component of $\boldsymbol{y}(t)$, we do observe discretely sampled paths
₁₇ of macroeconomic variables such as industrial production, unemployment or inflation. Let $\{M_{j,t}\}_{t=1,2,\cdots}$ be the
₁₈ discretely sampled path of the macroeconomic variable $M_{j,t}$ where, for example, $M_{j,t}$ can be the industrial production
₁₉ index available for month t, and $j = 1, \cdots, N_M$, where N_M is the number of observed macroeconomic factors. We
₂₀ assume, without loss of generality, that these observed macroeconomic factors are strictly positive, and that they are
₂₁ related to the state vector process in Eq. (29) by:

$$
₂₂ \quad \ln\left(M_{j,t}/M_{j,t-12}\right) = f_j\left(\boldsymbol{y}(t)\right), \quad j = 1, \cdots, N_M, \tag{30}
$$

₂₃ where the collection of functions $\{f_j\}$ determines how the factors dynamics impinge upon the observed macroeconomic
₂₄ variables. In terms of the model in the main text, the functions in Eq. (30) are $f_j\left(\boldsymbol{y}\right) \equiv \ln y_j$.

₂₅   We now turn to model asset prices. We assume that asset prices are related to the vector of factors $\boldsymbol{y}(t)$ in
₂₆ Eq. (29), and that some of these factors affect developments in macroeconomic conditions, through Eq. (30). For
₂₇ analytical convenience, we rule out that asset prices can feed back the real economy, although we acknowledge that
₂₈ the presence of frictions can make capital markets and the macroeconomy intimately related, as in the financial
₂₉ accelerator hypothesis reviewed by Bernanke, Gertler and Gilchrist (1999), or in the static model analyzed by
₃₀ Angeletos, Lorenzoni and Pavan (2008), where feedbacks arise due to asymmetric information and learning between
₃₁ agents acting within the real and the financial spheres of the economy.

    The Arrow-Debreu density we consider is exactly that in Eq. (5), with the sole exception that the vector Brownian motion $\boldsymbol{W}$ is the d-dimensional one in Eq. (29). Consider, then, the following "essentially affine" specification for the dynamics of the factors in Eq. (29), and the risk-premiums. Let $\boldsymbol{V}^-\left(\boldsymbol{y}\right)$ be a $d \times d$ diagonal matrix with elements

$$
V^-\left(\boldsymbol{y}\right)_{(ii)} = \begin{cases} \frac{1}{V\left(\boldsymbol{y}\right)_{(ii)}} & \text{if } \Pr\{V\left(\boldsymbol{y}(t)\right)_{(ii)} > 0 \text{ all } t\} = 1 \\ 0 & \text{otherwise} \end{cases}
$$

₃₂ and set, $\boldsymbol{\Lambda}\left(\boldsymbol{y}\right) = \boldsymbol{V}\left(\boldsymbol{y}\right)\boldsymbol{\lambda}_1 + \boldsymbol{V}^-\left(\boldsymbol{y}\right)\boldsymbol{\lambda}_2\boldsymbol{y}$, for some d-dimensional vector $\boldsymbol{\lambda}_1$ and some $d \times n$ matrix $\boldsymbol{\lambda}_2$.

₃₃   By the definition of the dividends in Eq. (2), the stock price follows:

$$
₃₄ \quad \frac{\mathrm{d}S(t)}{S(t)} = \left(r - \frac{G(t)\,\delta\left(\boldsymbol{y}(t)\right)}{S\left(\boldsymbol{y}(t)\right)}\right)\mathrm{d}t + \frac{s_y\left(\boldsymbol{y}(t)\right)^{\top}\boldsymbol{\Sigma}\boldsymbol{V}\left(\boldsymbol{y}(t)\right)}{s\left(\boldsymbol{y}(t)\right)}\mathrm{d}\hat{\boldsymbol{W}}(t) + \sigma_G\mathrm{d}\hat{W}_G(t), \tag{31}
$$

where $\hat{\boldsymbol{W}}$ and $\hat{W}_G$ are Brownian motions defined under the risk-neutral probability Q. Under regularity conditions provided below, and in the absence of bubbles, Eq. (31) implies that the stock price is,

$$S(G, \boldsymbol{y}) = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} G(s)\, \delta(\boldsymbol{y}(s))\, \mathrm{d}s \,\middle|\, G(t) = G, \boldsymbol{y}(t) = \boldsymbol{y}\right], \tag{32}$$

where $\mathbb{E}$ is the expectation taken under the risk-neutral probability Q.

We are only left with specifying how the instantaneous dividend relates to the state vector $\boldsymbol{y}$. Let

$$\delta(\boldsymbol{y}) = \delta_0 + \boldsymbol{\delta}^\top \boldsymbol{y}, \tag{33}$$

for some scalar δ_0 and some vector $\boldsymbol{\delta}$.

We have:

Proposition A1: *Let the risk-premiums be as in Eq. (6), and the instantaneous dividend rate be as in Eqs. (2) and (33). Then, under a technical regularity condition (condition (38)), we have that: (i) Eq. (32) holds; and (ii) the rational stock price function $S(G, \boldsymbol{y}) = G \cdot s(\boldsymbol{y})$, where $s(\boldsymbol{y})$ is affine in the state vector $\boldsymbol{y}$, viz*

$$s(\boldsymbol{y}) = \frac{\delta_0 + \boldsymbol{\delta}^\top \left(\boldsymbol{D} + (r - g + \sigma_G \lambda_G) \boldsymbol{I}_{n \times n}\right)^{-1} \boldsymbol{c}}{r - g + \sigma_G \lambda_G} + \boldsymbol{\delta}^\top \left(\boldsymbol{D} + (r - g + \sigma_G \lambda_G) \boldsymbol{I}_{n \times n}\right)^{-1} \boldsymbol{y}, \tag{34}$$

where

$$\boldsymbol{c} = \boldsymbol{\kappa}\boldsymbol{\mu} - \boldsymbol{\Sigma} \left(\begin{array}{ccc} \alpha_1 \lambda_{1(1)} & \cdots & \alpha_d \lambda_{1(d)} \end{array}\right)^\top \tag{35}$$

$$\boldsymbol{D} = \boldsymbol{\kappa} + \boldsymbol{\Sigma} \left[\left(\begin{array}{ccc} \lambda_{1(1)} \boldsymbol{\beta}_1^\top & \cdots & \lambda_{1(d)} \boldsymbol{\beta}_d^\top \end{array}\right)^\top + \boldsymbol{I}^- \boldsymbol{\lambda}_2\right], \tag{36}$$

$\boldsymbol{I}^-$ is a $d \times d$ diagonal matrix with elements $I^-_{(ii)} = 1$ if $\Pr\{V(\boldsymbol{y}(t))_{(ii)} > 0 \text{ all } t\} = 1$ and 0 otherwise; and, finally $\{\lambda_{1(j)}\}_{j=1}^d$ are the components of $\boldsymbol{\lambda}_1$.

Existence of a strong solution to Eq. (29)

Consider the following conditions: for all i,

(i) For all $\boldsymbol{y} : V(\boldsymbol{y})_{(ii)} = 0$, $\boldsymbol{\beta}_i^\top (-\boldsymbol{\kappa}\boldsymbol{y} + \boldsymbol{\kappa}\boldsymbol{\mu}) > \frac{1}{2}\boldsymbol{\beta}_i^\top \boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \boldsymbol{\beta}_i$

(ii) For all j, if $\left(\boldsymbol{\beta}_i^\top \boldsymbol{\Sigma}\right)_j \neq 0$, then $V_{ii} = V_{jj}$.

Then, by Duffie and Kan (1996) (unnumbered theorem, p. 388), there exists a unique strong solution to Eq. (29) for which $V(\boldsymbol{y}(t))_{(ii)} > 0$ for all t almost surely.

We apply these conditions to the diffusion in Eq. (1). Condition (i) collapses to,

$$\text{For all } y_i : \alpha_i + \beta_i y_i = 0, \quad \beta_i \left[\kappa_i (\mu_i - y_i) + \bar{\kappa}_i (\mu_j - y_j)\right] > \frac{1}{2}\beta_i^2, \quad i \neq j,$$

with $\bar{\kappa}_3 \equiv 0$. That is, ruling out the trivial case $\beta_i = 0$,

$$\kappa_i (\mu_i \beta_i + \alpha_i) + \bar{\kappa}_i \beta_i \left(\mu_j + \frac{\alpha_j}{\beta_j}\right) > \frac{1}{2}\beta_i^2, \quad i \neq j. \tag{37}$$

Proof of Proposition A1

The technical condition in Proposition A1 is,

$$E\left[\int_t^T \left\|\frac{\boldsymbol{\eta}^\top \boldsymbol{\Sigma} \boldsymbol{V}(\boldsymbol{y}(\tau))}{\gamma + \boldsymbol{\eta}^\top \boldsymbol{y}(\tau)} - \boldsymbol{\Lambda}(\tau)^\top\right\|^2 \mathrm{d}\tau\right] < \infty, \tag{38}$$

for some constants γ and $\boldsymbol{\eta}$ in Eq. (47) below.

We proceed as follows. First, we determine the solution to the stock price, in the absence of secular growth, i.e. when

$$g = \sigma_G \equiv 0. \tag{39}$$

1 Then, we generalize, by elaborating on Eq. (32), as in Eq. (48) below.

2 When Eq. (39) holds true, define the Arrow-Debreu adjusted asset price process as, $s^\xi(t) \equiv e^{-rt}\xi(t)s(\boldsymbol{y}(t))$,

3 $t > 0$. By Itô's lemma, it satisfies,

$$\frac{\mathrm{d}s^\xi(t)}{s^\xi(t)} = \mathrm{Dr}\left(\boldsymbol{y}(t)\right)\mathrm{d}t + \left(\boldsymbol{Q}\left(\boldsymbol{y}(t)\right)^\top - \boldsymbol{\Lambda}\left(\boldsymbol{y}(t)\right)^\top\right)\mathrm{d}\boldsymbol{W}(t), \tag{40}$$

5 where

$$\mathrm{Dr}\left(\boldsymbol{y}\right) = -r + \frac{\mathcal{A}s\left(\boldsymbol{y}\right)}{s\left(\boldsymbol{y}\right)} - \boldsymbol{Q}\left(\boldsymbol{y}\right)^\top\boldsymbol{\Lambda}\left(\boldsymbol{y}\right),$$

$$\mathcal{A}s\left(\boldsymbol{y}\right) = s_y\left(\boldsymbol{y}\right)^\top\boldsymbol{\kappa}\left(\boldsymbol{\mu} - \boldsymbol{y}\right) + \frac{1}{2}\mathrm{Tr}\left(\left[\boldsymbol{\Sigma}\boldsymbol{V}\left(\boldsymbol{y}\right)\right]\left[\boldsymbol{\Sigma}\boldsymbol{V}\left(\boldsymbol{y}\right)\right]^\top s_{yy}\left(\boldsymbol{y}\right)\right), \quad \boldsymbol{Q}\left(\boldsymbol{y}\right)^\top = \frac{s_y\left(\boldsymbol{y}\right)^\top\boldsymbol{\Sigma}\boldsymbol{V}\left(\boldsymbol{y}\right)}{s\left(\boldsymbol{y}\right)},$$

6 and s_y and s_{yy} denote the gradient and the Hessian of s with respect to $\boldsymbol{y}$. By absence of arbitrage opportunities,

7 for any $T < \infty$,

$$s^\xi(t) = E\left[\int_t^T \delta^\xi(h)\,\mathrm{d}h\,\middle|\,\mathbb{F}(t)\right] + E[s^\xi(T) \mid \mathbb{F}(t)], \tag{41}$$

9 where $\delta^\xi(t)$ is the current Arrow-Debreu value of the dividend to be paid off at time t, viz $\delta^\xi(t) = e^{-rt}\xi(t)\delta(t)$.

10 Below, we show that the following transversality condition holds,

$$\lim_{T\to\infty} E[s^\xi(T) \mid \mathbb{F}(t)] = 0, \tag{42}$$

12 from which Eq. (32) follows, once we show that $\int_t^\infty E[\delta^\xi(h)]\mathrm{d}h < \infty$.

13 Next, by Eq. (41),

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}E[s^\xi(\tau) \mid \mathbb{F}(t)]\Big|_{\tau=t} + \delta^\xi(t). \tag{43}$$

15 Below, we show that

$$E[s^\xi(T) \mid \mathbb{F}(t)] = s^\xi(t) + \int_t^T \mathrm{D}\left(\boldsymbol{y}(h)\right)s^\xi(h)\,\mathrm{d}h. \tag{44}$$

17 Therefore, by the assumptions on $\boldsymbol{\Lambda}$, Eq. (43) can be rearranged to yield the following ordinary differential equation,

$$\text{For all } \boldsymbol{y}, \quad s_y\left(\boldsymbol{y}\right)^\top\left(\boldsymbol{c} - \boldsymbol{D}\boldsymbol{y}\right) + \frac{1}{2}\mathrm{Tr}\left(\left[\boldsymbol{\Sigma}\boldsymbol{V}\left(\boldsymbol{y}\right)\right]\left[\boldsymbol{\Sigma}\boldsymbol{V}\left(\boldsymbol{y}\right)\right]^\top s_{yy}\left(\boldsymbol{y}\right)\right) + \delta\left(\boldsymbol{y}\right) - rs\left(\boldsymbol{y}\right) = 0, \tag{45}$$

19 where $\boldsymbol{c}$ and $\boldsymbol{D}$ are defined in the proposition.

20 Assume that the price function is affine in $\boldsymbol{y}$,

$$s\left(\boldsymbol{y}\right) = \gamma + \boldsymbol{\eta}^\top\boldsymbol{y}, \tag{46}$$

for some scalar γ and some vector $\boldsymbol{\eta}$. By plugging this guess back into Eq. (45) we obtain,

$$\text{For all } \boldsymbol{y}, \quad \boldsymbol{\eta}^\top\boldsymbol{c} + \delta_0 - r\gamma - \left[\boldsymbol{\eta}^\top\left(\boldsymbol{D} + r\boldsymbol{I}_{n\times n}\right) - \boldsymbol{\delta}^\top\right]\boldsymbol{y} = 0.$$

That is,

$$\boldsymbol{\eta}^\top\boldsymbol{c} + \delta_0 - r\gamma = 0 \quad \text{and} \quad \left[\boldsymbol{\eta}^\top\left(\boldsymbol{D} + r\boldsymbol{I}_{n\times n}\right) - \boldsymbol{\delta}^\top\right] = \boldsymbol{0}_{1\times n}.$$

22 The solution to this system is,

$$\gamma = \frac{\delta_0 + \boldsymbol{\eta}^\top\boldsymbol{c}}{r} \quad \text{and} \quad \boldsymbol{\eta}^\top = \boldsymbol{\delta}^\top\left(\boldsymbol{D} + r\boldsymbol{I}_{n\times n}\right)^{-1}. \tag{47}$$

24 We are left to show that Eq. (42) and (44) hold true, when Eq. (39) also holds true.

As for Eq. (42), we have:

$$\lim_{T \to \infty} E[s^\xi(T) \mid \mathbb{F}(t)] = \lim_{T \to \infty} E[e^{-r(T-t)} \xi(T) s(\boldsymbol{y}(T)) \mid \mathbb{F}(t)]$$

$$= \gamma \lim_{T \to \infty} e^{-r(T-t)} E[\xi(T) \mid \mathbb{F}(t)] + \lim_{T \to \infty} e^{-r(T-t)} E[\xi(T) \boldsymbol{\eta}^\top \boldsymbol{y}(T) \mid \mathbb{F}(t)]$$

$$= \xi(t) \lim_{T \to \infty} e^{-r(T-t)} \mathbb{E}[\boldsymbol{\eta}^\top \boldsymbol{y}(T) \mid \mathbb{F}(t)],$$

where the second line follows by Eq. (46), and the third line holds because $E[\xi(T) \mid \mathbb{F}(t)] = 1$ for all T, and by a change of measure. Eq. (42) follows because $\boldsymbol{y}$ is stationary mean-reverting under the risk-neutral probability.

To show that Eq. (44) holds, we need to show that the diffusion part of s^ξ in Eq. (40) is a martingale, not only a local martingale, which it does whenever for all T,

$$E\left[\int_t^T \left\| \boldsymbol{Q}(\boldsymbol{y}(\tau))^\top - \boldsymbol{\Lambda}(\tau)^\top \right\|^2 \mathrm{d}\tau\right] < \infty,$$

which is the condition in (38). This ends the proof of Proposition A1, in the case $g = \sigma_G \equiv 0$.

For the general case of Proposition A1, note that by Eq. (32):

$$S(G, \boldsymbol{y})$$

$$= G \cdot \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} \mathbb{E}\left(\frac{G(s)}{G} \delta(\boldsymbol{y}(s)) \mathrm{d}s \,\middle|\, G(t) = G\right) \,\middle|\, G(t) = G, \boldsymbol{y}(t) = \boldsymbol{y}\right]$$

$$= G \cdot \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} \mathbb{E}\left(e^{\left(g - \frac{1}{2}\sigma_G^2 - \lambda_G \sigma_G\right)(s-t) + \sigma_G \left(\tilde{W}_G(s) - \tilde{W}_G(t)\right)} \,\middle|\, G(t) = G\right) \delta(\boldsymbol{y}(s)) \mathrm{d}s \,\middle|\, \boldsymbol{y}(t) = \boldsymbol{y}\right]$$

$$= G \cdot \mathbb{E}\left[\int_t^\infty e^{-(r - g + \lambda_G \sigma_G)(s-t)} \delta(\boldsymbol{y}(s)) \mathrm{d}s \,\middle|\, \boldsymbol{y}(t) = \boldsymbol{y}\right], \tag{48}$$

where the first equality follows by the law of iterated expectations, the second by the independence of G and $\boldsymbol{y}$, and the definition of G in Eq. (4), and the third from a simple computation. The term in the brackets is the same as the RHS of Eq. (32), for $G(s) \equiv 1$, $s \in (t, \infty)$. Therefore, the solution for the term in the brackets is the same as that provided in the case of absence of secular growth, i.e. when Eq. (39) holds true, but with $r - g + \lambda_G \sigma_G$ replacing r.

B.   Supplemental material for Section 3

Remarks on notation: Hereafter, we let Avar and Acov denote the limits of the variance and covariance operators, respectively. Let $\boldsymbol{u}$ be a $n \times 1$ vector, where each element depends on some $m \times 1$ parameter vector $\boldsymbol{\theta}$. We define: the $m \times n$ matrix $\nabla_\theta \boldsymbol{u} = \frac{\partial \boldsymbol{u}^\top}{\partial \boldsymbol{\theta}}$; $\|\boldsymbol{u}\|^p = \left(\sqrt{\boldsymbol{u}^\top \boldsymbol{u}}\right)^p$, for some scalar $p > 0$; and $|\boldsymbol{u}|_2 = \boldsymbol{u}\boldsymbol{u}^\top$, the outer product of $\boldsymbol{u}$. Finally, for any $n \times m$ matrix $\boldsymbol{A}$, we set $|\boldsymbol{A}| = \sum_{i=1}^n \sum_{j=1}^m |a_{i,j}|$.

B.1.   Asymptotic theory for the estimators in Section 3

The sets $\boldsymbol{\Phi}$ and $\boldsymbol{\Theta}$ in Sections 3.1 and 3.2 are defined as:

$$\boldsymbol{\Phi} = \left\{\boldsymbol{\phi} : \text{The inequality in (37) holds}, \ \kappa_i > 0, \ \text{and} \ \kappa_i \kappa_j - \bar{\kappa}_i \bar{\kappa}_j > 0, \ i, j = 1, 2 \ \text{and} \ i \neq j\right\},$$

and

$$\boldsymbol{\Theta} = \left\{\boldsymbol{\theta} : \text{The inequality in (37) holds for } i = 3, \text{ and } \kappa_3 > 0\right\}.$$

Furthermore, we let $\boldsymbol{\phi}_0$ and $\boldsymbol{\theta}_0$ be the solutions to the two limit problems,

$$\boldsymbol{\phi}_0 = \arg\min_{\boldsymbol{\phi} \in \boldsymbol{\Phi}_0} \operatorname*{plim}_{T \to \infty, \Delta \to 0} \left\| \frac{1}{H} \sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\boldsymbol{\phi}) - \tilde{\boldsymbol{\varphi}}_T \right\|^2, \tag{49}$$

and

$$\boldsymbol{\theta}_0 = \arg\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} \operatorname*{plim}_{T \to \infty, \Delta \to 0} \left\| \frac{1}{H} \sum_{h=1}^H \hat{\boldsymbol{\vartheta}}_{T,\Delta,h}(\boldsymbol{\theta}) - \tilde{\boldsymbol{\vartheta}}_T \right\|^2, \tag{50}$$

where $\boldsymbol{\Phi}_0$ and $\boldsymbol{\Theta}_0$ are compact sets of $\boldsymbol{\Phi}$ and $\boldsymbol{\Theta}$, respectively. Finally, we define the limit problem for the estimator of the risk-premium parameters,

$$\boldsymbol{\lambda}_0 = \arg\min_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}_0} \plim_{\mathcal{T}\to\infty,\Delta\to 0} \left\| \frac{1}{H}\sum_{h=1}^{H} \hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}(\hat{\boldsymbol{\phi}}_T,\hat{\boldsymbol{\theta}}_T,\boldsymbol{\lambda}) - \tilde{\boldsymbol{\psi}}_{\mathcal{T}} \right\|^2. \tag{51}$$

We are now ready to analyze the asymptotic behavior of these estimators. The following assumption summarizes the properties of the data generating mechanism we rely on.

Assumption B1: (i) Conditions (i) and (ii) in Appendix A hold for $i = 1, 2, 3$; (ii) The sample observations for the macroeconomic factors $y_1(t), y_2(t)$ are generated by Eq. (1) for $j = 1, 2$; (iii) As for Eq. (1), for $i, j = 1, 2$ $i \neq j$, $\kappa_i \kappa_j - \bar{\kappa}_i \bar{\kappa}_j > 0$ and for all $i = 1, 2, 3$ $\kappa_i > 0$; (iv) The sample observations for the stock market index $s(t)$ are generated by Eq. (8); (v) The risk-premium vector $\boldsymbol{\pi}(\boldsymbol{y})$ and the dividend vector $\delta(\boldsymbol{y})$ are defined as in Eqs. (7) and (3).

The estimator of $\hat{\boldsymbol{\phi}}_T$ in Eq. (19)

We have:

Proposition B1: *Under regularity conditions (Assumption B1(i)-(iii) in Appendix B), as $T \to \infty$ and $\Delta\sqrt{T} \to 0$,*

$$\sqrt{T}\left(\hat{\boldsymbol{\phi}}_T - \boldsymbol{\phi}_0\right) \xrightarrow{d} \mathrm{N}\left(\boldsymbol{0}, \boldsymbol{V}_1\right), \quad \boldsymbol{V}_1 = \left(1 + \frac{1}{H}\right)\left(\boldsymbol{D}_1^\top \boldsymbol{D}_1\right)^{-1} \boldsymbol{D}_1^\top \boldsymbol{J}_1 \boldsymbol{D}_1 \left(\boldsymbol{D}_1^\top \boldsymbol{D}_1\right)^{-1},$$

where $\boldsymbol{\phi}_0$ is as in Eq. (49), and the two matrices, $\boldsymbol{D}_1$ and $\boldsymbol{J}_1$, are defined in the proof below.

Proof: By the conditions in Assumptions B1(i) and B1(ii), $(y_1(t), y_2(t))$ admits a unique strong solution, and has a positive-definite covariance matrix with probability one. Assumption B1(iii) ensures that $(y_1(t), y_2(t))$ is geometrically ergodic and the skeleton $(y_{1,t}, y_{2,t})$ is geometrically β-mixing. Further, by Glasserman and Kim (2010), the stationary distribution of $(y_1(t), y_2(t))$ and $(y_{1,t}, y_{2,t})$ has exponential tails, which ensures that there are enough finite moments for the uniform law of large numbers and the central limit theorem to apply. By the same argument, for any $\boldsymbol{\phi} \in \boldsymbol{\Phi}_0$, the simulated skeleton $(y_{1,t,\Delta,h}^\phi, y_{2,t,\Delta,h}^\phi)$ is also geometrically β-mixing, with stationary distribution having exponential tails. Finally, given Eq. (1), $(y_{1,t,\Delta,h}^\phi, y_{2,t,\Delta,h}^\phi)$ is at least twice continuously differentiable in any open neighborhood of $\boldsymbol{\phi}_0$.

We have that $\hat{\boldsymbol{\phi}}_T - \boldsymbol{\phi}_0 = o_p(1)$, because of the uniform law of large numbers and unique identifiability. Next, by the first order conditions and a mean-value expansion around $\boldsymbol{\phi}_0$,

$$0 = \nabla_\phi \left(\frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\hat{\boldsymbol{\phi}}_T)\right)^\top \left(\frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\hat{\boldsymbol{\phi}}_T) - \tilde{\boldsymbol{\varphi}}_T\right)$$

$$= \nabla_\phi \left(\frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\hat{\boldsymbol{\phi}}_T)\right)^\top \left(\frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\boldsymbol{\phi}_0) - \tilde{\boldsymbol{\varphi}}_T\right)$$

$$+ \nabla_\phi \left(\frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\hat{\boldsymbol{\phi}}_T)\right)^\top \nabla_\phi \left(\frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\bar{\boldsymbol{\phi}}_T)\right)\left(\hat{\boldsymbol{\phi}}_T - \boldsymbol{\phi}_0\right),$$

where $\bar{\boldsymbol{\phi}}_T$ is some convex combination of $\hat{\boldsymbol{\phi}}_T$ and $\boldsymbol{\phi}_0$. Let

$$\boldsymbol{D}_1(\boldsymbol{\phi}_0) \equiv \boldsymbol{D}_1 = \plim \nabla_\phi \left(\frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\boldsymbol{\phi}_0)\right).$$

By the uniform law of large numbers, $\sup_{\phi\in\boldsymbol{\Phi}_0} \left|\nabla_\phi \left(\frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\boldsymbol{\phi})\right) - \boldsymbol{D}_1(\boldsymbol{\phi})\right| = o_p(1)$, and as $\hat{\boldsymbol{\phi}}_T - \boldsymbol{\phi}_0 = o_p(1)$, $\nabla_\phi \left(\frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\bar{\boldsymbol{\phi}}_T)\right) - \boldsymbol{D}_1 = o_p(1)$. Hence,

$$\sqrt{T}\left(\hat{\boldsymbol{\phi}}_T - \boldsymbol{\phi}_0\right) = -\left(\boldsymbol{D}_1^\top \boldsymbol{D}_1\right)^{-1}\boldsymbol{D}_1^\top \left(\sqrt{T}\left(\frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\boldsymbol{\phi}_0) - \boldsymbol{\varphi}_0\right) - \sqrt{T}\left(\tilde{\boldsymbol{\varphi}}_T - \boldsymbol{\varphi}_0\right)\right) + o_p(1).$$

Let $\hat{\boldsymbol{\varphi}}_{T,h}(\boldsymbol{\phi}_0)$ be the unfeasible estimator, obtained by simulating continuous paths for $y_j(t)$, i.e. $y_{j,t,h}^{\phi_0}$, $j = 1, 2$. We claim that for $h = 1, \cdots, H$,

$$\sqrt{T}\left(\hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\boldsymbol{\phi}_0) - \hat{\boldsymbol{\varphi}}_{T,h}(\boldsymbol{\phi}_0)\right) = o_p(1).$$

Let $\boldsymbol{Y}_{t,\Delta,h}^{\phi_0}$ be the vector containing all the regressors in Eq. (18), and let $\hat{\boldsymbol{\varphi}}_{1,T,\Delta,h}(\boldsymbol{\phi}_0)$ be the parameter estimator

of the OLS regression of $y_{1,t,\Delta,h}^{\phi_0}$ on $\boldsymbol{Y}_{t,\Delta,h}^{\phi_0}$. We have:

$$
\begin{aligned}
&\sqrt{T}\left(\hat{\boldsymbol{\varphi}}_{1,T,\Delta,h}\left(\boldsymbol{\phi}_0\right) - \hat{\boldsymbol{\varphi}}_{1,T,h}\left(\boldsymbol{\phi}_0\right)\right) \\
&= \left(\frac{1}{T}\sum_{t=25}^{T}\boldsymbol{Y}_{t,h}^{\phi_0}\boldsymbol{Y}_{t,h}^{\phi_0\top}\right)^{-1}\sqrt{T}\left(\frac{1}{T}\sum_{t=25}^{T}\left(\boldsymbol{Y}_{t,\Delta,h}^{\phi_0}y_{1,t,\Delta,h}^{\phi_0} - \boldsymbol{Y}_{t,h}^{\phi_0}y_{1,t,h}^{\phi_0}\right)\right) \\
&\quad + \sqrt{T}\left(\left(\frac{1}{T}\sum_{t=25}^{T}\boldsymbol{Y}_{t,\Delta,h}^{\phi_0}\boldsymbol{Y}_{t,\Delta,h}^{\phi_0\top}\right)^{-1} - \left(\frac{1}{T}\sum_{t=25}^{T}\boldsymbol{Y}_{t,h}^{\phi_0}\boldsymbol{Y}_{t,h}^{\phi_0\top}\right)^{-1}\right)\left(\frac{1}{T}\sum_{t=25}^{T}\boldsymbol{Y}_{t,\Delta,h}^{\phi_0}y_{1,t,\Delta,h}^{\phi_0}\right).
\end{aligned} \tag{52}
$$

As for the first term on the RHS of (52), $\left(\frac{1}{T}\sum_{t=25}^{T}\boldsymbol{Y}_{t,h}^{\phi_0}\boldsymbol{Y}_{t,h}^{\phi_0\top}\right)^{-1} = O_p(1)$, and by Theorem 2.3 in Pardoux and Talay (1985), we have, for $\varepsilon > 0$ and $\sqrt{T}\Delta \to 0$,

$$
\Pr\left(\left|\frac{1}{\sqrt{T}}\sum_{t=25}^{T}\left(\boldsymbol{Y}_{t,\Delta,h}^{\phi_0}y_{1,t,\Delta,h}^{\phi_0} - \boldsymbol{Y}_{t,h}^{\phi_0}y_{1,t,h}^{\phi_0}\right)\right| > \varepsilon\right) < \frac{1}{\varepsilon}\sqrt{T}\mathrm{E}\left(\left|\boldsymbol{Y}_{t,\Delta,h}^{\phi_0}y_{1,t,\Delta,h}^{\phi_0} - \boldsymbol{Y}_{t,h}^{\phi_0}y_{1,t,h}^{\phi_0}\right|\right) = \sqrt{T}O\left(\Delta\right) = o(1).
$$

The second term on the RHS of Eq. (52) can be dealt with similarly. Thus, we have:

$$
\mathrm{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\phi}}_T - \boldsymbol{\phi}_0\right)\right) = \left(\boldsymbol{D}_1^\top\boldsymbol{D}_1\right)^{-1}\boldsymbol{D}_1^\top\mathrm{Avar}\left(\sqrt{T}\left(\frac{1}{H}\sum_{h=1}^{H}\hat{\boldsymbol{\varphi}}_{T,h}\left(\boldsymbol{\phi}_0\right) - \boldsymbol{\varphi}_0\right) - \sqrt{T}\left(\tilde{\boldsymbol{\varphi}}_T - \boldsymbol{\varphi}_0\right)\right)\boldsymbol{D}_1\left(\boldsymbol{D}_1^\top\boldsymbol{D}_1\right)^{-1},
$$

where,

$$
\begin{aligned}
&\mathrm{Avar}\left(\sqrt{T}\left(\tfrac{1}{H}\sum_{h=1}^{H}\hat{\boldsymbol{\varphi}}_{T,h}\left(\boldsymbol{\phi}_0\right) - \boldsymbol{\varphi}_0\right) - \sqrt{T}\left(\tilde{\boldsymbol{\varphi}}_T - \boldsymbol{\varphi}_0\right)\right) \\
&= \mathrm{Avar}\left(\sqrt{T}\left(\tfrac{1}{H}\sum_{h=1}^{H}\hat{\boldsymbol{\varphi}}_{T,h}\left(\boldsymbol{\phi}_0\right) - \boldsymbol{\varphi}_0\right)\right) + \mathrm{Avar}\left(\sqrt{T}\left(\tilde{\boldsymbol{\varphi}}_T - \boldsymbol{\varphi}_0\right)\right) \\
&\quad - 2\mathrm{Acov}\left(\sqrt{T}\left(\tfrac{1}{H}\sum_{h=1}^{H}\hat{\boldsymbol{\varphi}}_{T,h}\left(\boldsymbol{\phi}_0\right) - \boldsymbol{\varphi}_0\right),\ \sqrt{T}\left(\tilde{\boldsymbol{\varphi}}_T - \boldsymbol{\varphi}_0\right)\right).
\end{aligned}
$$

The last term of the RHS of this equality is zero, because the simulated paths are independent of the sample paths. Moreover, the simulated paths are independent and identically distributed across all simulation replications and, hence,

$$
\mathrm{Avar}\left(\sqrt{T}\left(\tfrac{1}{H}\sum_{h=1}^{H}\hat{\boldsymbol{\varphi}}_{T,h}\left(\boldsymbol{\phi}_0\right) - \boldsymbol{\varphi}_0\right)\right) = \frac{1}{H}\mathrm{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\varphi}}_{T,h}\left(\boldsymbol{\phi}_0\right) - \boldsymbol{\varphi}_0\right)\right),\ \text{for all } h.
$$

Finally, given Assumption B1(ii),

$$
\boldsymbol{J}_1 \equiv \mathrm{Avar}\left(\sqrt{T}\left(\tilde{\boldsymbol{\varphi}}_T - \boldsymbol{\varphi}_0\right)\right) = \mathrm{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\varphi}}_{T,\Delta,h}\left(\boldsymbol{\phi}_0\right) - \boldsymbol{\varphi}_0\right)\right),\ \text{for all } h,
$$

and so

$$
\mathrm{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\phi}}_T - \boldsymbol{\phi}_0\right)\right) = \left(1 + \frac{1}{H}\right)\left(\boldsymbol{D}_1^\top\boldsymbol{D}_1\right)^{-1}\boldsymbol{D}_1^\top\boldsymbol{J}_1\boldsymbol{D}_1\left(\boldsymbol{D}_1^\top\boldsymbol{D}_1\right)^{-1}.
$$

The proposition follows by the central limit theorem for geometrically strong mixing processes.

The estimator of $\hat{\boldsymbol{\theta}}_T$ in Eq. (24)

We have:

Proposition B2: *Under regularity conditions (Assumption B1(i)-(iv) in Appendix B), as $T \to \infty$ and $\Delta\sqrt{T} \to 0$,*

$$
\sqrt{T}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right) \xrightarrow{d} \mathrm{N}\left(\boldsymbol{0}, \boldsymbol{V}_2\right), \quad \boldsymbol{V}_2 = \left(\boldsymbol{D}_2^\top\boldsymbol{D}_2\right)^{-1}\boldsymbol{D}_2^\top\left(\left(1 + \frac{1}{H}\right)\left(\boldsymbol{J}_2 - \boldsymbol{K}_2\right) + \boldsymbol{P}_2\right)\boldsymbol{D}_2\left(\boldsymbol{D}_2^\top\boldsymbol{D}_2\right)^{-1},
$$

where $\boldsymbol{\theta}_0$ is as in Eq. (50), and the four matrices, $\boldsymbol{D}_2$, $\boldsymbol{J}_2$, $\boldsymbol{K}_2$ and $\boldsymbol{P}_2$, are defined in the proof below.

As discussed in the main text, the matrix $\boldsymbol{P}_2$ arises due to parameter estimation error, as the stock price in Eq. (20), is simulated with parameters $\boldsymbol{\theta}$ fixed at their estimates, $\hat{\boldsymbol{\theta}}_{G,T}$. Moreover, the matrix $\boldsymbol{K}_2$ captures the covariance of the structural parameter estimates over all the simulation replications, as well as the covariance between actual and simulated paths, thereby resulting in an improved efficiency, if compared to estimators based on unconditional (simulated) inference.

Proof of Proposition B2: By the same arguments utilized in the proof of Proposition B1,

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right) = -\left(\boldsymbol{D}_2^\top \boldsymbol{D}_2\right)^{-1} \boldsymbol{D}_2^\top \left(\sqrt{T}\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\vartheta}}_{T,\Delta,h}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right) + \boldsymbol{C}_2^\top \sqrt{T}\left(\hat{\boldsymbol{\theta}}_{G,T} - \boldsymbol{\theta}_G\right)\right.$$
$$\left.-\sqrt{T}\left(\tilde{\boldsymbol{\vartheta}}_T - \boldsymbol{\vartheta}_0\right)\right) + o_p(1),$$

where for $\bar{\boldsymbol{\theta}}_{G,T} \in (\hat{\boldsymbol{\theta}}_{G,T}, \boldsymbol{\theta}_G)$,

$$\boldsymbol{D}_2 = \text{plim } \nabla_\theta \left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\vartheta}}_{T,\Delta,h}\left(\boldsymbol{\theta}_0, \bar{\boldsymbol{\theta}}_{G,T}\right)\right), \quad \boldsymbol{C}_2 = \text{plim } \nabla_{\theta_G} \left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\vartheta}}_{T,\Delta,h}\left(\boldsymbol{\theta}_0, \bar{\boldsymbol{\theta}}_{G,T}\right)\right).$$

Therefore:

$$\text{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right)\right) = \left(\boldsymbol{D}_2^\top \boldsymbol{D}_2\right)^{-1} \boldsymbol{D}_2^\top \boldsymbol{J}_0 \boldsymbol{D}_2 \left(\boldsymbol{D}_2^\top \boldsymbol{D}_2\right)^{-1},$$

where

$$\boldsymbol{J}_0 = \text{Avar}\left(\sqrt{T}\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\vartheta}}_{T,\Delta,h}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right)\right) + \text{Avar}\left(\sqrt{T}\left(\tilde{\boldsymbol{\vartheta}}_T - \boldsymbol{\vartheta}_0\right)\right)$$
$$+ \boldsymbol{C}_2^\top \text{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{G,T} - \boldsymbol{\theta}_G\right)\right) \boldsymbol{C}_2 + 2\boldsymbol{C}_2^\top \text{Acov}\left(\sqrt{T}\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\vartheta}}_{T,\Delta,h}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right), \sqrt{T}\left(\hat{\boldsymbol{\theta}}_{G,T} - \boldsymbol{\theta}_G\right)\right)$$
$$- 2\text{Acov}\left(\sqrt{T}\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\vartheta}}_{T,\Delta,h}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right), \sqrt{T}\left(\tilde{\boldsymbol{\vartheta}}_T - \boldsymbol{\vartheta}_0\right)\right)$$
$$- 2\boldsymbol{C}_2^\top \text{Acov}\left(\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{G,T} - \boldsymbol{\theta}_G\right), \sqrt{T}\left(\tilde{\boldsymbol{\vartheta}}_T - \boldsymbol{\vartheta}_0\right)\right). \tag{53}$$

Let $\hat{\boldsymbol{\vartheta}}_{T,h}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right)$ be the infeasible estimator, obtained by simulating continuous paths for the unobservable factor $y_3(t)$ and for $G^{\hat{\boldsymbol{\theta}}_{G,T}}(t)$. By the same arguments as those in the proof of Proposition B1,

$$\text{Avar}\left(\sqrt{T}\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\vartheta}}_{T,\Delta,h}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right)\right) = \text{Avar}\left(\sqrt{T}\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\vartheta}}_{T,h}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right)\right).$$

Paths of the model-implied stock price are obtained through the sample paths of the observable factors $y_{1,t}$ and $y_{2,t}$. Therefore, simulated paths are not independent across simulations, and are not independent of the actual sample paths of stock price and volatility. We have:

$$\text{Avar}\left(\sqrt{T}\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\vartheta}}_{T,h}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right)\right)$$
$$= \frac{1}{H}\text{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\vartheta}}_{T,1}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right)\right)$$
$$+ \frac{1}{H^2}\sum_{h=1}^H \sum_{h'=1, h'\neq h}^H \text{Acov}\left(\sqrt{T}\left(\hat{\boldsymbol{\vartheta}}_{T,h}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right), \sqrt{T}\left(\hat{\boldsymbol{\vartheta}}_{T,h'}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right)\right)$$
$$= \frac{1}{H}\boldsymbol{J}_2 + \frac{H(H-1)}{H^2}\boldsymbol{K}_2,$$

where

$$\boldsymbol{J}_2 = \text{Avar}\left(\sqrt{T}\left(\tilde{\boldsymbol{\vartheta}}_T - \boldsymbol{\vartheta}_0\right)\right) = \text{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\vartheta}}_{T,\Delta,h}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right)\right), \quad \text{for all } h,$$

and

$$\boldsymbol{K}_2 = \frac{1}{H(H-1)}\sum_{h=1}^H \sum_{h'=1, h'\neq h}^H \text{Acov}\left(\sqrt{T}\left(\hat{\boldsymbol{\vartheta}}_{T,h}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right), \sqrt{T}\left(\hat{\boldsymbol{\vartheta}}_{T,h'}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right)\right)$$
$$= \text{Acov}\left(\sqrt{T}\left(\hat{\boldsymbol{\vartheta}}_{T,1}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right), \sqrt{T}\left(\hat{\boldsymbol{\vartheta}}_{T,2}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}_G\right) - \boldsymbol{\vartheta}_0\right)^\top\right).$$

Therefore, using the fact that $\text{Avar}\left(\sqrt{T}\left(\tilde{\boldsymbol{\vartheta}}_T - \boldsymbol{\vartheta}_0\right)\right) = \text{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\vartheta}}_{T,1}\left(\boldsymbol{\theta}_0\right) - \boldsymbol{\vartheta}_0\right)\right) = \boldsymbol{J}_2$, letting $\boldsymbol{P}_2$ denoting the sum of the third, fourth and sixth terms in Eq. (53), and exploiting the expression for $\boldsymbol{J}_0$, we obtain:

$$\boldsymbol{J}_0 = \frac{1}{H}\boldsymbol{J}_2 + \frac{H(H-1)}{H^2}\boldsymbol{K}_2 + \boldsymbol{J}_2 - 2\boldsymbol{K}_2 + \boldsymbol{P}_2 = \left(1 + \frac{1}{H}\right)(\boldsymbol{J}_2 - \boldsymbol{K}_2) + \boldsymbol{P}_2,$$

and, hence:

$$\mathrm{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right)\right) = \left(\boldsymbol{D}_2^\top \boldsymbol{D}_2\right)^{-1} \boldsymbol{D}_2^\top \left(\left(1 + \frac{1}{H}\right)(\boldsymbol{J}_2 - \boldsymbol{K}_2) + \boldsymbol{P}_2\right) \boldsymbol{D}_2 \left(\boldsymbol{D}_2^\top \boldsymbol{D}_2\right)^{-1}.$$

12 *Details on the simulations of the VIX index predicted by the model*

We construct a simulated series of length $\mathcal{T}$ for the VIX index, at a monthly frequency. Since we do not have a closed-form formula for the VIX index, we need to resort to numerical methods aiming to approximate it. We address this issue by simulating the three factors at a daily frequency, which we then use to numerically integrate the daily volatilities. For each simulation draw $h = 1, \cdots, H$, we initialize each monthly path at the values taken by the observable macroeconomic factors, i.e. at $y_{1,t}$, $y_{2,t}$, $t = T_*, \cdots, T_* + \mathcal{T} - 1$, where T_* is the first date where the VIX is available, and at the monthly unconditional mean of the unobservable factor. In the additional experiments of Appendix C, we initialize each monthly path of the unobservable factor at the values taken by (i) the model-implied factor and (ii) the University of Michigan Consumer Sentiment index to generate the statistics, as defined, respectively, by Eq. (28) and by Eq. (64) below. For $i = 1, 2, 3$, $h = 1, \cdots, H$, $k = 0, \cdots, \hat{\Delta}^{-1} - 1$, let $\hat{y}^\lambda_{i,t+k\hat{\Delta},h}$ be the value of the i-th factor, at time $t + k\hat{\Delta}$, for the h-th simulation under the risk-neutral probability, performed with parameter $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0$ and remaining parameters fixed at their estimates obtained in the first and second step of our estimation procedure. $\hat{\Delta}$ will be defined in a moment. Simulations are obtained through a Milstein approximation to the risk-neutral version of Eq. (1),

$$\mathrm{d}y_i(t) = [\kappa_i(\mu_i - y_i(t)) + \bar{\kappa}_i(\bar{\mu}_i - \bar{y}_i(t)) - \pi(y_i)]\mathrm{d}t + \sqrt{\alpha_i + \beta_i y_i(t)}\mathrm{d}\tilde{W}_i(t), \quad i = 1, 2, 3,$$

1 where $\pi(y_i)$ denotes the i-th element of the vector $\boldsymbol{\pi}(\boldsymbol{y})$ in Eq. (7), and $\tilde{W}_i$ is a standard Brownian motion under
2 the risk-neutral probability. We use the discretization step $\hat{\Delta} = \Delta/22$, where Δ is the discretization step used in
3 the first and the second step of our estimation procedure Given Eqs. (8)-(11), the model-based volatility under the
4 risk-neutral measure, at the j-th simulation, is:

5
$$\sigma^2_{t+k\hat{\Delta},h}(\hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\phi}}_T, \hat{\sigma}_{G,T}, \boldsymbol{\lambda}) = \hat{\sigma}^2_G + \frac{\sum_{i=1}^3 \hat{s}^2_{i,T}\left(\hat{\alpha}_{i,T} + \hat{\beta}_{i,T}\hat{y}^\lambda_{i,t+k\hat{\Delta},h}\right)}{\tilde{s}^2_{t+k\hat{\Delta},h}(\hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\phi}}_T, \hat{\sigma}_{G,T}, \boldsymbol{\lambda})}, \tag{54}$$

6 where

7
$$\tilde{s}_{t+k\hat{\Delta},h}(\hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\phi}}_T, \hat{\sigma}_{G,T}, \boldsymbol{\lambda}) = \hat{s}_{0,T} + \sum_{i=1}^3 \hat{s}_{i,T}\hat{y}^\lambda_{i,t+k\hat{\Delta},h}, \tag{55}$$

8 and $\hat{\sigma}_{G,T}$ and $\hat{s}_{l,T}$ $l = 0, \cdots, 3$ are the standard deviation of stochastic secular growth and the reduced-form parameters
9 obtained in the second step of the estimation procedure. Finally, we compute the simulated value of the model-based
10 VIX, $\mathrm{VIX}_{t,\hat{\Delta},h}(\hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\phi}}_T, \boldsymbol{\lambda})$, by integrating volatility over each month, as follows:

11
$$\mathrm{VIX}_{t,\hat{\Delta},h}(\hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\phi}}_T, \hat{\sigma}_{G,T}, \boldsymbol{\lambda}) = \sqrt{\frac{1}{\hat{\Delta}}\sum_{k=0}^{\hat{\Delta}^{-1}-1} \sigma^2_{t+(k+1)\hat{\Delta},h}(\hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\phi}}_T, \hat{\sigma}_{G,T}, \boldsymbol{\lambda})}. \tag{56}$$

12 By repeating the same procedure outlined above H times, we can then generate H paths of length $\mathcal{T}$. From now on,
13 we simplify notation and index all parameter estimators and simulated factors by Δ, rather than $\hat{\Delta}$.

14 *The estimator of $\hat{\boldsymbol{\lambda}}_{\mathcal{T}}$ in Eq. (26)*

15 We have:
16

Proposition B3: *Under regularity conditions (Assumption B1 in Appendix B), if for some $\pi \in (0,1)$, $T, \mathcal{T}$, $\Delta\sqrt{T} \to 0$, $\Delta T \to \infty$, and $\mathcal{T}/T \to \pi$, then:*

$$\sqrt{\mathcal{T}}\left(\hat{\boldsymbol{\lambda}}_{\mathcal{T}} - \boldsymbol{\lambda}_0\right) \xrightarrow{d} \mathrm{N}(\boldsymbol{0}, \boldsymbol{V}_3), \quad \boldsymbol{V}_3 = \left(\boldsymbol{D}_3^\top \boldsymbol{D}_3\right)^{-1} \boldsymbol{D}_3^\top \left(\left(1 + \frac{1}{H}\right)(\boldsymbol{J}_3 - \boldsymbol{K}_3) + \boldsymbol{P}_3\right) \boldsymbol{D}_3 \left(\boldsymbol{D}_3^\top \boldsymbol{D}_3\right)^{-1},$$

17 *where $\boldsymbol{\lambda}_0$ is as in Eq. (51), and the four matrices, $\boldsymbol{D}_3$, $\boldsymbol{J}_3$, $\boldsymbol{K}_3$ and $\boldsymbol{P}_3$, are defined in the proof below.*
18

Proof: Given Assumptions B1(i) and B1(iii), for any $\boldsymbol{\lambda}$ in a compact set $\boldsymbol{\Lambda}_0$, $y^\lambda_{i,t+(k+1)\Delta,h}$, $i = 1, 2, 3$, $h = 1, \cdots, H$, is geometrically β-mixing, and has a stationary distribution with exponential tails. Thus, by Eqs. (54), (55) and (56), $\text{VIX}_{t,\Delta,h}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \sigma_G, \boldsymbol{\lambda}_0)$ is also geometrically β-mixing with exponential tails. Therefore, $\text{VIX}_{t,\Delta,h}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \sigma_G, \boldsymbol{\lambda}_0)$ has enough finite moments to satisfy sufficient conditions for the law of large numbers and the central limit theorem to apply. Next, note that $\text{VIX}_{t,\Delta,h}(\boldsymbol{\theta}, \boldsymbol{\phi}, \sigma_G, \boldsymbol{\lambda})$ is continuously differentiable in the interior of $\boldsymbol{\Phi}_0 \times \boldsymbol{\Theta}_0 \times \boldsymbol{\Sigma}_G \times \boldsymbol{\Lambda}_0$ (for some compact set $\boldsymbol{\Sigma}_G$) and, hence, the uniform law of large numbers also applies. Similarly as in the proof of Propositions B2, we take into account the contribution of parameter estimation error, arising because the risk-neutral paths of the factors are generated using $\hat{\boldsymbol{\phi}}_T, \hat{\boldsymbol{\theta}}_T$ and $\hat{\sigma}_{G,T}$, instead of the unknown $\boldsymbol{\phi}_0, \boldsymbol{\theta}_0$ and σ_G.

By an argument similar to that in the proof of Proposition B1,

$$\sqrt{\mathcal{T}}\left(\hat{\boldsymbol{\lambda}}_{\mathcal{T}} - \boldsymbol{\lambda}_0\right)$$
$$= -\left(\boldsymbol{D}_3^\top \boldsymbol{D}_3\right)^{-1} \boldsymbol{D}_3^\top \left(\sqrt{\mathcal{T}}\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta h}(\hat{\boldsymbol{\phi}}_T, \hat{\boldsymbol{\theta}}_T, \hat{\sigma}_{G,T}, \boldsymbol{\lambda}_0) - \boldsymbol{\psi}_0\right) - \sqrt{\mathcal{T}}\left(\tilde{\boldsymbol{\psi}}_{\mathcal{T}} - \boldsymbol{\psi}_0\right)\right) + o_p(1),$$

where

$$\boldsymbol{D}_3 = \operatorname*{plim}_{\mathcal{T}\to\infty} \nabla_\lambda \left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}(\boldsymbol{\phi}_0, \boldsymbol{\theta}_0, \sigma_G, \boldsymbol{\lambda}_0)\right),$$

and along the same lines as those in the proof of Proposition B2,

$$\sqrt{\mathcal{T}}\left(\frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}(\hat{\boldsymbol{\phi}}_T, \hat{\boldsymbol{\theta}}_T, \hat{\sigma}_{G,T}, \boldsymbol{\lambda}_0) - \boldsymbol{\psi}_0\right)$$
$$= \sqrt{\mathcal{T}}\left(\frac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}(\boldsymbol{\phi}_0, \boldsymbol{\theta}_0, \sigma_G, \boldsymbol{\lambda}_0) - \boldsymbol{\psi}_0\right) + \sqrt{\pi}\boldsymbol{F}_{\theta_0}^\top \sqrt{T}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right) + \sqrt{\pi}\boldsymbol{F}_{\phi_0}^\top \sqrt{T}\left(\hat{\boldsymbol{\phi}}_T - \boldsymbol{\phi}_0\right)$$
$$+ \sqrt{\pi}\boldsymbol{F}_{\sigma_G}^\top \sqrt{T}\left(\hat{\sigma}_{G,T} - \sigma_G\right) + o_p(1),$$

where $\pi = \lim_{T,\mathcal{T}\to\infty} \mathcal{T}/T$, and, with $\bar{\boldsymbol{\theta}}_T, \bar{\boldsymbol{\phi}}_T$ and $\bar{\sigma}_{G,T}$ denoting convex combinations of $(\hat{\boldsymbol{\theta}}_T, \boldsymbol{\theta}_0)$, $(\hat{\boldsymbol{\phi}}_T, \boldsymbol{\phi}_0)$ and $(\hat{\boldsymbol{\sigma}}_{G,T}, \sigma_G)$, respectively,

$$\boldsymbol{F}_{\theta_0}^\top = \operatorname*{plim}_{T,\mathcal{T}\to\infty} \nabla_\theta \left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}(\bar{\boldsymbol{\phi}}_T, \bar{\boldsymbol{\theta}}_T, \bar{\sigma}_{G,T}, \boldsymbol{\lambda}_0)\right),$$

with $\boldsymbol{F}_{\phi_0}^\top$ and $\boldsymbol{F}_{\sigma_G}^\top$ defined analogously. Therefore, by the same argument as those in the proofs of Propositions B1 and B2,

$$\text{Avar}\left(\sqrt{\mathcal{T}}\left(\hat{\boldsymbol{\lambda}}_{\mathcal{T}} - \boldsymbol{\lambda}_0\right)\right) = \left(\boldsymbol{D}_3^\top \boldsymbol{D}_3\right)^{-1} \boldsymbol{D}_3^\top \left(\left(1 + \frac{1}{H}\right)(\boldsymbol{J}_3 - \boldsymbol{K}_3) + \boldsymbol{P}_3\right) \boldsymbol{D}_3 \left(\boldsymbol{D}_3^\top \boldsymbol{D}_3\right)^{-1},$$

where

$$\boldsymbol{J}_3 = \text{Avar}\left(\sqrt{\mathcal{T}}\left(\tilde{\boldsymbol{\psi}}_{\mathcal{T}} - \boldsymbol{\psi}_0\right)\right) = \text{Avar}\left(\sqrt{\mathcal{T}}\left(\hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}(\boldsymbol{\phi}_0, \boldsymbol{\theta}_0, \sigma_G, \boldsymbol{\lambda}_0) - \boldsymbol{\psi}_0\right)\right), \quad \text{for all } h,$$

$$\boldsymbol{K}_3 = \text{Acov}\left(\sqrt{\mathcal{T}}\left(\hat{\boldsymbol{\psi}}_{\mathcal{T},1}(\boldsymbol{\phi}_0, \boldsymbol{\theta}_0, \sigma_G, \boldsymbol{\lambda}_0) - \boldsymbol{\psi}_0\right), \sqrt{\mathcal{T}}\left(\hat{\boldsymbol{\psi}}_{\mathcal{T},2}(\boldsymbol{\phi}_0, \boldsymbol{\theta}_0, \sigma_G, \boldsymbol{\lambda}_0) - \boldsymbol{\psi}_0\right)^\top\right),$$

and

$$\boldsymbol{P}_3 = \pi\boldsymbol{F}_{\theta_0}^\top \text{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right)\right)\boldsymbol{F}_{\theta_0} + \pi\boldsymbol{F}_{\phi_0}^\top \text{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\phi}}_T - \boldsymbol{\phi}_0\right)\right)\boldsymbol{F}_{\phi_0}$$
$$+ \pi\boldsymbol{F}_{\sigma_G}^\top \text{Avar}\left(\sqrt{T}\left(\hat{\sigma}_{G,T} - \sigma_G\right)\right)\boldsymbol{F}_{\sigma_G}$$
$$+ 2\sqrt{\pi}\text{Acov}\left(\sqrt{\mathcal{T}}\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}(\boldsymbol{\phi}_0, \boldsymbol{\theta}_0, \sigma_G, \boldsymbol{\lambda}_0) - \boldsymbol{\psi}_0\right), \boldsymbol{F}_{\phi_0}^\top \sqrt{T}\left(\hat{\boldsymbol{\phi}}_T - \boldsymbol{\phi}_0\right)\right)$$
$$+ 2\sqrt{\pi}\text{Acov}\left(\sqrt{\mathcal{T}}\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}(\boldsymbol{\phi}_0, \boldsymbol{\theta}_0, \boldsymbol{\lambda}_0) - \boldsymbol{\psi}_0\right), \boldsymbol{F}_{\theta_0}^\top \sqrt{T}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right)\right)$$
$$+ 2\sqrt{\pi}\text{Acov}\left(\sqrt{\mathcal{T}}\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}(\boldsymbol{\phi}_0, \boldsymbol{\theta}_0, \boldsymbol{\lambda}_0) - \boldsymbol{\psi}_0\right), \boldsymbol{F}_{\sigma_G}^\top \sqrt{T}\left(\hat{\sigma}_{G,T} - \sigma_G\right)\right)$$
$$- 2\sqrt{\pi}\text{Acov}\left(\sqrt{\mathcal{T}}\left(\tilde{\boldsymbol{\psi}}_{\mathcal{T}} - \boldsymbol{\psi}_0\right), \boldsymbol{F}_{\phi_0}^\top \sqrt{T}\left(\hat{\boldsymbol{\phi}}_T - \boldsymbol{\phi}_0\right)\right)$$
$$- 2\sqrt{\pi}\text{Acov}\left(\sqrt{\mathcal{T}}\left(\tilde{\boldsymbol{\psi}}_{\mathcal{T}} - \boldsymbol{\psi}_0\right), \boldsymbol{F}_{\theta_0}^\top \sqrt{T}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right)\right)$$
$$- 2\sqrt{\pi}\text{Acov}\left(\sqrt{\mathcal{T}}\left(\tilde{\boldsymbol{\psi}}_{\mathcal{T}} - \boldsymbol{\psi}_0\right), \boldsymbol{F}_{\sigma_G}^\top \sqrt{T}\left(\hat{\sigma}_{G,T} - \sigma_G\right)\right).$$

8 *B.2. Bootstrap estimates of the standard errors*

We develop bootstrap standard errors consistent for $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, and $\boldsymbol{V}_3$ of Propositions B1, B2, and B3. We draw B overlapping blocks of length l, with $T = Bl$, of

$$\boldsymbol{X}_t = (y_{1,t}, \cdots, y_{1,t-k_1}, y_{2,t}, \cdots, y_{2,t-k_2}, S_t, \cdots, S_{t-k_3}),$$

where k_1, k_2, k_3 depend on the lags we use in the auxiliary models. The resampled observations are:

$$\boldsymbol{X}_t^* = (y_{1,t}^*, \cdots, y_{1,t-k_1}^*, y_{2,t}^*, \cdots, y_{2,t-k_2}^*, S_t^*, \cdots, S_{t-k_3}^*).$$

1 Let P^* be the probability measure governing the resampled series, $\boldsymbol{X}_t^*$, and let E^*, var^* denote the mean and the
2 variance taken with respect to P^*, respectively. Further $O_p^*(1)$ and $o_p^*(1)$ denote, respectively, a term bounded in
3 probability, and converging to zero in probability, under P^*, conditional on the sample and for all samples but a set
4 of probability measure approaching zero.
5   For the implementation, we use block sizes of approximately $T^{1/4}$ and $T^{1/3}$ (see Lahiri (2003)), which give similar
6 results. The standard errors reported in the main text are based on block sizes of approximately $T^{1/4}$. (Note: Whilst
7 S_t^* does not not necessarily mimic the dependence of S_t, we just use S_t^* to compute R_t^* and Vol_t^*, which indeed mimic
8 the dependence of R_t and Vol_t.)

9 *Bootstrap Standard Errors for $\boldsymbol{\phi}$*

10   The simulated samples for $y_{1,t}$ and $y_{2,t}$ are independent of the actual samples and are also independent across
11 simulation replications. Also, as stated in Proposition B1, the estimators of the auxiliary model parameters, based on
12 actual and simulated samples, have the same asymptotic variance. Hence, there is no need to resample the simulated
13 series.
14   Given that the number of auxiliary model parameters and moment conditions is larger than the number of
15 parameters to be estimated, we need to use an appropriate re-centering term. In the over-identified case, even if the
16 population moment conditions have mean zero, the bootstrap moment conditions do not have mean zero, and hence
17 a proper re-centering term is necessary (see, e.g., Hall and Horowitz (1996)).
       Let $\tilde{\boldsymbol{\varphi}}_{T,i}^*$ be the bootstrap analog to $\tilde{\boldsymbol{\varphi}}_T$ at draw i, and define:

$$\hat{\boldsymbol{\phi}}_{T,i}^* = \arg \min_{\phi \in \boldsymbol{\Phi}_0} \left\| \left(\frac{1}{H} \sum_{h=1}^{H} \left(\hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\phi) - \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\hat{\boldsymbol{\phi}}_T) \right) - \left(\tilde{\boldsymbol{\varphi}}_{T,i}^* - \tilde{\boldsymbol{\varphi}}_T \right) \right) \right\|^2, \quad i = 1, \cdots, B.$$

We compute the bootstrap covariance matrix, as follows:

$$\hat{\boldsymbol{V}}_{1,T,B} = \frac{T}{B} \sum_{i=1}^{B} \left| \hat{\boldsymbol{\phi}}_{T,i}^* - \frac{1}{B} \sum_{i=1}^{B} \hat{\boldsymbol{\phi}}_{T,i}^* \right|_2.$$

18   The next proposition shows that $\left(1 + \frac{1}{H} \right) \hat{\boldsymbol{V}}_{1,T,B}$, is a consistent estimator of $\boldsymbol{V}_1$, thereby allowing to compute
19 asymptotically valid bootstrap standard errors.

Proposition B4: *Under the same assumptions of Proposition B1, if $l/T^{1/2} \to 0$ as $T, B, l \to \infty$, then for all $\varepsilon > 0$,*

$$\Pr\left(\omega : P^*\left(\left\|\left(1 + \frac{1}{H}\right)\hat{\mathbf{V}}_{1,T,B} - \mathbf{V}_1\right\| > \varepsilon\right)\right) \to 0.$$

Proof: By the first order conditions and a mean value expansion around $\hat{\phi}_T$,

$$\mathbf{0} = \nabla_\phi\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\hat{\phi}_T^*)\right)^\top \left(\tfrac{1}{H}\sum_{h=1}^H\left(\hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\hat{\phi}_T^*) - \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\hat{\phi}_T)\right) - (\tilde{\boldsymbol{\varphi}}_T^* - \tilde{\boldsymbol{\varphi}}_T)\right)$$

$$= \nabla_\phi\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\hat{\phi}_T^*)\right)^\top (\tilde{\boldsymbol{\varphi}}_T - \tilde{\boldsymbol{\varphi}}_T^*)$$

$$+ \nabla_\phi\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\hat{\phi}_T^*)\right)^\top \nabla_\phi\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,\Delta,h}(\bar{\phi}_T^*)\right)\left(\hat{\phi}_T^* - \hat{\phi}_T\right),$$

where $\bar{\phi}_T^*$ is some convex combination of $\hat{\phi}_T^*$ and $\hat{\phi}_T$. Hence,

$$\sqrt{T}\left(\hat{\phi}_T^* - \hat{\phi}_T\right)$$

$$= \left(\nabla_\phi\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,h}(\hat{\phi}_T^*)\right)^\top \nabla_\phi\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,h}(\bar{\phi}_T^*)\right)\right)^{-1} \nabla_\phi\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,h}(\hat{\phi}_T^*)\right)^\top \sqrt{T}\,(\tilde{\boldsymbol{\varphi}}_T^* - \tilde{\boldsymbol{\varphi}}_T).$$

The Proposition follows, once we show that:

$$\mathrm{E}^*\left(\sqrt{T}\,(\tilde{\boldsymbol{\varphi}}_T^* - \tilde{\boldsymbol{\varphi}}_T)\right) = o_p(1), \tag{57}$$

$$\mathrm{var}^*\left(\sqrt{T}\,(\tilde{\boldsymbol{\varphi}}_T^* - \tilde{\boldsymbol{\varphi}}_T)\right) = \mathrm{var}\left(\sqrt{T}\,(\tilde{\boldsymbol{\varphi}}_T - \boldsymbol{\varphi}_0)\right) + O_p(l/\sqrt{T}), \tag{58}$$

and for $\varepsilon > 0$,

$$\mathrm{E}^*\left(\left(\sqrt{T}\,\|\tilde{\boldsymbol{\varphi}}_T^* - \tilde{\boldsymbol{\varphi}}_T\|\right)^{2+\varepsilon}\right) = O_p(1). \tag{59}$$

Indeed, under conditions (57)-(58), we have that by the uniform law of large numbers, $\left|\nabla_\phi\left(\tfrac{1}{H}\sum_{h=1}^H \hat{\boldsymbol{\varphi}}_{T,h}(\hat{\phi}_T^*)\right) - \mathbf{D}_1\right| = o_p^*(1)$. Hence,

$$\sqrt{T}\left(\hat{\phi}_T^* - \hat{\phi}_T\right) = \left(\mathbf{D}_1^\top \mathbf{D}_1\right)^{-1}\mathbf{D}_1^\top \sqrt{T}\,(\tilde{\boldsymbol{\varphi}}_T - \tilde{\boldsymbol{\varphi}}_T^*) + o_p^*(1).$$

and, given (58), and recalling that $l/\sqrt{T} \to 0$,

$$\mathrm{var}^*\left(\sqrt{T}\,(\tilde{\boldsymbol{\varphi}}_T^* - \tilde{\boldsymbol{\varphi}}_T)\right) = \mathrm{Avar}\left(\sqrt{T}\,(\tilde{\boldsymbol{\varphi}}_T - \boldsymbol{\varphi}_0)\right) + o_p(1).$$

Given (59), the statement follows by Theorem 1 in Goncalves and White (2005).
Let us show (57), (58) and (59). We have,

$$\sqrt{T}\,(\tilde{\boldsymbol{\varphi}}_T^* - \tilde{\boldsymbol{\varphi}}_T) = \sqrt{T}\left(\left(\tilde{\boldsymbol{\varphi}}_{1,T}^* - \tilde{\boldsymbol{\varphi}}_{1,T}\right), \left(\tilde{\boldsymbol{\varphi}}_{2,T}^* - \tilde{\boldsymbol{\varphi}}_{2,T}\right), (\bar{y}_1^* - \bar{y}_1), (\bar{y}_2^* - \bar{y}_2), \left(\hat{\sigma}_1^{*2} - \hat{\sigma}_1^2\right), \left(\hat{\sigma}_2^{*2} - \hat{\sigma}_2^2\right)\right)^\top.$$

Since each component of $\sqrt{T}\,(\tilde{\boldsymbol{\varphi}}_T^* - \tilde{\boldsymbol{\varphi}}_T)$ can be dealt with in the same way, we only consider $\sqrt{T}\,(\tilde{\boldsymbol{\varphi}}_{1,T}^* - \tilde{\boldsymbol{\varphi}}_{1,T})$. Let $\mathbf{Y}_t$ be the vector containing all the regressors in Eq. (17), and $\mathbf{Y}_t^*$ be its bootstrap counterpart. By the first order conditions,

$$\sqrt{T}\,(\tilde{\boldsymbol{\varphi}}_{1,T}^* - \tilde{\boldsymbol{\varphi}}_{1,T}) = \left(\frac{1}{T}\sum_{t=25}^T \mathbf{Y}_t^* \mathbf{Y}_t^{*\top}\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=25}^T \mathbf{Y}_t^*\left(y_{1,t}^* - \mathbf{Y}_t^{*\top}\tilde{\boldsymbol{\varphi}}_{1,T}\right)$$

$$= \left(\mathrm{E}(\mathbf{Y}_t\mathbf{Y}_t^\top)\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=25}^T \mathbf{Y}_t^*\left(y_{1,t}^* - \mathbf{Y}_t^{*\top}\tilde{\boldsymbol{\varphi}}_{1,T}\right) + o_p^*(1),$$

as $\frac{1}{T}\sum_{t=25}^{T} \boldsymbol{Y}_t^* \boldsymbol{Y}_t^{*\top} - \mathrm{E}^* \left(\frac{1}{T}\sum_{t=25}^{T} \boldsymbol{Y}_t^* \boldsymbol{Y}_t^{*\top} \right) = o_p^*(1)$, and $\mathrm{E}^* \left(\frac{1}{T}\sum_{t=25}^{T} \boldsymbol{Y}_t^* \boldsymbol{Y}_t^{*\top} \right) = \frac{1}{T}\sum_{t=25}^{T} \boldsymbol{Y}_t \boldsymbol{Y}_t^{\top} + O_p(l/T) = \mathrm{E}\left(\boldsymbol{Y}_t \boldsymbol{Y}_t^{\top} \right) + o_p(1)$. We have,

$$\mathrm{E}^* \left(\sqrt{T} \left(\tilde{\boldsymbol{\varphi}}_{1,T}^* - \tilde{\boldsymbol{\varphi}}_{1,T} \right) \right) = \mathrm{E}(\boldsymbol{Y}_t \boldsymbol{Y}_t^{\top}) \frac{1}{T}\sum_{t=25}^{T} \boldsymbol{Y}_t \left(y_{1,t} - \boldsymbol{Y}_t^{\top} \tilde{\boldsymbol{\varphi}}_{1,T} \right) + O_p(l/\sqrt{T}) = o_p(1).$$

This proves (57). Next,

$$
\begin{aligned}
&\mathrm{var}^* \left(\sqrt{T} \left(\tilde{\boldsymbol{\varphi}}_{1,T}^* - \tilde{\boldsymbol{\varphi}}_{1,T} \right) \right) \\
&= \left(\mathrm{E}^*(\boldsymbol{Y}_t^* \boldsymbol{Y}_t^{*\top}) \right)^{-1} \mathrm{var}^* \left(\frac{1}{T}\sum_{t=25}^{T} \boldsymbol{Y}_t^* \left(y_{1,t}^* - \boldsymbol{Y}_t^{*\top} \tilde{\boldsymbol{\varphi}}_{1,T} \right) \right) \left(\mathrm{E}^*(\boldsymbol{Y}_t^* \boldsymbol{Y}_t^{*\top}) \right)^{-1} + o_p(1) \\
&= \left(\mathrm{E}(\boldsymbol{Y}_t \boldsymbol{Y}_t^{\top}) \right)^{-1} \left(\frac{1}{T}\sum_{j=-l}^{l}\sum_{t=25+l}^{T-l} \boldsymbol{Y}_t \boldsymbol{Y}_{t-j}^{\top} \tilde{\epsilon}_{1,t} \tilde{\epsilon}_{1,t-j} \right) \left(\mathrm{E}(\boldsymbol{Y}_t \boldsymbol{Y}_t^{\top}) \right)^{-1} + o_p(1) \\
&= \mathrm{Avar} \left(\sqrt{T} \left(\tilde{\boldsymbol{\varphi}}_{1,T} - \boldsymbol{\varphi}_{1,0} \right) \right) + o_p(1),
\end{aligned}
$$

where $\tilde{\epsilon}_{1,t} = y_{1,t} - \boldsymbol{Y}_t^{\top} \tilde{\boldsymbol{\varphi}}_{1,T}$. This proves (58). Finally, as $\frac{1}{T}\sum_{t=25}^{T} \boldsymbol{Y}_t \boldsymbol{Y}_t^{\top}$ is full rank, for a generic constant C, and $\varepsilon > 0$,

$$\mathrm{E}^* \left(\left(\sqrt{T} \left\| \tilde{\boldsymbol{\varphi}}_{1,T}^* - \tilde{\boldsymbol{\varphi}}_{1,T} \right\| \right)^{2+\varepsilon} \right) \leq C \mathrm{E}^* \left\| \frac{1}{\sqrt{T}}\sum_{t=25}^{T} \boldsymbol{Y}_t^{*\top} \left(y_{1,t}^* - \boldsymbol{Y}_t^{*\top} \tilde{\boldsymbol{\varphi}}_{1,T} \right) \right\|^{2+\varepsilon}.$$

By Lemma 2.1 in Goncalves and White (2005), $\mathrm{E}\left(\mathrm{E}^* \left\| \frac{1}{\sqrt{T}}\sum_{t=25}^{T} \boldsymbol{Y}_t^{*\top} \left(y_{1,t}^* - \boldsymbol{Y}_t^{*\top} \tilde{\boldsymbol{\varphi}}_{1,T} \right) \right\|^{2+\varepsilon} \right) = O(1)$. Hence, (59) follows by Markov inequality.

Bootstrap Standard Errors for $\boldsymbol{\theta}$

The model-based stock price series is simulated using the actual samples of the observable factors, and simulated samples for the unobservable factor and secular growth, $\ln G_{t,\Delta,h}^{\hat{\theta}_{G,T}}$. Thus, we need to take into account the contribution of $\boldsymbol{K}_2$, the covariance between simulated and sample paths, as well as the contribution of $\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{G,T} - \boldsymbol{\theta}_G \right)$. To construct the resampled simulated stock prices through Eq. (20), we need to resample secular growth, $\ln G_{t,\Delta,h}^{\theta_G}$ through $\hat{\boldsymbol{\theta}}_{G,T}^*$, the bootstrap analog to $\hat{\boldsymbol{\theta}}_{G,T}$. As secular growth is a geometric Brownian motion, we cannot use the block bootstrap to obtain $\hat{\boldsymbol{\theta}}_{G,T}^*$. Instead, we rely on the residual-based bootstrap of Paparoditis and Politis (2003). Let $\hat{\epsilon}_t = \left(\ln G_t - \ln G_{t-1} - \hat{g}_T + \frac{1}{2}\hat{\sigma}_{G,T}^2 \right)/\hat{\sigma}_{G,T}$, where G_t is the secular growth, extracted through the Hodrick-Prescott filter, as discussed in the main text. Resample from $\hat{\epsilon}_t - \frac{1}{T}\sum_{t=1}^{T} \hat{\epsilon}_t$, to obtain $\hat{\epsilon}_1^*, \cdots, \hat{\epsilon}_T^*$. Next, define

$$
\ln G_t^* = \begin{cases} \ln G_1, & \text{for } t = 1 \\ \ln G_{t-1}^* + \hat{g}_T - \frac{1}{2}\hat{\sigma}_{G,T}^2 + \hat{\sigma}_{G,T}\hat{\epsilon}_1^*, & \text{for } t = 2, \cdots, T \end{cases}
$$

Use $\ln G_t^*$ to get the bootstrap estimator, $\hat{\boldsymbol{\theta}}_{G,T}^* = \left(\hat{g}_T^*, \hat{\sigma}_{G,T}^* \right)$. Use Eq. (4), to generate $\ln G_{t,\Delta,h}^{\hat{\theta}_{G,T}^*}$, and resample blocks from it, to obtain $\ln G_{t,\Delta,h}^{*\hat{\theta}_{G,T}^*}$. Construct the resampled simulated stock price series as:

$$\ln S_{t,\Delta,h}^{*\theta}(\hat{\boldsymbol{\theta}}_{G,T}^*) = \ln G_{t,\Delta,h}^{*\hat{\theta}_{G,T}^*} + \ln \left(s_0 + s_1 y_{1,t}^* + s_2 y_{2,t}^* + Z_{t,\Delta,h}^{\theta_u,*} \right), \tag{60}$$

where $Z_{t,\Delta,h}^{\theta_u,*}$ is resampled from the simulated unobservable process $Z_{t,\Delta,h}^{\theta_u}$, and use $S_{t,\Delta,h}^{*\theta}(\hat{\boldsymbol{\theta}}_{G,T}^*)$ to construct $R_{t,\Delta,h}^*(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}_{G,T}^*)$ and $\mathrm{Vol}_{t,\Delta,h}^*(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}_{G,T}^*)$. Define,

$$\tilde{\boldsymbol{\vartheta}}_T^* = \left(\tilde{\boldsymbol{\vartheta}}_{1,T}^*, \tilde{\boldsymbol{\vartheta}}_{2,T}^*, \bar{R}^*, \overline{\mathrm{Vol}}^* \right)^{\top},$$

where $\tilde{\boldsymbol{\vartheta}}_{1,T}^*, \tilde{\boldsymbol{\vartheta}}_{2,T}^*$ are the estimators of the auxiliary models obtained using resampled observations, and $\overline{R}^*, \overline{\mathrm{Vol}}^*$ are the sample means of $R_t^* = \ln(S_t^*/S_{t-12}^*)$ and $\mathrm{Vol}_t^* = \sqrt{6\pi} \cdot \frac{1}{12}\sum_{i=1}^{12} \left| \ln \left(S_{t+1-i}^*/S_{t-i}^* \right) \right|$, with S_t^* being the resampled

series of the observable stock price process S_t, and

$$\hat{\boldsymbol{\vartheta}}^*_{T,\Delta,h}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T}) = \left(\hat{\boldsymbol{\vartheta}}^*_{1,T,\Delta,h}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T}), \hat{\boldsymbol{\vartheta}}^*_{2,T,\Delta,h}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T}), \bar{R}^*_{\Delta,h}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T}), \overline{\mathrm{Vol}}^*_{\Delta,h}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T})\right)^\top,$$

where $\hat{\boldsymbol{\vartheta}}^*_{1,T,\Delta,h}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T})$ and $\hat{\boldsymbol{\vartheta}}^*_{2,T,\Delta,h}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T})$ are the parameters of the auxiliary models estimated using resampled simulated observations, and $\bar{R}^*_{\Delta,h}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T})$, $\overline{\mathrm{Vol}}^*_{\Delta,h}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T})$ are the sample means of $R^*_{t,\Delta,h}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T})$ and $\mathrm{Vol}^*_{t,\Delta,h}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T})$. Define:

$$\hat{\boldsymbol{\theta}}^*_{T,i} = \arg\min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0}\left\|\frac{1}{H}\sum_{h=1}^H\left(\hat{\boldsymbol{\vartheta}}^*_{T,\Delta,h,i}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T,i}) - \hat{\boldsymbol{\vartheta}}^\Delta_{T,h}(\hat{\boldsymbol{\theta}}_T,\hat{\boldsymbol{\theta}}_{G,T,i})\right) - \left(\tilde{\boldsymbol{\vartheta}}^*_{T,i} - \tilde{\boldsymbol{\vartheta}}_T\right)\right\|^2,\quad i=1,\cdots,B,$$

where $\hat{\boldsymbol{\vartheta}}^*_{T,\Delta,h,i}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T,i})$ and $\tilde{\boldsymbol{\vartheta}}^*_{T,i}$ denote the values of $\hat{\boldsymbol{\vartheta}}^*_{T,\Delta,h}\left(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}^*_{G,T,i}\right)$ and $\tilde{\boldsymbol{\vartheta}}^*_T$ at the i-th bootstrap replication. The bootstrap covariance matrix is:

$$\hat{\boldsymbol{V}}_{2,T,B} = \frac{T}{B}\sum_{i=1}^B\left|\hat{\boldsymbol{\theta}}^*_{T,i} - \frac{1}{B}\sum_{i=1}^B\hat{\boldsymbol{\theta}}^*_{T,i}\right|_2.$$

The next proposition shows that $\hat{\boldsymbol{V}}_{2,T,B}$ is a consistent estimator of $\boldsymbol{V}_2$, and can then be used to obtain asymptotically valid bootstrap standard errors.

Proposition B5: *Under the same assumptions of Proposition B2, if $l/T^{1/2}\to 0$ as $T,B,l\to\infty$, then, for all $\varepsilon > 0$,*

$$\Pr\left(\omega : P^*\left(\left|\hat{\boldsymbol{V}}_{2,T,B} - \boldsymbol{V}_2\right| > \varepsilon\right)\right) \to 0.$$

Proof: By a similar argument as that in the proof of Proposition B4,

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}^*_T - \hat{\boldsymbol{\theta}}_T\right)$$

$$= -\left(\boldsymbol{D}_2^\top\boldsymbol{D}_2\right)^{-1}\boldsymbol{D}_2^\top\sqrt{T}\left(\frac{1}{H}\sum_{h=1}^H\left(\hat{\boldsymbol{\vartheta}}^*_{T,\Delta,h}(\hat{\boldsymbol{\theta}}_T,\hat{\boldsymbol{\theta}}_{G,T}) - \hat{\boldsymbol{\vartheta}}_{T,\Delta,h}(\hat{\boldsymbol{\theta}}_T,\hat{\boldsymbol{\theta}}_{G,T})\right) - \left(\tilde{\boldsymbol{\vartheta}}^*_T - \tilde{\boldsymbol{\vartheta}}_T\right)\right) + o_{p^*}(1)$$

$$= -\left(\boldsymbol{D}_2^\top\boldsymbol{D}_2\right)^{-1}\boldsymbol{D}_2^\top\left(\sqrt{T}\left(\frac{1}{H}\sum_{h=1}^H\left(\hat{\boldsymbol{\vartheta}}^*_{T,\Delta,h}(\hat{\boldsymbol{\theta}}_T,\hat{\boldsymbol{\theta}}_{G,T}) - \hat{\boldsymbol{\vartheta}}_{T,\Delta,h}(\hat{\boldsymbol{\theta}}_T,\hat{\boldsymbol{\theta}}_{G,T})\right) - \left(\tilde{\boldsymbol{\vartheta}}^*_T - \tilde{\boldsymbol{\vartheta}}_T\right)\right)$$

$$+\boldsymbol{C}_2^\top\sqrt{T}\left(\hat{\boldsymbol{\theta}}^*_{G,T} - \hat{\boldsymbol{\theta}}_{G,T}\right)\right) + o_{p^*(1)}.$$

Moreover, along the lines of the proof of Proposition B4, we can show that

$$\mathrm{E}^*\left(\sqrt{T}\left(\hat{\boldsymbol{\theta}}^*_T - \hat{\boldsymbol{\theta}}_T\right)\right) = o_{p^*}(1),$$

and:

$$\mathrm{Var}^*\left(\sqrt{T}\left(\hat{\boldsymbol{\theta}}^*_T - \hat{\boldsymbol{\theta}}_T\right)\right)$$

$$= \left(\boldsymbol{D}_2^\top\boldsymbol{D}_2\right)^{-1}\boldsymbol{D}_2^\top\mathrm{Var}^*\left(\left(\sqrt{T}\frac{1}{H}\sum_{h=1}^H\left(\hat{\boldsymbol{\vartheta}}^*_{T,\Delta,h}(\hat{\boldsymbol{\theta}}_T,\hat{\boldsymbol{\theta}}_{G,T}) - \hat{\boldsymbol{\vartheta}}_{T,\Delta,h}(\hat{\boldsymbol{\theta}}_T,\hat{\boldsymbol{\theta}}_{G,T})\right)\right)\right.$$

$$\left.-\sqrt{T}\left(\tilde{\boldsymbol{\vartheta}}^*_T - \tilde{\boldsymbol{\vartheta}}_T\right) + \boldsymbol{C}_2^\top\sqrt{T}\left(\hat{\boldsymbol{\theta}}^*_{G,T} - \hat{\boldsymbol{\theta}}_{G,T}\right)\right)\boldsymbol{D}_2\left(\boldsymbol{D}_2^\top\boldsymbol{D}_2\right)^{-1} + o_p(1)$$

$$= \left(\boldsymbol{D}_2^\top\boldsymbol{D}_2\right)^{-1}\boldsymbol{D}_2^\top\mathrm{Avar}\left(\left(\sqrt{T}\frac{1}{H}\sum_{h=1}^H\left(\hat{\boldsymbol{\vartheta}}_{T,\Delta,h}(\boldsymbol{\theta}_0,\boldsymbol{\theta}_G) - \boldsymbol{\vartheta}_0\right)\right)\right.$$

$$\left.-\sqrt{T}\left(\tilde{\boldsymbol{\vartheta}}_T - \boldsymbol{\vartheta}_0\right) + \boldsymbol{C}_2^\top\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{G,T} - \boldsymbol{\theta}_G\right)\right)\boldsymbol{D}_2\left(\boldsymbol{D}_2^\top\boldsymbol{D}_2\right)^{-1} + o_p(1).$$

Hence, $\mathrm{Var}^*\left(\sqrt{T}\left(\hat{\boldsymbol{\theta}}^*_T - \hat{\boldsymbol{\theta}}_T\right)\right) = \mathrm{Avar}\left(\sqrt{T}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right)\right) + o_p(1)$.

Finally, under the parameter restrictions in Assumptions B1(i) and B1(iii), Minkowski's inequality ensures that

$$\mathrm{E}^*\left(\left(\sqrt{T}\left\|\hat{\boldsymbol{\theta}}^*_T - \hat{\boldsymbol{\theta}}_T\right\|\right)^{2+\varepsilon}\right) = O_p(1).$$

4   The statement then follows from Theorem 1 in Goncalves and White (2005).

1   *Bootstrap Standard Errors for $\boldsymbol{\lambda}$*

As mentioned in the main text, the model free VIX index series is available only from 1990 and so in the third step we have a sample of length $\mathcal{T}$, instead of length T. Thus, we need to resample $y_{1,t}, y_{2,t}, S_t$ and VIX_t from the shorter sample, using blocksize l and number of blocks B, so that $lB = \mathcal{T}$. Also, we need to resample the unobservable factor from a sample of length $\mathcal{T}$, at the parameter estimate of $\boldsymbol{\theta}_u$ obtained in the previous step, $\hat{y}_{3,t,\Delta,h}^{\theta_u}$ say. Let $\text{VIX}_{t,\Delta,h}^*(\boldsymbol{y}_t^*; \hat{\boldsymbol{\phi}}_T^*, \hat{\boldsymbol{\theta}}_T^*, \hat{\sigma}_{G,T}^*, \boldsymbol{\lambda})$ be the resampled model-based VIX, according to Eq. (55). Finally, let

$$\tilde{\boldsymbol{\psi}}_{\mathcal{T}}^* = \left(\tilde{\boldsymbol{\psi}}_{1,\mathcal{T}}^*, \overline{\text{VIX}}^*, \hat{\sigma}_{\text{VIX}}^* \right)^\top,$$

2   where $\tilde{\boldsymbol{\psi}}_{1,\mathcal{T}}^*$ is the parameter vector for the auxiliary model, estimated using $y_{1,t}^*, y_{2,t}^*$, and VIX_t^*, with VIX_t^* being
3   the resampled series of the model-free VIX, and $\overline{\text{VIX}}^*, \hat{\sigma}_{\text{VIX}}^*$ are the sample mean and standard deviation of VIX_t^*,
4   and:

$$\hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}^*(\hat{\boldsymbol{\theta}}_T^*, \hat{\boldsymbol{\phi}}_T^*, \hat{\sigma}_{G,T}^*, \boldsymbol{\lambda})$$
$$= \left(\hat{\boldsymbol{\psi}}_{1,\mathcal{T},\Delta,h}^*(\hat{\boldsymbol{\theta}}_T^*, \hat{\boldsymbol{\phi}}_T^*, \hat{\sigma}_{G,T}^*, \boldsymbol{\lambda}), \overline{\text{VIX}}_{\Delta,h}^*(\hat{\boldsymbol{\theta}}_T^*, \hat{\boldsymbol{\phi}}_T^*, \hat{\sigma}_{G,T}^*, \boldsymbol{\lambda}), \tilde{\sigma}_{\Delta,h,\text{VIX}}^*(\hat{\boldsymbol{\theta}}_T^*, \hat{\boldsymbol{\phi}}_T^*, \hat{\sigma}_{G,T}^*, \boldsymbol{\lambda}) \right)^\top,$$

where $\hat{\boldsymbol{\psi}}_{1,\mathcal{T},\Delta,h}^*(\hat{\boldsymbol{\phi}}_T^*, \hat{\boldsymbol{\theta}}_T^*, \hat{\sigma}_{G,T}^*, \boldsymbol{\lambda})$ is the parameter vector for the auxiliary model, estimated using $y_{1,t}^*, y_{2,t}^*$, and $\overline{\text{VIX}}_{\Delta,h}^*(\hat{\boldsymbol{\theta}}_T^*, \hat{\boldsymbol{\phi}}_T^*, \hat{\sigma}_{G,T}^*, \boldsymbol{\lambda})$ and $\tilde{\sigma}_{\Delta,h,\text{VIX}}^*(\hat{\boldsymbol{\theta}}_T^*, \hat{\boldsymbol{\phi}}_T^*, \hat{\sigma}_{G,T}^*, \boldsymbol{\lambda})$ are the sample mean and standard deviation of $\text{VIX}_{t,\Delta,h}^*(\hat{\boldsymbol{\phi}}_T^*, \hat{\boldsymbol{\theta}}_T^*, \hat{\sigma}_{G,T}^*, \boldsymbol{\lambda})$. Define,

$$\hat{\boldsymbol{\lambda}}_{\mathcal{T}}^* = \arg\min_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0} \left\| \left(\frac{1}{H} \sum_{h=1}^H \left(\hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}^*(\hat{\boldsymbol{\phi}}_T^*, \hat{\boldsymbol{\theta}}_T^*, \hat{\sigma}_{G,T}^*, \boldsymbol{\lambda}) - \hat{\boldsymbol{\psi}}_{\mathcal{T},\Delta,h}(\hat{\boldsymbol{\phi}}_T, \hat{\boldsymbol{\theta}}_T, \hat{\sigma}_{G,T}, \hat{\boldsymbol{\lambda}}_{\mathcal{T}}) \right) - \left(\tilde{\boldsymbol{\psi}}_{\mathcal{T}}^* - \tilde{\boldsymbol{\psi}}_{\mathcal{T}} \right) \right) \right\|^2.$$

Construct the bootstrap covariance matrix, as

$$\hat{\boldsymbol{V}}_{3,\mathcal{T},B} = \frac{\mathcal{T}}{B} \sum_{i=1}^B \left| \hat{\boldsymbol{\lambda}}_{\mathcal{T},i}^* - \frac{1}{B} \sum_{i=1}^B \hat{\boldsymbol{\lambda}}_{\mathcal{T},i}^* \right|_2,$$

5   where $\hat{\boldsymbol{\lambda}}_{\mathcal{T},i}^*$ denotes the value of $\hat{\boldsymbol{\lambda}}_{\mathcal{T}}^*$ at the i-th bootstrap replication.
6   The next proposition is the counterpart to Propositions B4 and B5. It shows that $\hat{\boldsymbol{V}}_{3,\mathcal{T},B}$ is a consistent estimator
7   of $\boldsymbol{V}_3$, and can then provide asymptotically valid bootstrap standard errors.
8

Proposition B6: *Under the same assumptions of Proposition B3, if $l/\mathcal{T}^{1/2} \to 0$ as $T, \mathcal{T}, B, l \to \infty$, then, for all $\varepsilon > 0$,*
$$\Pr\left(\omega : P^*\left(\left| \hat{\boldsymbol{V}}_{3,\mathcal{T},B} - \boldsymbol{V}_3 \right| > \varepsilon \right) \right) \to 0.$$

9   *Proof:* Follows by arguments nearly identical to those in the proof of Proposition B5.

C.   Supplemental material for Section 4

11 We provide additional results aiming to: (i) ascertain how the model-implied statistics match those of the data
12 (in Section C.1), and (ii) assess some implications of the model estimates for the dynamics of dividends (in Section
13 C.2). Finally, (iii) we conduct experiments to check whether our variance decompositions and out-of-sample results
14 are robust to the modification of the methodology we use to integrate out the unobserved factor (in Section C.3).

C.1.   *Model-implied dividends dynamics*

16 We explore the implications our three-step estimation procedure has on the dynamics of the dividends. Consider
17 the system of equations $s_i = f_i(\boldsymbol{s}_{\neq i}, \boldsymbol{\lambda}, \boldsymbol{\phi}, \boldsymbol{\theta}_G, \boldsymbol{\theta}; \boldsymbol{\delta}, \lambda_G)$, $i = 0, 1, 2, 3$, for four functions $f_i(\boldsymbol{s}_{\neq i}, \boldsymbol{\lambda}, \boldsymbol{\phi}, \boldsymbol{\theta}_G, \boldsymbol{\theta}; \boldsymbol{\delta}, \lambda_G)$
18 given by Eqs. (9)-(10)-(11), where $\boldsymbol{s}_{\neq i}$ denotes the parameter vector that includes all the elements s_j except s_i, and
19 the remaining notation is as in the main text. Given the estimates of $\boldsymbol{s}, \boldsymbol{\lambda}, \boldsymbol{\phi}, \boldsymbol{\theta}_G$, and $\boldsymbol{\theta}$, say $\hat{\boldsymbol{s}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}_G$ and $\hat{\boldsymbol{\theta}}$
20 reported in Section 4 (see Tables 1, 2 and 4), we set $r = 0.01$, and search for values of $\boldsymbol{\delta}$ and λ_G that jointly minimize

the criterion,

$$\Xi\left(\boldsymbol{\delta}, \lambda_G\right) \equiv \sum_{i=0}^{3} (\hat{s}_i - f_i(\hat{\boldsymbol{s}}_{\neq i}, \hat{\boldsymbol{\lambda}}, \hat{\phi}, \hat{\boldsymbol{\theta}}_G, \hat{\boldsymbol{\theta}}; \boldsymbol{\delta}, \lambda_G))^2, \tag{61}$$

as well as moment conditions relying on an auxiliary model for the real dividends (see Eq. (63) below), along with the mean and the standard deviation of real dividends, say $\bar{\delta}$ and $\hat{\sigma}_\delta$.

To generate moment conditions for the dividends, we simulate H paths of length T of the unobservable factor $y_3(t)$, and the unobservable secular growth, $G(t)$, using a Milstein approximation with discrete interval Δ, using the parameter estimates of $\boldsymbol{\theta}_u$ and $\boldsymbol{\theta}_G$ in Table 2, and sample them at the same frequency as the data, obtaining the series $y_{3,t,\Delta,h}^{\theta_u}$ and $G_{t,\Delta,h}^{\theta_G}$. Likewise, let $\mathrm{Div}_{t,\Delta,h}^{\theta_u,\theta_G,\delta}$ be the simulated series of the dividends, when the parameters are fixed at $\boldsymbol{\theta}$:

$$\ln \mathrm{Div}_{t,\Delta,h}^{\theta_u,\theta_G,\delta} = \ln G_{t,\Delta,h}^{\theta_G} + \ln\left(\delta_0 + \delta_1 y_{1,t} + \delta_2 y_{2,t} + \delta_3 y_{3,t,\Delta,h}^{\theta_u}\right), \tag{62}$$

where $y_{1,t}$ and $y_{2,t}$ denote gross inflation and gross industrial production growth. Eq. (62) is, naturally, the simulated counterpart to Eqs. (2)-(3), with $G_{0,\Delta,h}^{\theta_G} \equiv 1$, as in Eq. (4). We fix the intercept, δ_0, so as to make the model-implied average of the detrended dividends match its empirical counterpart: $\delta_0 = \overline{\mathrm{Div}}^d - \delta_1 \bar{y}_1 - \delta_2 \bar{y}_2 - \delta_3$, where $\overline{\mathrm{Div}}^d$ denotes the sample mean of the detrended dividends, $\mathrm{Div}_t^d \equiv e^{-gt}\mathrm{Div}_t$, Div_t is the observed real dividend at time t, and finally, $\bar{y}_1$ and $\bar{y}_2$ are the sample means of two macroeconomic factors gross inflation and gross industrial production growth, $y_{1,t}$ and $y_{2,t}$, depicted in Figure 1. Real dividends are defined as dividends divided by the consumer price index, and dividend data are obtained from Robert Shiller's website (`http://www.econ.yale.edu/~shiller/`), covering monthly data for the period from January 1950 to December 2006. Next, define yearly dividend growth as $\mathrm{DG}_t \equiv \ln(\mathrm{Div}_t/\mathrm{Div}_{t-12})$, and let $\mathrm{DG}_{t,\Delta,h}^{\theta_u,\theta_G,\delta}$ be the simulated counterparts to DG_t. The auxiliary model for dividend growth is:

$$\mathrm{DG}_t = a^{\mathrm{D}} + \sum_{i\in\{12,24\}} b_{1,i}^{\mathrm{D}} y_{1,t-i} + \sum_{i\in\{12,24\}} b_{2,i}^{\mathrm{D}} y_{2,t-i} + \epsilon_t^{\mathrm{D}}, \tag{63}$$

and is, naturally, the same as that we use to fit the model-implied dividend growth, $\mathrm{DG}_{t,\Delta,h}^{\theta_u,\theta_G,\delta}$. Our optimization leads us to find that $\lambda_G = 1.3901$ and the values of $\boldsymbol{\delta}$ reported in Table C.1 below.

Table C.1

Calibrated values of $\boldsymbol{\delta}$, the parameter vector relating to the dividend process in Eqs. (2)-(3), as implied by the three-step estimation procedure.

δ_0	δ_1	δ_2	δ_3
0.0382	-0.0302	0.0291	0.0006

Table C.2 (Panel E) reports parameter estimates for the auxiliary model for the real dividends in Eq. (63). Figure C.1 depicts the dynamics of real dividend growth, DG_t, as well as its simulated counterparts, obtained by feeding Eq. (62) with the realization of the two macroeconomic factors, gross inflation, $y_{1,t}$, and gross industrial production growth, $y_{2,t}$, and by averaging over the cross-section of 1000 simulations of secular growth and the unobserved factor and, finally, by fixing the parameters to $\hat{\boldsymbol{\theta}}_u$ and $\hat{\boldsymbol{\theta}}_G$ (as reported in Table 2 of Section 4), and the calibrated values for $\boldsymbol{\delta}$ reported in Table C.1.

The calibrated parameter values in Table C.1 suggest real dividend growth is procyclical in our model, at least because it positively links to gross industrial production growth, through the positive coefficient value, δ_2. While it also negatively relates to inflation (due to $\delta_1 < 0$), it is overall procyclical, in that our model-implied dividend growth drops over *all* NBER recession episodes of our sample, mimicking the behavior of their observed counterparts, both qualitatively (especially over the last ten years in the sample, and quantitatively (as seen from the parameter estimates for the auxiliary model in Panel E of Table C.2). Note that the negative coefficient estimates of (the sum of) $b_{2,12}^{\mathrm{D}}$ and $b_{2,24}^{\mathrm{D}}$ are consistent with a procyclical dividends behavior, given a mean-reverting behavior of the industrial production growth. Intuitively, bad times (when industrial production is low) are followed by good. But good times are those where detrended dividends also increase. Therefore, a slowdown in industrial production growth is a predictor of high dividend growth. (A similar explanation holds for the negative values of the macroeconomic loadings resulting from the estimates of the auxiliary regressions relating to the asset returns in Panel B, and the positive values of the short-term macroeconomic loadings relating to the auxiliary regressions for volatility reported in Panel C.)

To illustrate with a simple example, consider a case where detrended dividends positively link to a single state variable tracking the business cycle conditions $x(t)$, say, such that $\delta(t) = \delta_0 + \delta_x x(t)$, for two constant δ_0 and δ_x, and where $\delta_x > 0$. Assume, then, and critically, that $x(t)$ is mean-reverting, with unconditional expectation μ, speed of adjustment $\kappa > 0$, and some volatility coefficient $\sigma(x)$,

$$dx(t) = \kappa(\mu - x(t))\,dt + \sigma(x(t))\,dW(t),$$

where $W(t)$ is a Brownian motion. Then, it is straightforward to show that $E_{t-12}(\delta(t) - \delta(t-12)) = a_0 - a_1 x(t-12)$, where E_t denotes the expectation taken conditionally upon the information set as of time t, and $a_0 \equiv \delta_x (1 - e^{-12\kappa})\mu$ and $a_1 \equiv \delta_x (1 - e^{-12\kappa})$. That is, if $x(t)$ is mean-reverting, $\kappa > 0$, and $\delta(t)$ is procyclical, $\delta_x > 0$, expected changes in detrended dividends negatively link to past values of $x(t)$, i.e. $a_1 > 0$. This reasoning generalizes to a multivariate case, although the presence of feedbacks between macroeconomic variables might then dilute the contribution of each variable as a predictor of future detrended dividends.

Real dividend growth
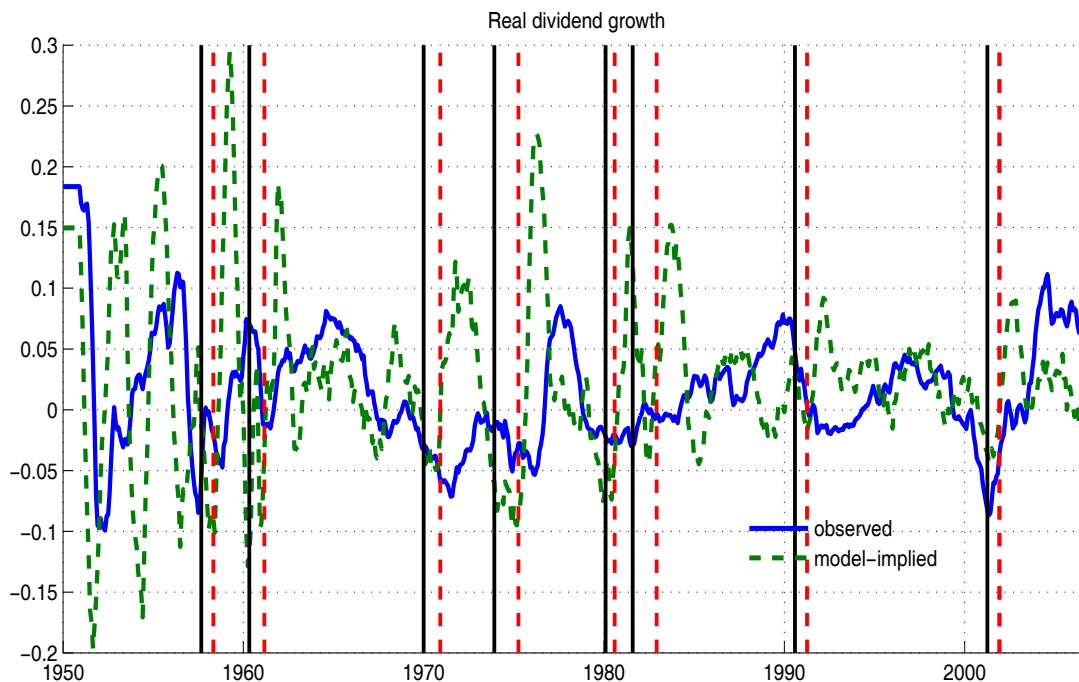
Figure C.1 – Dividend growth, observed and implied by the model estimates, with NBER dated recession periods. This figure plots year-to-year dividend growth, along with its counterpart implied by the model estimates, obtained through our three-step procedure. Dividend growth is defined as $\text{DG}_t = \ln(\text{Div}_t/\text{Div}_{t-12})$, where Div_t is the dividend observed at time t. The model-implied dividend growth is obtained by feeding Eq. (62) with the two macroeconomic factors depicted in Figure 1 (inflation and growth), by averaging over the cross-section of 1000 simulations of secular growth and the unobserved factor and, finally, by fixing the parameters to $\hat{\boldsymbol{\theta}}_u$ and $\hat{\boldsymbol{\theta}}_G$ (as reported in Table 2 of Section 4), and the calibrated values for $\boldsymbol{\delta}$ reported in Table C.1. The sample covers monthly data for the period from January 1950 to December 2006. Vertical solid lines (in black) track the beginning of NBER-dated recessions, and vertical dashed lines (in red) indicate the end of NBER-dated recessions.

C.2. Model-implied predictions of reduced-form regressions

This section contains details concerning parameter estimates for the auxiliary models utilized to implement the three-step estimation procedure in the main paper—Table C.2, Panel A through D. The values of $\boldsymbol{\delta}$ in Table C.1, which simultaneously minimize the criterion function $\Xi(\boldsymbol{\delta}, \lambda_G)$ in Eq. (61) and moment conditions for the auxiliary models for the real dividends in Eq. (63), lead to the parameter estimates for the auxiliary models reported in Panel E of Table C.2.

Table C.2

Parameter estimates for the auxiliary models fitted on both data and data generated by the model, and relating to (i) the macroeconomic factors (Eqs. (17)-(18)), in Panel A; (ii) asset returns (Eq. (22)), in Panel B; (iii) asset volatility (Eq. (23)), in Panel C; and the VIX index (Eq. (25)), in Panel D. Panel E reports parameter estimates for the auxiliary model relating to the real dividends (Eq. (63) in this Appendix). For each of these auxiliary models, we report R^2 as well as the residual variances, denoted with c_1 and c_2 (Panel A), $\sigma^2_{\epsilon R}$ (Panel B), $\sigma^2_{\epsilon V}$ (Panel C), $\sigma^2_{\epsilon VIX}$ (Panel D), and $\sigma^2_{\epsilon D}$ (Panel E). The parameter estimates w_i, a_{ij} and c_i in Panel A refer to the vector $\boldsymbol{w}$, the matrix $\boldsymbol{A}$ and the diagonal elements of the variance-covariance matrix $\boldsymbol{C}$ in Eqs. (17)-(18), and $R^2_{y_i}$ is the R^2 of the regression for the macroeconomic factor y_i, $i = 1, 2$. Remaining notation is as in the main text.

Panel A
Auxiliary regressions relating to macroeconomic factors

	Data	Model-implied
w_1	-0.0197	-0.0232
w_2	0.1640	0.2134
a_{11}	0.9982	0.9970
a_{21}	-0.1063	-0.1255
a_{12}	0.0208	0.0254
a_{22}	0.9483	0.9199
$R^2_{y_1}$	0.9516	0.9203
c_1	$1.41 \cdot 10^{-5}$	$2.03 \cdot 10^{-5}$
$R^2_{y_2}$	0.9394	0.9114
c_2	0.0002	0.0004
$\bar{y}_1$	1.0385	1.0386
$\bar{y}_2$	1.0367	1.0375
$\hat{\sigma}_{y_1}$	0.0296	0.0296
$\hat{\sigma}_{y_2}$	0.0571	0.0572

Panel B
Auxiliary regressions relating to asset returns

	Data	Model-implied
a^R	2.4293	3.4159
$b^R_{1,12}$	-1.2560	-1.0303
$b^R_{2,12}$	-1.0609	-2.2459
R^2	0.1707	0.5221
$\sigma^2_{\epsilon R}$	0.0204	0.0139
$\bar{S}$	2.8843	2.7915

20 **Table C.2**—*continued*

1

Panel C
Auxiliary regressions relating to asset volatility

	Data	Model-implied
a^{V}	-0.3745	0.0628
b_6^{V}	1.0162	0.8525
b_{12}^{V}	-0.6893	-0.2889
b_{18}^{V}	0.5311	0.3741
b_{24}^{V}	-0.3272	-0.1515
b_{36}^{V}	0.0541	0.0269
b_{48}^{V}	-0.0205	0.0167
$b_{1,12}^{\mathrm{V}}$	0.0813	0.2969
$b_{1,24}^{\mathrm{V}}$	-0.1191	-0.1023
$b_{1,36}^{\mathrm{V}}$	0.1831	-0.1114
$b_{1,48}^{\mathrm{V}}$	-0.0035	-0.0649
$b_{2,12}^{\mathrm{V}}$	0.0435	-0.0957
$b_{2,24}^{\mathrm{V}}$	0.0910	0.0517
$b_{2,36}^{\mathrm{V}}$	0.1022	-0.0215
$b_{2,48}^{\mathrm{V}}$	0.0312	0.0036
R^2	0.6184	0.7551
$\sigma_{\epsilon^{\mathrm{V}}}^2$	0.0006	0.0005
$\overline{\mathrm{Vol}}$	0.1150	0.1305
$\hat{\sigma}_{\mathrm{Vol}}$	0.0401	0.0314

2

Panel D
Auxiliary regressions relating to the VIX index

	Data	Model-implied
a^{VIX}	-0.4316	-0.0037
b^{VIX}	0.5532	0.9788
$b_{1,36}^{\mathrm{VIX}}$	-0.2800	-0.1318
$b_{1,48}^{\mathrm{VIX}}$	-0.0357	0.0989
$b_{2,36}^{\mathrm{VIX}}$	0.3320	0.0518
$b_{2,48}^{\mathrm{VIX}}$	0.4865	-0.0107
R^2	0.6969	0.9512
$\sigma_{\epsilon^{\mathrm{VIX}}}^2$	0.0013	$3.97 \cdot 10^{-5}$
$\overline{\mathrm{VIX}}$	0.1894	0.2326
$\hat{\sigma}_{\mathrm{VIX}}$	0.0636	0.0278

3

Panel E
Auxiliary regressions relating to detrended dividends

	Data	Model-implied
a^{D}	0.6753	1.6282
$b_{1,12}^{\mathrm{D}}$	-0.5057	-1.0051
$b_{1,24}^{\mathrm{D}}$	-0.0753	0.6234
$b_{2,12}^{\mathrm{D}}$	0.1043	-1.0894
$b_{2,24}^{\mathrm{D}}$	-0.1598	-0.0780
R^2	0.2352	0.5659
$\sigma_{\epsilon^{\mathrm{D}}}^2$	0.0014	0.0026
$\bar{\delta}$	0.0781	0.0879
$\hat{\sigma}_{\delta}$	0.0136	0.0363

C.3. *Variance decomposition and out-of-sample statistics*

The experiments of this section produce variance decomposition and out-of-sample statistics under two alternative methodologies regarding the treatment of the unobserved factor. In the main text, these statistics originate from simulations of the unobservable factor. Under the first methodology of this section, we feed the model through the time series of the model-implied unobservable factor $\hat{y}_3(t)$, calculated as in Eq. (28) of the main text. Figure C.2 depicts conditional variance decompositions, calculated through Eq. (27), and obtained by feeding the model with the two macroeconomic factors, inflation and growth in Figure 1, and replacing the unobserved factor $y_3(t)$ with $\hat{y}_3(t)$. Table C.3 (Panel A) summarizes results applying to the overall sample as well as selected subsamples. Panel B of Table C.3 reports the statistics corresponding to the VIX index. Sample periods are as indicated in Figure C.2.

Under the second methodology, we use the University of Michigan Consumer Sentiment (UMCSENT) index to generate the statistics. Note that the UMCSENT index has a different scale than that of the model-implied unobserved factor. Therefore, we re-scale the index in a way that it has the same average and standard deviation as the model-implied factor over any given sampling period where we conduct the experiments, as follows. Let Sent_t and $\hat{y}_3(t)$ be the UMCSENT index and, as usual, the model-implied unobserved factor. When calculating statistics through the UMCSENT index, we utilize the time series $-\overline{\text{Sent}}_t$, where we define:

$$\overline{\text{Sent}}_t \equiv -E_T(\hat{y}_3) + \frac{\sigma_T(\hat{y}_3)}{\sigma_T(\text{Sent})}(\text{Sent}_t - E_T(\text{Sent})), \tag{64}$$

with $E_T(x)$ and $\sigma_T(x)$ denoting the average and the standard deviation of a given time series x over a certain sampling period of size T. The rationale behind the term $-E_T(\hat{y}_3)$ is that higher realizations of $\hat{y}_3$ are bad news to the stock market, given the negative sign of the s_3 estimate reported in Table 4, as explained in the main text. Therefore, according to Eq. (64), sample periods where the extracted factor is on average high correspond to periods where our rescaled index, $\overline{\text{Sent}}_t$, is on average low. The sampling periods we consider are (i) from January 1978 to December 2006, for the variance decompositions relating to stock volatility, where $E_T(\hat{y}_3) = 2.302$, $\sigma_T(\hat{y}_3) = 7.915$, and $\sigma_T(\text{Sent}) = 12.2747$, and (ii) from January 1990 to December 2006, for the decomposition statistics relating to the VIX index, where $E_T(\hat{y}_3) = -2.341$, $\sigma_T(\hat{y}_3) = 7.1305$ and $\sigma_T(\text{Sent}) = 12.2184$. Figure C.3 and Table C.4 contain variance decomposition statistics of these experiments, which are the counterparts to those relating to Figure C.2 and Table C.3.

The findings summarized in Tables C.3-C4 and Figures C.2-C.3 closely match those relying on simulations, and reported in the main text (Table 3 and Figure 3). The contributions are quite comparable across all the factors, quantitatively. For example, industrial production growth makes the most important contribution to the overall variation in both realized volatility and the VIX index, with its conditional properties being basically the same as those in Figure 3. The contribution of secular growth is, at times, more important than that of the unobserved factor. These cases arise, for example, during the dotcom bubble of the late 1990s, in the experiments relating to the use of the model-implied factor (Figure C.2), or during the subprime events and the 2007 recession (Figure C.3), in the experiments of the UMCSENT index.

The explanations of these cases link to the fact that the variance of secular growth is constant, such that the contribution of growth, $C_G(t)$ in Eq. (27), is inversely related to volatility, $\sigma^2(t)$. The volatility predicted by the model for the dotcom bubble is quite low, and especially so in the experiments based on the model-implied factor, which explains the findings in Figure C.2. Instead, a higher contribution of growth over the subprime events in Figure C.3, relates to the failure of the model to fully capture the surge in realized volatility experienced over that period, once our model is fed wih the UMCSENT index. This fact is confirmed by out-of-sample experiments, discussed below (see Table C.5, and Figures C.4 and C.5), where our model delivers much better results relating to realized volatility predictions, once we use as an input the model-implied unobserved factor rather than the UMCSENT index, and even controlling for a different window over which to evaluate $E_T(\hat{y}_3)$ and $\sigma_T(\hat{y}_3)$ in Eq. (64). (Model predictions based on the UMCSENT index remain, however, better than predictions based on OLS.) At the same time, the model delivers the best out-of-sample predictions in terms of the VIX index, once we utilize the UMCSENT index as an input.
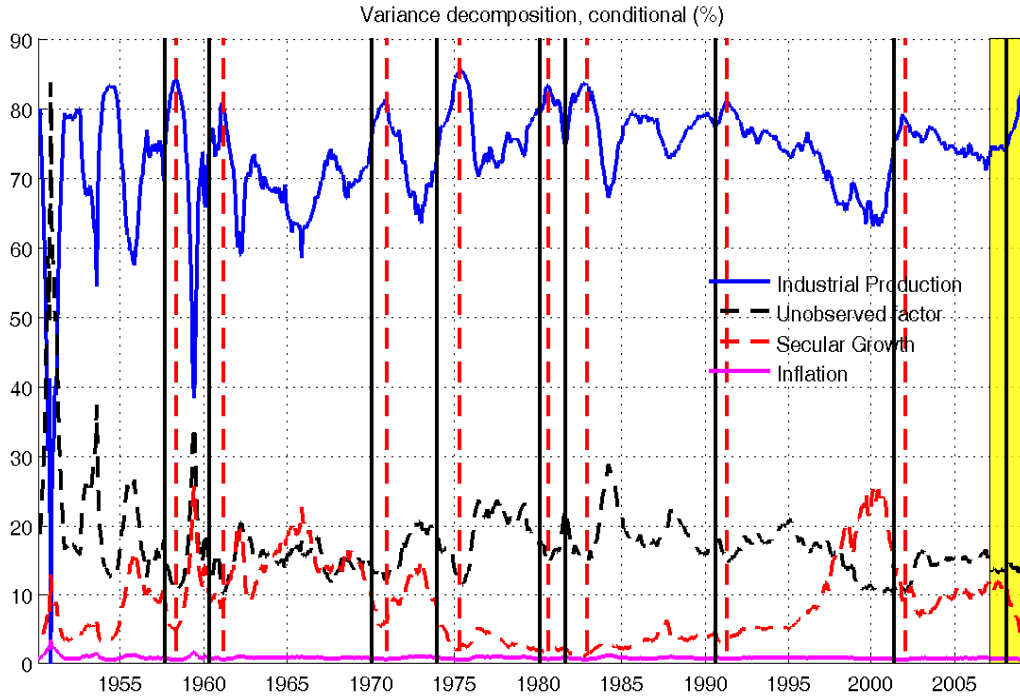
Figure C.2 – Contributions to total stock volatility made by macroeconomic and the model-implied unobservable factors, with NBER dated recession periods. This figure plots the contributions to stock volatility, $C_j(t)$ and $C_G(t)$ in Eq. (27), obtained as the ratios of the instantaneous stock return variance due to factor y_j to the total instantaneous variance, $\sigma^2(t)$, $C_j(t)$ $(j = 1, 2, 3)$, as well as the ratio of the instantaneous variance of secular growth to $\sigma^2(t)$, $C_G(t)$. From top to bottom, "Industrial Production" is $C_2(t)$, "Unobservable factor" is $C_3(t)$, "Secular Growth" is $C_G(t)$, and "Inflation" is $C_1(t)$. Each prediction at each point in time is obtained by feeding the model with the two macroeconomic factors depicted in Figure 1 (inflation and growth), and replacing the unobserved factor with the model-implied factor $\hat{y}_3(t)$, as defined in Eq. (28). The sample covers monthly data for the period from January 1950 to December 2006. Vertical solid lines (in black) track the beginning of NBER-dated recessions, and vertical dashed lines (in red) indicate the end of NBER-dated recessions. The shaded area (in yellow) covers the out-of-sample period, from January 2007 to March 2009.

Table C.3

Variance decomposition statistics for (i) realized volatility (Panel A) and expected volatility under the risk-neutral probability (Panel B). Panel A reports averages and standard deviations of the contributions $C_j(t)$ and $C_G(t)$ to the total variance, $\sigma^2(t)$ in Eq. (12), made by: (i) the two macroeconomic factors, gross inflation, $y_1(t)$, and gross industrial production growth, $y_2(t)$, as defined in Table 1, (ii) the model-implied unobserved factor, $\hat{y}_3(t)$, estimated as in Eq. (28),

$$\hat{y}_{3,t} \equiv \frac{1}{\hat{s}_3}\left(\frac{S_t}{\hat{G}_t} - \hat{s}_0 - \hat{s}_1 y_{1,t} - \hat{s}_2 y_{2,t}\right),$$

and (iii) secular growth, defined respectively, as:

$$C_j(t)\,\bar{s}^2\left(\boldsymbol{y}(t)\right) \equiv \frac{\hat{s}_j^2(\hat{\alpha}_j + \hat{\beta}_j y_j(t))}{\sigma^2(t)},\ \ j=1,2,3, \quad \text{and} \quad C_G(t) \equiv \frac{\sigma_G^2}{\sigma^2(t)},$$

where S_t is the real stock price at time t, $\bar{s}(\boldsymbol{y}) \equiv \hat{s}_0 + \sum_{j=1}^3 \hat{s}_j y_j$, $\hat{G}_t$ is the cross-sectional average of 1000 simulations of secular growth, $(\hat{\alpha}_j, \hat{\beta}_j)_{j=1}^3$ and $(\hat{s}_j)_{j=0}^3$ are the parameter estimates, as reported in Tables 1 and 2, $y_3(t) \equiv \hat{y}_{3t}$, and the total variance in Eq. (12) is obtained fixing $y_3(t) \equiv \hat{y}_{3t}$. The sample covers monthly data for the period from January 1950 to December 2006. Panel B reports statistics for the risk-neutral counterparts to the average paths of $C_j(t)$ and $C_G(t)$. The sample covers monthly data for the period from January 1990 to December 2006.
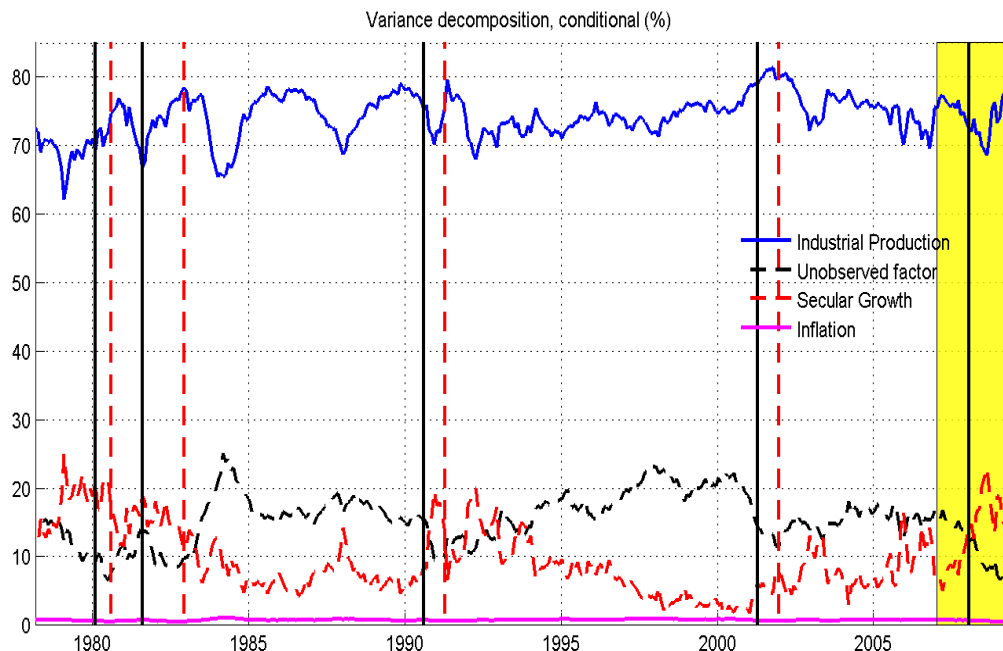
Panel A: Contributions of factors to stock volatility

Averages

	1950-2006	1950-1981	1960-1981	1982-2006
Gross inflation	0.87%	0.91%	0.86%	0.82%
Gross growth	73.20%	71.59%	72.63%	74.98%
Unobserved factor	17.35%	18.23%	16.63%	16.39%
Secular growth	8.56%	9.24%	9.85%	7.79%

Standard deviations

	1950-2006	1950-1981	1960-1981	1982-2006
Gross inflation	0.22%	0.28%	0.12%	0.09%
Gross growth	7.64%	9.37%	5.87%	4.52%
Unobserved factor	5.94%	7.46%	3.14%	3.36%
Secular growth	5.69%	5.46%	5.71%	5.85%

Panel B: Contributions of factors to the VIX Index

	Averages	Standard deviations
Gross inflation	3.13%	0.03%
Gross growth	85.94%	2.38%
Unobserved factor	8.22%	1.22%
Secular growth	2.71%	0.17%

Figure C.3 – Contributions to total stock volatility made by macroeconomic factors and the UMCSENT index, with NBER dated recession periods. This figure plots the contributions to stock volatility, $C_j(t)$ and $C_G(t)$ in Eq. (27), obtained as the ratios of the instantaneous stock return variance due to factor y_j to the total instantaneous variance, $\sigma^2(t)$, $C_j(t)$ ($j = 1, 2, 3$), as well as the ratio of the instantaneous variance of secular growth to $\sigma^2(t)$, $C_G(t)$. From top to bottom, "Industrial Production" is $C_2(t)$, "Unobservable factor" is $C_3(t)$, "Secular Growth" is $C_G(t)$, and "Inflation" is $C_1(t)$. Each prediction at each point in time is obtained by feeding the model with the two macroeconomic factors depicted in Figure 1 (inflation and growth), and replacing the unobserved factor with the UMCSENT index, rescaled as in Eq. (64). The sample covers monthly data for the period from January 1978 to December 2006. Vertical solid lines (in black) track the beginning of NBER-dated recessions, and vertical dashed lines (in red) indicate the end of NBER-dated recessions. The shaded area (in yellow) covers the out-of-sample period, from January 2007 to March 2009.

Table C.4

Variance decomposition statistics for (i) realized volatility (Panel A) and expected volatility under the risk-neutral probability (Panel B). Panel A reports averages and standard deviations of the contributions $C_j(t)$ and $C_G(t)$ to the total variance, $\sigma^2(t)$ in Eq. (12), made by: (i) the two macroeconomic factors, gross inflation, $y_1(t)$, and gross industrial production growth, $y_2(t)$, as defined in Table 1, (ii) the UMCSENT index, and (iii) secular growth, defined respectively, as:

$$C_j(t)\,\bar{s}^2\left(\boldsymbol{y}(t)\right) \equiv \frac{\hat{s}_j^2(\hat{\alpha}_j + \hat{\beta}_j y_j(t))}{\sigma^2(t)}, \; j = 1, 2, 3, \quad \text{and} \quad C_G(t) \equiv \frac{\sigma_G^2}{\sigma^2(t)},$$

where $\bar{s}(\boldsymbol{y}) \equiv \hat{s}_0 + \sum_{j=1}^{3}\hat{s}_j y_j$, $y_3(t) \equiv -\overline{\text{Sent}}_t$, and $\overline{\text{Sent}}_t$ is the UMCSENT index, rescaled as in Eq. (64). The sample covers monthly data for the period from January 1978 to December 2006. Panel B reports statistics for the risk-neutral counterparts to the average paths of $C_j(t)$ and $C_G(t)$. The sample covers monthly data for the period from January 1990 to December 2006.

Panel A: Contributions of factors to stock volatility

	Averages	Standard deviations
Gross inflation	0.83%	0.11%
Gross growth	74.11%	3.17%
Unobserved factor	15.40%	3.78%
Secular growth	9.66%	4.92%

Panel B: Contributions of factors to the VIX Index

	Averages	Standard deviations
Gross inflation	1.37%	0.02%
Gross growth	86.80%	2.47%
Unobserved factor	9.13%	0.92%
Secular growth	2.70%	0.15%

Finally, we provide out-of-sample results pertaining to the two methodologies of this appendix to integrate out the unobserved factor. Table C.5 reports Root Mean Squared Errors (RMSE) for the two cases where the model is fed with (i) the series of the model-implied unobserved factor, $\hat{y}_{3t}$, as calculated through Eq. (28) (second column), and (ii) the series $-\overline{\text{Sent}}_t$, where $\overline{\text{Sent}}_t$ is the UMCSENT index, rescaled as in Eq. (64) (third column). We rescale the UMCSENT index through Eq. (64), using the average and standard deviation of the model-implied unobserved factor over the five years prior to the out-of-sample experiments (January 2007 to March 2009), where $E_T(\hat{y}_3) = -2.612$ and $\sigma_T(\hat{y}_3) = 2.072$, and fixing $\sigma_T(\text{Sent}) = 10.6134$, which is the standard deviation of the UMCSENT index over the period from January 1990 to December 2006. For comparison, we also report RMSE for the benchmark cases considered in the main text (Section 4.2.4), namely the case of simulations of the unobserved factor (first column), and the OLS (fourth column).

As anticipated at the beginning of Section C.3, the model generates better predictions than those stemming from OLS, in all the experiments. To summarize, (i) predictions relating to realized volatility are best performed when we feed the model with the model-implied factor; (ii) predictions relating to the VIX index are the best when we utilize the UMCSENT index. While predictions based on simulations of the unobserved factors never rank first, they rank in a stable way, compared to the predictions from alternative methodologies regarding the treatment of the unobservable factor.
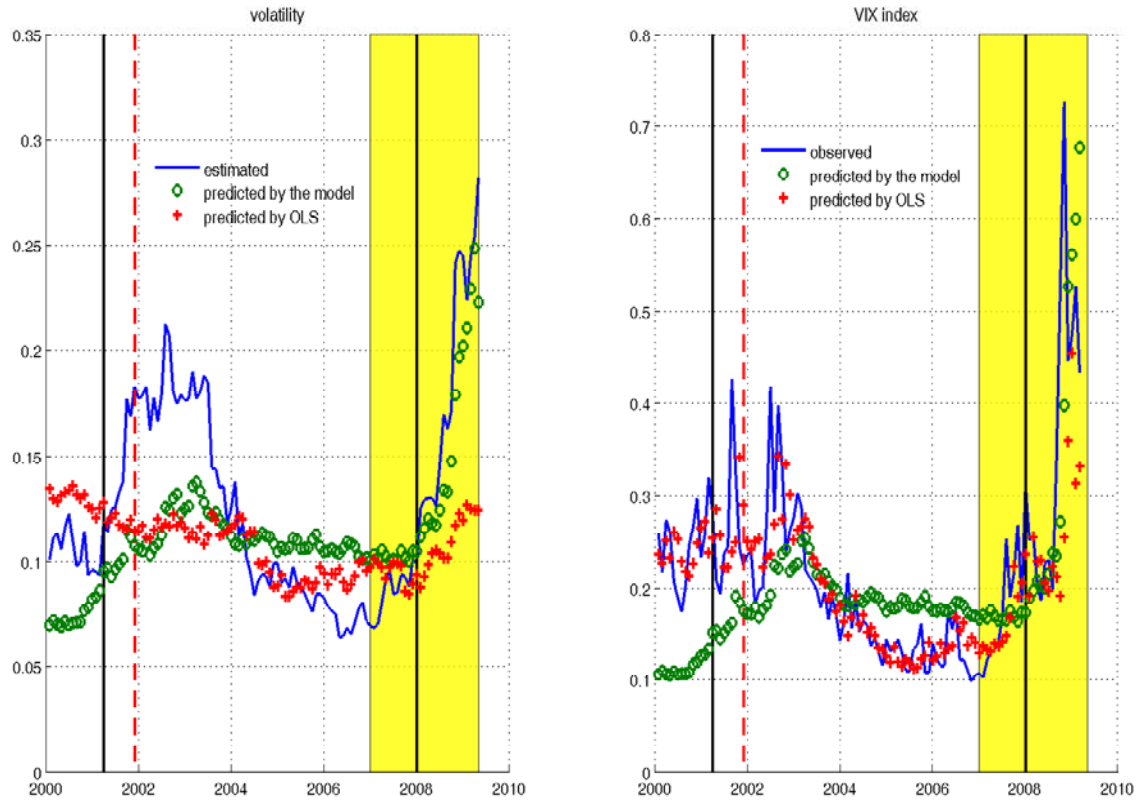
Table C.5

Out-of-sample statistics under different assumptions about the treatment of the unobserved factor.

RMSE for the model and OLS

	Model (simulations)	Model (Extracted factor)	Model (UMCSENT index)	OLS
Volatility	0.0478	0.0262	0.0607	0.0700
VIX Index	0.1119	0.1121	0.1034	0.1319

Figure C.4 – Out of sample predictions through the model-implied unobserved factor, and the subprime crisis. This figure plots one-year return volatility and the VIX index, along with its counterparts predicted by the model and by an OLS regression. The left panel depicts smoothed return volatility, defined as $\text{Vol}_t \equiv \sqrt{6\pi} \cdot 12^{-1} \sum_{i=1}^{12} |\ln (S_{t+1-i}/S_{t-i})|$, where S_t is the real stock price as of month t, along with the instantaneous standard deviation predicted by (i) the model, through Eq. (12), and (ii) the predictive part of an OLS regression of Vol_t on to past values of Vol_t, inflation and industrial production growth. The right panel depicts the VIX index, along with the VIX index predicted by (i) the model; and (ii) the predictive part of an OLS regression of the VIX index on to past values of the VIX index, inflation and industrial production growth. Each prediction is obtained by feeding the model and the predictive part of the OLS regression with the two macroeconomic factors depicted in Figure 1 (inflation and growth) and, for the model, the model-implied unobserved factor estimated as in Eq. (28), $\hat{y}_{3,t} \equiv \hat{s}_3^{-1}(\frac{S_t}{\hat{G}_t} - \hat{s}_0 - \hat{s}_1 y_{1,t} - \hat{s}_2 y_{2,t})$, where $\hat{G}_t$ is the cross-sectional average of 1000 simulations of secular growth. The sample depicted in the figure spans the period from January 2000 to March 2009. The estimation of both the model and the OLS regressions relates to the period from January 1950 to December 2006. Vertical solid lines (in black) track the beginning of NBER-dated recessions, and the vertical dashed line (in red) indicates the end of the NBER-dated recession, occurred in November 2001. The shaded area (in yellow) covers the out-of-sample period, from January 2007 to March 2009, which includes the NBER recession announced to have occurred in December 2007, and the subprime crisis, which started in June 2007.
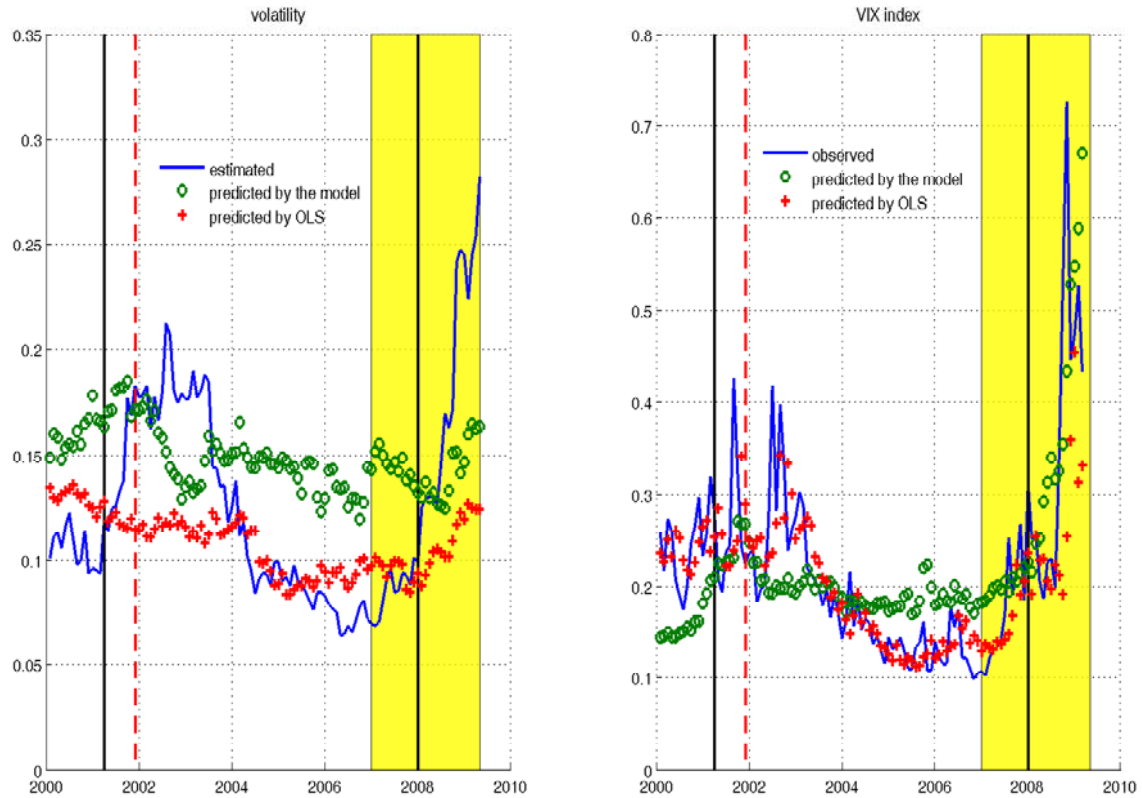
Figure C.5 – Out of sample predictions through the UMCSENT index, and the subprime crisis. This figure plots one-year return volatility and the VIX index, along with its counterparts predicted by the model and by an OLS regression. The left panel depicts smoothed return volatility, defined as $\text{Vol}_t \equiv \sqrt{6\pi} \cdot 12^{-1} \sum_{i=1}^{12} |\ln(S_{t+1-i}/S_{t-i})|$, where S_t is the real stock price as of month t, along with the instantaneous standard deviation predicted by (i) the model, through Eq. (12), and (ii) the predictive part of an OLS regression of Vol_t on to past values of Vol_t, inflation and industrial production growth. The right panel depicts the VIX index, along with the VIX index predicted by (i) the model; and (ii) the predictive part of an OLS regression of the VIX index on to past values of the VIX index, inflation and industrial production growth. Each prediction is obtained by feeding the model and the predictive part of the OLS regression with the two macroeconomic factors depicted in Figure 1 (inflation and growth) and, for the model, using $y_3(t) \equiv -\overline{\text{Sent}}_t$, where $\overline{\text{Sent}}_t$ is the UMCSENT index, rescaled as in Eq. (64). The sample depicted in the figure spans the period from January 2000 to March 2009. The estimation of both the model and the OLS regressions relates to the period from January 1950 to December 2006. Vertical solid lines (in black) track the beginning of NBER-dated recessions, and the vertical dashed line (in red) indicates the end of the NBER-dated recession, occurred in November 2001. The shaded area (in yellow) covers the out-of-sample period, from January 2007 to March 2009, which includes the NBER recession announced to have occurred in December 2007, and the subprime crisis, which started in June 2007.

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