THE ECONOMICS OF BARGAINING

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Abstract. This article presents the main principles of bargaining theory, along with some examples to illustrate the potential applicability of this theory to a variety of real-life bargaining situations. The roles of various key factors on the outcome of bargaining will be discussed and explored in the context of various canonical models of bargaining. It will be shown that such models can be adapted, extended and modified in order to explore other issues concerning bargaining situations.

1. Introduction

Bargaining is ubiquitous. Married couples negotiate over a variety of matters such as who will do which domestic chores. Government policy is typically the outcome of negotiations amongst cabinet ministers. Whether or not a particular piece of legislation meets with the legislature's approval may depend on the outcome of negotiations amongst the dominant political parties. National governments are often engaged in a variety of international negotiations on matters ranging from economic issues (such as the removal of trade restrictions) to global security (such as the reduction in the stockpiles of conventional armaments, and nuclear non-proliferation and test ban), and environmental and related issues (such as carbon emissions trading, bio-diversity conservation and intellectual property rights). Much economic interaction involves negotiations on a variety of issues. Wages, and prices of other commodities (such as oil, gas and computer chips) are often the outcome of negotiations amongst the concerned parties. Mergers and acquisitions require negotiations over, amongst other issues, the price at which such transactions are to take place.

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What variables (or factors) determine the outcome of negotiations such as those mentioned above? What are the sources of bargaining power? What strategies can help improve one's bargaining power? What variables determine whether parties to a territorial dispute will reach a negotiated settlement, or engage in military war? How can one enhance the likelihood that parties in such negotiations will strike an agreement quickly so as to minimise the loss of life through war? What strategies should one adopt to maximise the negotiated sale price of one's house? How can one negotiate a better deal (such as a wage increase) from ones employers?

Bargaining theory seeks to address the above and many similar real-life questions concerning bargaining situations.

1.1. **Bargaining Situations and Bargaining.** Consider the following situation. An individual, called Aruna, owns a house that she is willing to sell at a minimum price of £50,000; that is, she 'values' her house at £50,000. Another individual, called Mohan, is willing to pay up to £70,000 for Aruna's house; that is, he values her house at £70,000. If trade occurs — that is, if Aruna sells the house to Mohan — at a price that lies between £50,000 and £70,000, then both Aruna (the 'seller') and Mohan (the 'buyer') would become better off. This means that in this situation these two individuals have a common interest to trade. At the same time, however, they have conflicting (or divergent) interests over the price at which to trade: Aruna, the seller, would like to trade at a high price, while Mohan, the buyer, would like to trade at a low price.

Any exchange situation, such as the one just described, in which a pair of individuals (or organisations) can engage in mutually beneficial trade but have conflicting interests over the terms of trade is a *bargaining situation*. Stated in general terms, a bargaining situation is a situation in which two or more players[1] have a common interest to co-operate, but have conflicting interests over exactly how to co-operate.

There are two main reasons for being interested in bargaining situations. The first, practical reason is that many important and interesting human (economic, social and political) interactions are bargaining situations. As mentioned above, exchange situations (which characterise much of human economic interaction) are bargaining situations. In the arena of social interaction, a married couple, for example, is involved in many bargaining situations throughout the relationship. In the political arena, a bargaining situation exists, for example, when no

[1]A 'player' can be either an individual, or an organisation (such as a firm, a political party or a country).

single political party on its own can form a government (such as when there is a hung parliament); the party that has obtained the most votes will typically find itself in a bargaining situation with one or more of the other parties. The second, theoretical reason for being interested in bargaining situations is that understanding such situations is fundamental to the development of an understanding of the workings of markets and the appropriateness, or otherwise, of prevailing monetary and fiscal policies.

The main issue that confronts the players in a bargaining situation is the need to reach agreement over exactly how to co-operate. Each player would like to reach some agreement rather than to disagree and not reach any agreement, but each player would also like to reach an agreement that is as favourable to her as possible. It is thus possible that the players will strike an agreement only after some costly delay, or indeed fail to reach any agreement — as is witnessed by the history of disagreements and costly delayed agreements in many real-life situations (as exemplified by the occurrences of trade wars, military wars, strikes and divorce).

Bargaining is any process through which the players try to reach an agreement. This process is typically time consuming, and involves the players making offers and counteroffers to each other. A main focus of any theory of bargaining is on the *efficiency* and *distribution* properties of the outcome of bargaining. The former property relates to the possibility that the players fail to reach an agreement, or that they reach an agreement after some costly delay. Examples of costly delayed agreements include: when a wage agreement is reached after lost production due to a long strike, and when a peace settlement is negotiated after the loss of life through war. The distribution property relates to the issue of exactly how the gains from co-operation are divided between the players.

The principles of bargaining theory set out in this article determine the roles of various key factors (or variables) on the bargaining outcome (and its efficiency and distribution properties). As such, they determine the sources of a player's bargaining power.

1.2. **An Outline of this Article.** If the bargaining process is 'frictionless' — by which I mean that neither player incurs any cost from haggling — then each player may continuously demand that agreement be struck on terms that are most favourable to her.[2] In such a circumstance the negotiations are likely to end up in an impasse (or deadlock), since the negotiators would have no incentive to compromise and reach an agreement. Indeed, if it did not matter *when* the negotiators agree, then it would not matter *whether* they agreed at all. In most real-life situations the bargaining process is not frictionless. A basic source of a player's cost from haggling comes from the twin facts that bargaining is time consuming and that time is valuable to the player. In section 3 below, I shall discuss the role of the players' degrees of *impatience* on the outcome of bargaining. A key principle that will be discussed is that a player's bargaining power is higher the less impatient she is relative to the other negotiator. For example, in the exchange situation described above, the price at which Aruna sells her house will be higher the less impatient she is relative to Mohan. Indeed, patience confers bargaining power.

A person who has been unemployed for a long time is typically quite desperate to find a job, and, may thus be willing to accept work at almost any wage. The high degree of impatience of the long-term unemployed can be exploited by potential employers, who may thus obtain most of the gains from employment. As such, an important role of minimum wage legislation would seem to be to strengthen the bargaining power of the long-term unemployed. In general, since a player who is poor is typically more eager to strike a deal in any negotiations, poverty (by inducing a larger degree of impatience) adversely affects bargaining power. No wonder, then, that the richer nations of the world often obtain relatively better deals than the poorer nations in international trade negotiations.

Another potential source of friction in the bargaining process comes from the possibility that the negotiations might breakdown into disagreement because of some exogenous and uncontrollable factors. Even if the possibility of such an occurrence is small, it nevertheless may provide appropriate incentives to the players to compromise and reach an agreement. The role of such a *risk of breakdown* on the bargaining outcome is discussed in section 4.

In many bargaining situations the players may have access to 'outside' options and/or 'inside' options. For example, in the exchange situation described above, Aruna may have a non-negotiable (fixed) price offer on her house from another buyer; and, she may derive some

[2]For example, in the exchange situation described above, Aruna may continuously demand that trade take place at the price of £69,000, while Mohan may continuously demand that it take place at the price of £51,000.

'utility' (or benefit) while she lives in it. The former is her outside option, while the latter her inside option. When, and if, Aruna exercises her outside option, the negotiations between her and Mohan terminate forever in disagreement. In contrast, her inside option is the utility per day that she derives by living in her house while she temporarily disagrees with Mohan over the price at which to trade. As another example, consider a married couple who are bargaining over a variety of issues. Their outside options are their payoffs from divorce, while their inside options are their payoffs from remaining married but without much co-operation within their marriage. The role of *outside options* on the bargaining outcome is discussed in section 5, while the role of *inside options* is discussed in section 6.

An important set of questions addressed in sections 3–6 are *why, when and how* to apply Nash's bargaining solution, where the latter is described and studied in section 2. It is shown that under some circumstances, when appropriately applied, Nash's bargaining solution describes the outcome of a variety of bargaining situations. These results are especially important and useful in applications, since it is often convenient for applied economic and political theorists to describe the outcome of a bargaining situation — which may be one of many ingredients of their economic models — in a *simple* (and tractable) manner.

An important determinant of the outcome of bargaining is the extent to which information about various variables (or factors) are known to all the parties in the bargaining situation. For example, the outcome of union-firm wage negotiations will typically be influenced by whether or not the current level of the firm's revenue is known to the union. The role of such *asymmetric information* on the bargaining outcome is studied in section 7.

In the preceding chapters the focus is on 'one-shot' bargaining situations. In section 8 I study repeated bargaining situations in which the players have the opportunity to be involved in a sequence of bargaining situations. I conclude in section 9.[3]

[3]It may be noted that a bargaining situation is a game situation in the sense that the outcome of bargaining depends on *both* players' bargaining strategies: whether or not an agreement is struck, and the terms of the agreement (if one is struck), depends on both players' actions during the bargaining process. It is therefore natural to study bargaining situations using the methodology of game theory **(see Game Theory)**.

2. The Nash Bargaining Solution

A bargaining solution may be interpreted as a formula that determines a unique outcome for each bargaining situation in some class of bargaining situations. In this section I introduce the bargaining solution created by John Nash.[4] The Nash bargaining solution is defined by a fairly simple formula, and it is applicable to a large class of bargaining situations — these features contribute to its attractiveness in applications. However, the most important of reasons for studying and applying the Nash bargaining solution is that it possesses sound strategic foundations: several plausible (game-theoretic) models of bargaining vindicate its use. These strategic bargaining models will be studied in later sections where I shall address the issues of why, when and how to use the Nash bargaining solution. A prime objective of the current section, on the other hand, is to develop a thorough understanding of the definition of the Nash bargaining solution, which should, in particular, facilitate its characterization and use in any application.

Two players, A and B, bargain over the partition of a cake (or surplus) of size π, where $\pi > 0$. The set of possible agreements is $X = \{(x_A, x_B) : 0 \leq x_A \leq \pi \text{ and } x_B = \pi - x_A\}$, where x_i is the share of the cake to player i ($i = A, B$). For each $x_i \in [0, \pi]$, $U_i(x_i)$ is player i's utility from obtaining a share x_i of the cake, where player i's utility function $U_i : [0, \pi] \to \Re$ is differentiable, strictly increasing and concave. If the players fail to reach agreement, then player i obtains a utility of d_i, where $d_i \geq U_i(0)$. There exists an agreement $x \in X$ such that $U_A(x) > d_A$ and $U_B(x) > d_B$, which ensures that there exists a mutually beneficial agreement.

The utility pair $d = (d_A, d_B)$ is called the *disagreement point*. In order to define the Nash bargaining solution of this bargaining situation, it is useful to first define the set Ω of *possible utility pairs* obtainable through agreement. For the bargaining situation described above, $\Omega = \{(u_A, u_B) : \text{there exists } x \in X \text{ such that } U_A(x_A) = u_A \text{ and } U_B(x_B) = u_B\}$.

Fix an arbitrary utility u_A to player A, where $u_A \in [U_A(0), U_A(\pi)]$. From the strict monotonicity of U_i, there exists a unique share $x_A \in [0, \pi]$ such that $U_A(x_A) = u_A$; i.e., $x_A = U_A^{-1}(u_A)$, where U_A^{-1} denotes the inverse of U_A. Hence

$$g(u_A) \equiv U_B(\pi - U_A^{-1}(u_A))$$

[4] Nash's bargaining solution and the concept of a Nash equilibrium are unrelated concepts, other than the fact that both concepts are the creations of the same individual.

is the utility player B obtains when player A obtains the utility u_A. It immediately follows that $\Omega = \{(u_A, u_B) : U_A(0) \le u_A \le U_A(\pi) \text{ and } u_B = g(u_A)\}$.

The *Nash bargaining solution* (NBS) of the bargaining situation described above is the unique pair of utilities, denoted by (u_A^N, u_B^N), that solves the following maximization problem:

$$\max_{(u_A, u_B) \in \Theta} (u_A - d_A)(u_B - d_B),$$

where $\Theta \equiv \{(u_A, u_B) \in \Omega : u_A \ge d_A \text{ and } u_B \ge d_B\} \equiv \{(u_A, u_B) : U_A(0) \le u_A \le U_A(\pi), u_B = g(u_A), u_A \ge d_A \text{ and } u_B \ge d_B\}$.

The maximization problem stated above has a unique solution, because the maximand $(u_A - d_A)(u_B - d_B)$ — which is referred to as the Nash product — is continuous and strictly quasiconcave, g is strictly decreasing and concave, and the set Θ is non-empty.

Hence, the Nash bargaining solution is the unique solution to the following pair of equations

$$-g'(u_A) = \frac{u_B - d_B}{u_A - d_A} \quad \text{and} \quad u_B = g(u_A), \tag{1}$$

where g' denotes the derivative of g.

Example 1 (Split-The-Difference Rule)**.** *Suppose $U_A(x_A) = x_A$ for all $x_A \in [0, \pi]$ and $U_B(x_B) = x_B$ for all $x_B \in [0, \pi]$. This means that for each $u_A \in [0, \pi]$, $g(u_A) = \pi - u_A$, and $d_i \ge 0$ $(i = A, B)$. It follows from equation 1 that*

$$u_A^N = d_A + \frac{1}{2}\Big(\pi - d_A - d_B\Big) \quad \textit{and} \quad u_B^N = d_B + \frac{1}{2}\Big(\pi - d_A - d_B\Big),$$

which may be given the following interpretation. The players agree first of all to give player i $(i = A, B)$ a share d_i of the cake (which gives her a utility equal to the utility she obtains from not reaching agreement), and then they split equally the remaining cake $\pi - d_A - d_B$. Notice that player i's share is strictly increasing in d_i and strictly decreasing in d_j $(j \ne i)$.

Example 2 (Risk Aversion)**.** *Suppose $U_A(x_A) = x_A^\gamma$ for all $x_A \in [0, \pi]$, where $0 < \gamma < 1$, $U_B(x_B) = x_B$ for all $x_B \in [0, \pi]$ and $d_A = d_B = 0$. This means that for each $u_A \in [0, \pi]$, $g(u_A) = \pi - u_A^{1/\gamma}$. Using equation 1, it is easy to show that the shares x_A^N and x_B^N of the cake in the NBS to players A and B respectively are as follows:*

$$x_A^N = \frac{\gamma\pi}{1+\gamma} \quad \textit{and} \quad x_B^N = \frac{\pi}{1+\gamma}.$$

As γ decreases, x_A^N decreases and x_B^N increases. In the limit, as $\gamma \to 0$, $x_A^N \to 0$ and $x_B^N \to 1$. Player B may be considered risk neutral (since her utility function is linear), while player A risk averse (since her utility function is strictly concave), where the degree of her risk aversion is decreasing in γ. Given this interpretation of the utility functions, it has been shown that player A's share of the cake decreases as she becomes more risk averse.

2.1. **An Application to Bribery and the Control of Crime.** An individual C decides whether or not to steal a fixed amount of money π, where $\pi > 0$. If she steals the money, then with probability ζ she is caught by a policeman P. The policeman is corruptible, and bargains with the criminal over the amount of bribe b that C gives P in return for not reporting her to the authorities. The set of possible agreements is the set of possible divisions of the stolen money, which (assuming money is perfectly divisible) is $\{(\pi - b, b) : 0 \leq b \leq \pi\}$. The policeman reports the criminal to the authorities if and only if they fail to reach agreement. In that eventuality, the criminal pays a monetary fine. The disagreement point $(d_C, d_P) = (\pi(1-\nu), 0)$, where $\nu \in (0, 1]$ is the penalty rate. The utility to each player from obtaining x units of money is x.

It immediately follows that the NBS is $u_C^N = \pi[1 - (\nu/2)]$ and $u_P^N = \pi\nu/2$. The bribe associated with the NBS is $b^N = \pi\nu/2$. Notice that, although the penalty is never paid to the authorities, the penalty rate influences the amount of bribe that the criminal pays the corruptible policeman.

Given this outcome of the bargaining situation, I now address the issue of whether or not the criminal commits the crime. The expected utility to the criminal from stealing the money is $\zeta\pi[1-(\nu/2)]+(1-\zeta)\pi$, because with probability ζ she is caught by the policeman (in which case her utility is u_C^N) and with probability $1-\zeta$ she is not caught by the policeman (in which case she keeps all of the stolen money). Since her utility from not stealing the money is zero, the crime is not committed if and only if $\pi[1 - (\zeta\nu/2)] \leq 0$. That is, since $\pi > 0$, the crime is not committed if and only if $\zeta\nu \geq 2$. Since $\zeta < 1$ and $0 < \nu \leq 1$ implies that $\zeta\nu < 1$, for any penalty rate $\nu \in (0, 1]$ and any probability $\zeta < 1$ of being caught, the crime is committed. This analysis thus vindicates the conventional wisdom that if penalties are evaded through bribery, then they have no role in preventing crime.

2.2. **Asymmetric Nash Bargaining Solutions.** The NBS depends upon the set Ω of possible utility pairs and the disagreement point d. However, the outcome of a bargaining situation may be influenced by other factors, such as the tactics employed by the bargainers, the procedure through which negotiations are conducted, the information structure and the players' discount rates. However, none of these factors seem to affect the two objects upon which the NBS is defined, and yet it seems reasonable not to rule out the possibility that such factors may have a significant impact on the bargaining outcome. I now state generalizations of the NBS which possess a facility to take into account additional factors that may be deemed relevant for the bargaining outcome.

For each $\tau \in (0, 1)$, an *asymmetric* (or, *generalized*) Nash bargaining solution of the bargaining problem (Ω, d) stated above is the unique solution to the following maximization problem

$$\max_{(u_A,u_B)\in\Theta} (u_A - d_A)^\tau (u_B - d_B)^{1-\tau},$$

where Θ is stated above. Note that if and only if $\tau = 1/2$ is the asymmetric NBS *identical* to the NBS.[5] For any $\tau \in (0, 1)$, the asymmetric NBS is the unique solution to the following pair of equations

$$(2) \qquad -h'(u_A) = \left(\frac{\tau}{1-\tau}\right)\left[\frac{u_B - d_B}{u_A - d_A}\right] \quad \text{and} \quad u_B = h(u_A).$$

Notice that as τ increases, player A's utility increases while player B's utility decreases. As such τ captures player A's bargaining power, while $1 - \tau$ player B's.

3. The Rubinstein Model

In this section I study Rubinstein's (alternating-offers) model of bargaining. A key feature of this model is that it specifies a rather attractive procedure of bargaining: the players take turns to make offers to each other until agreement is secured. This model has much intuitive appeal, since making offers and counteroffers lies at the heart of many real-life negotiations.

Rubinstein's model provides several insights about bargaining situations. One insight is that frictionless bargaining processes are indeterminate. A bargaining process may be considered 'frictionless' if the players do not incur any costs by haggling (i.e., by making offers and counteroffers) — in which case there is nothing to prevent them from haggling for as long as they wish. It seems intuitive that for the players to have some incentive to reach agreement they should find it costly to haggle. Another insight is that a player's bargaining power

[5]Strategic models of bargaining studied in later sections provide guidance on what elements of a bargaining situation determine the value of τ.

depends on the relative magnitude of the players' respective costs of haggling, with the absolute magnitudes of these costs being irrelevant to the bargaining outcome.

An important reason for the immense influence that Rubinstein's model has had, and continues to have, is that it provides a basic framework, which can be adapted, extended and modified for the purposes of application. This will become evident in several later sections of this article.

3.1. **The Alternating-Offers Model.** Two players, A and B, bargain over the partition of a cake of size π (where $\pi > 0$) according to the following, alternating-offers, procedure. At time 0 player A makes an offer to player B. An offer is a proposal of a partition of the cake. If player B accepts the offer, then agreement is struck and the players divide the cake according to the accepted offer. On the other hand, if player B rejects the offer, then she makes a counteroffer at time $\Delta > 0$. If this counteroffer is accepted by player A, then agreement is struck. Otherwise, player A makes a counter-counteroffer at time 2Δ. This process of making offers and counteroffers continues until a player accepts an offer.

A precise description of this bargaining procedure now follows. Offers are made at discrete points in time: namely, at times $0, \Delta, 2\Delta, 3\Delta, \ldots, t\Delta, \ldots$, where $\Delta > 0$. An offer is a number greater than or equal to zero and less than or equal to π. I adopt the convention that an offer is the share of the cake to the proposer, and, therefore, π minus the offer is the share to the responder. At time $t\Delta$ when t is even (i.e., $t = 0, 2, 4, \ldots$) player A makes an offer to player B. If player B accepts the offer, then the negotiations end with agreement. On the other hand, if player B rejects the offer, then Δ time units later, at time $(t+1)\Delta$, player B makes an offer to player A. If player A accepts the offer, then the negotiations end with agreement. On the other hand, if player A rejects the offer, then Δ time units later, at time $(t+2)\Delta$, player A makes an offer to player B, and so on. The negotiations end if and only if a player accepts an offer.

The payoffs are as follows. If the players reach agreement at time $t\Delta$ ($t = 0, 1, 2, \ldots$) on a partition that gives player i ($i = A, B$) a share x_i ($0 \le x_i \le \pi$) of the cake, then player i's payoff is $x_i \exp(-r_i t\Delta)$, where $r_i > 0$ is player i's discount rate. On the other hand, if the players perpetually disagree (i.e., each player always rejects any offer made to her), then each player's payoff is zero.

This completes the description of the alternating-offers game. For notational convenience, define $\delta_i \equiv \exp(-r_i\Delta)$, where δ_i is player i's discount factor. Notice that $0 < \delta_i < 1$.

The subgame perfect equilibrium (SPE) concept will be employed to characterize the outcome of this game. In particular, answers to the following questions will be sought. In equilibrium, do the players reach agreement or do they perpetually disagree? In the former case, what is the agreed partition and at what time is agreement struck?

3.2. **The Unique Subgame Perfect Equilibrium.** Consider a SPE that satisfies the following two properties:

Property 1 (No Delay)**.** *Whenever a player has to make an offer, her equilibrium offer is accepted by the other player.*

Property 2 (Stationarity)**.** *In equilibrium, a player makes the same offer whenever she has to make an offer.*

Given Property 2, let x_i^* denote the equilibrium offer that player i makes whenever she has to make an offer. Consider an arbitrary point in time at which player A has to make an offer to player B. It follows from Properties 1 and 2 that player B's equilibrium payoff from rejecting any offer is $\delta_B x_B^*$. This is because, by Property 2, she offers x_B^* after rejecting any offer, which, by Property 1, is accepted by player A. Perfection requires that player B accept any offer x_A such that $\pi - x_A > \delta_B x_B^*$, and reject any offer x_A such that $\pi - x_A < \delta_B x_B^*$. Furthermore, it follows from Property 1 that $\pi - x_A^* \geq \delta_B x_B^*$. However, $\pi - x_A^* \not> \delta_B x_B^*$; otherwise player A could increase her payoff by instead offering x_A' such that $\pi - x_A^* > \pi - x_A' > \delta_B x_B^*$. Hence

$$\pi - x_A^* = \delta_B x_B^*. \tag{3}$$

Equation 3 states that player B is indifferent between accepting and rejecting player A's equilibrium offer. By a symmetric argument (with the roles of A and B reversed), it follows that player A is indifferent between accepting and rejecting player B's equilibrium offer. That is

$$\pi - x_B^* = \delta_A x_A^*. \tag{4}$$

Equations 3 and 4 have a unique solution, namely

$$x_A^* = \mu_A \pi \qquad \text{and} \qquad x_B^* = \mu_B \pi, \quad \text{where} \tag{5}$$

$$\mu_A = \frac{1-\delta_B}{1-\delta_A\delta_B} \qquad \text{and} \qquad \mu_B = \frac{1-\delta_A}{1-\delta_A\delta_B}. \tag{6}$$

The uniqueness of the solution to equations 3 and 4 means that there exists at most one SPE satisfying Properties 1 and 2. In that SPE, player A always offers x_A^* and always accepts an offer x_B if and only if $\pi - x_B \geq \delta_A x_A^*$, and player B always offers x_B^* and always accepts an offer x_A if and only if $\pi - x_A \geq \delta_B x_B^*$, where x_A^* and x_B^* are defined in (5). It is straightforward to verify that this pair of strategies is a subgame perfect equilibrium. Furthermore, there does not exist another SPE, and, hence we have the following result:

Proposition 1. *The following pair of strategies constitute the unique subgame perfect equilibrium of the alternating-offers game:*
• *player A always offers x_A^* and always accepts an offer x_B if and only if $x_B \leq x_B^*$,*
• *player B always offers x_B^* and always accepts an offer x_A if and only if $x_A \leq x_A^*$,*
where x_A^* and x_B^* are defined in (5).

In the unique SPE, agreement is reached at time 0, and the SPE is Pareto efficient. Since it is player A who makes the offer at time 0, the shares of the cake obtained by players A and B in the unique SPE are x_A^* and $\pi - x_A^*$, respectively, where $x_A^* = \mu_A \pi$ and $\pi - x_A^* = \delta_B \mu_B \pi$.

The equilibrium share to each player depends on both players' discount factors. In particular, the equilibrium share obtained by a player is strictly increasing in her discount factor, and strictly decreasing in her opponent's discount factor. Notice that if the players' discount rates are identical (i.e., $r_A = r_B = r > 0$), then player A's equilibrium share $\pi/(1+\delta)$ is strictly greater than player B's equilibrium share $\pi\delta/(1+\delta)$, where $\delta \equiv \exp(-r\Delta)$. This result suggests that there exists a 'first-mover' advantage, since if $r_A = r_B$ then the only asymmetry in the game is that player A makes the first offer, at time 0. However, note that this first-mover advantage disappears in the limit as $\Delta \to 0$: each player obtains one-half of the cake.

As is evident in the following corollary, the properties of the equilibrium shares (when $r_A \neq r_B$) are relatively more transparent in the limit as the time interval Δ between two consecutive offers tends to zero.

Corollary 1. *In the limit, as $\Delta \to 0$, the shares obtained by players A and B respectively in the unique SPE converge to $\eta_A \pi$ and $\eta_B \pi$, where*

$$\eta_A = \frac{r_B}{r_A + r_B} \quad \textit{and} \quad \eta_B = \frac{r_A}{r_A + r_B}.$$

Proof. For any $\Delta > 0$

$$\mu_A = \frac{1 - \exp(-r_B \Delta)}{1 - \exp\bigl(-(r_A + r_B)\Delta\bigr)}.$$

Since when $\Delta > 0$ *but small*, $\exp(-r_i\Delta) = 1 - r_i\Delta$, it follows that when $\Delta > 0$ but small, $\mu_A = r_B\Delta/(r_A + r_B)\Delta$ — that is, $\mu_A = r_B/(r_A + r_B)$. The corollary now follows immediately. □

In the limit, as $\Delta \to 0$, the relative magnitude of the players' discount rates critically influence the equilibrium partition of the cake: the equilibrium share obtained by a player depends on the ratio r_A/r_B. Notice that even in the limit, as both r_A and r_B tend to zero, the equilibrium partition depends on the ratio r_A/r_B. The following metaphor nicely illustrates the message contained in Corollary 1: In a boxing match, the winner is the relatively stronger of the two boxers; the absolute strengths of the boxers are irrelevant to the outcome.

3.3. **Properties of the Equilibrium.** It has been shown that if $r_A > 0$ and $r_B > 0$, then the basic alternating-offers game has a unique SPE, which is Pareto efficient. If, on the other hand, $r_A = r_B = 0$, then there exists many (indeed, a continuum) of subgame perfect equilibria, including equilibria which are Pareto inefficient. If $r_A = r_B = 0$, then neither player cares about the time at which agreement is struck. This means that the players do not incur any costs by haggling (i.e., by making offers and counteroffers) — which characterizes, what may be called, a 'frictionless' bargaining process. One important message, therefore, of the basic alternating-offers model is that frictionless bargaining processes are indeterminate. This seems intuitive, because, if the players do not care about the time at which agreement is struck, then there is nothing to prevent them from haggling for as long as they wish. On the other hand, frictions in the bargaining process may provide the players with some incentive to reach agreement.

It seems reasonable to assume that the share of the cake obtained by a player in the unique SPE reflects her 'bargaining power'. Thus, a player's bargaining power is decreasing in her discount rate, and increasing in her opponent's discount rate. Why does being relatively more patient confer greater bargaining power? To obtain some insight into this issue, I now identify the cost of haggling to each player, because it is intuitive that a player's bargaining power is decreasing in her cost of haggling, and increasing in her opponent's cost of haggling.

In the alternating-offers game, if a player does not wish to accept any particular offer and, instead, would like to make a counteroffer, then she is free to do so, but she has to incur a 'cost': this is the cost to her of waiting Δ time units. The smaller is her discount rate, the smaller is this cost. That is why being relatively more patient confers greater bargaining power.

Notice that even if both players' costs of haggling become arbitrarily small in absolute terms (for example, as $\Delta \to 0$), the equilibrium partition depends on the relative magnitude of these costs (as captured by the ratio r_A/r_B).

3.3.1. *Relationship with Nash's Bargaining Solution.* It is straightforward to verify that the limiting, as $\Delta \to 0$, SPE payoff pair $(\eta_A \pi, \eta_B \pi)$ — as stated in Corollary 1 — is identical to the asymmetric Nash bargaining solution of the bargaining problem (Ω, d) with $\tau = \eta_A$, where $\Omega = \{(u_A, u_B) : 0 \leq u_A \leq \pi \text{ and } u_B = \pi - u_A\}$ and $d = (0, 0)$ — where the asymmetric Nash bargaining solution is stated in section 2.2.

This remarkable result provides a strategic justification for Nash's bargaining solution. In particular, it provides answers to the questions of *why*, *when* and *how* to use Nash's bargaining solution. The asymmetric Nash bargaining solution is applicable because the bargaining outcome that it generates is identical to the (limiting) bargaining outcome that is generated by the basic alternating-offers model. However, it should only be used when Δ is arbitrarily small, which may be interpreted as follows: it should be used in those bargaining situations in which the *absolute* magnitudes of the frictions in the bargaining process are small. Furthermore, it should be defined on the bargaining problem (Ω, d), where Ω is the set of instantaneous utility pairs obtainable through agreement, and d is the payoff pair obtainable through perpetual disagreement (or what may be called the players' payoffs from an *impasse*).

3.3.2. *The Value and Interpretation of the Alternating-Offers Model.* The basic alternating-offers game is a stylized representation of the following two features that lie at the heart of most real-life negotiations:

- Players attempt to reach agreement by making offers and counteroffers.
- Bargaining imposes costs on both players.

As such the game is useful because it provides a basic framework upon which one can build richer models of bargaining — this is shown in later chapters where, for example, I shall study extensions of this game that incorporate other important features of real-life negotiations. Furthermore, since this game is relatively plausible and tractable, it is attractive to embed it (or, some extension of it) in larger economic models. An important value, therefore, of the basic alternating-offers game is in terms of the insights that its extensions deliver about bargaining situations, and also in terms of its usefulness in applications.

Another important contribution of this model is that it provides a justification for the use of Nash's bargaining solution. Furthermore, as is shown in later chapters, its extensions can guide the application of the Nash bargaining solution in relatively more complex bargaining situations.

The basic alternating-offers model incorporates several specific assumptions that I now interpret. First, I shall interpret the (infinite horizon) assumption that the players may make offers and counteroffers forever. Is this assumption plausible, especially since players have finite lives? I now argue that, when properly interpreted, this modelling assumption is compelling. The assumption is motivated by the observation that a player can always make a counteroffer immediately after rejecting an offer. This observation points towards the infinite horizon assumption, and against the finite horizon assumption.[6] Suppose, for example, a seller and a buyer are bargaining over the price of some object, and have to reach agreement within a single day. The players' strategic reasoning (and hence their bargaining behaviour) is typically influenced by their perception that after any offer is rejected there is room for at least one more offer. This suggests that the infinite horizon assumption is an appropriate *modelling* assumption — notwithstanding the *descriptive* reality that bargaining ends in finite time.

The time interval Δ between two consecutive offers is a parameter of the game, and it is assumed that $\Delta > 0$. I now argue that attention should, in general, focus on arbitrarily small values of Δ. The argument rests on the observation that since waiting to make a counteroffer is costly, a player will wish to make her counteroffer immediately after rejecting her opponent's offer. Unless there is some good reason that prevents her from doing so, the model with Δ arbitrarily small is the most compelling and least artificial. Why not then set $\Delta = 0$? There are two reasons for not doing so. The straightforward reason is that it does take some time, albeit very small, to reject an offer and make a counteroffer. The subtle reason is that such a model fails to capture any friction in the bargaining process. The model with $\Delta > 0$, on the other hand, does contain frictions — as captured by the players' positive costs of haggling. To further illustrate the difference between the model with $\Delta = 0$ and the model with Δ strictly positive but arbitrarily small, notice the following. In the limit as $\Delta \to 0$ the relative magnitude

[6]The finite horizon version of the alternating-offers game begs the following question. Why should the players stop bargaining after some exogenously given number of offers have been rejected? Since they would prefer to continue bargaining, somehow they have to be prevented from doing so. What, or who, prevents them from continuing to attempt to reach agreement? Unless a convincing answer is provided, the finite horizon assumption is implausible.

of the players' costs of haggling is well defined, but if $\Delta = 0$ then this relative magnitude is undetermined (since zero divided by zero is meaningless). This difference is of considerable significance, since, as I argued above, it is intuitive that the equilibrium partition depends critically on the players' relative bargaining powers — as captured by the relative magnitude of the players' costs of haggling — and not so much on absolute magnitudes.[7]

The discount factor may be interpreted more broadly as reflecting the costs of haggling — it need not be given a literal interpretation. For example, set $r_A = r_B = r$. The literal interpretation is that r is the players' common discount rate. An alternative interpretation is that r is the rate at which the cake (or, 'gains from trade') shrinks. For example, when bargaining over the partition of an ice cream, discounting future utilities is an insignificant factor. The friction in this bargaining process is that the ice cream is melting, where the rate r at which the ice cream is melting captures the magnitude of this friction.

4. Risk of Breakdown

While bargaining the players may perceive that the negotiations might break down in a *random* manner for one reason or another. A potential cause for such a *risk of breakdown* is that the players may get fed up as negotiations become protracted, and thus walk away from the negotiating table. This type of human behaviour is random, in the sense that the exact time at which a player walks away for such reasons is random. Another possible cause for the existence of a risk of breakdown is that 'intervention' by a third party results in the disappearance of the 'gains from co-operation' that exists between the two players. For example, while two firms bargain over how to divide the returns from the exploitation of a new technology, an outside firm may discover a superior technology that makes their technology obsolete. Another example is as follows: while a corruptible policeman and a criminal are bargaining over the bribe, an honest policeman turns up and reports the criminal to the authorities.

Two players, A and B, bargain over the partition of a cake of size π (where $\pi > 0$) according to the alternating-offers procedure, but with the following modification: immediately after any player rejects any offer at any time $t\Delta$, with probability p (where $0 < p < 1$) the negotiations break down in disagreement, and with probability $1 - p$

[7] In the limit as $\Delta \to 0$ the first-mover advantage disappears — hence this is an additional reason for focusing on arbitrarily small values of Δ.

the game proceeds to time $(t+1)\Delta$ — where the player makes her counteroffer.

The payoffs are as follows. If the players reach agreement at time $t\Delta$ ($t = 0, 1, 2, 3, \ldots$, and $\Delta > 0$) on a partition that gives player i a share x_i ($0 \le x_i \le \pi$) of the cake, then her payoff is x_i. If negotiations break down in disagreement at time $t\Delta$, then player i obtains a payoff of b_i, where $0 \le b_i < \pi$. The payoff pair (b_A, b_B) is called the *breakdown point*. Assume that $b_A + b_B < \pi$, which ensures that there exist mutually beneficial partitions of the cake.

If the players perpetually disagree (i.e., each player always rejects any offer made to her), then player i's payoff is

$$pb_i \sum_{t=0}^{\infty} (1-p)^t,$$

which equals b_i. Since player i can guarantee a payoff of b_i by always asking for a share b_i and always rejecting all offers, it follows that in any subgame perfect equilibrium, player i's ($i = A, B$) payoff is greater than or equal to b_i. The following proposition characterizes the unique subgame perfect equilibrium.

Proposition 2. *The unique subgame perfect equilibrium of the model with a risk of breakdown is as follows:*
• *player A always offers x_A^* and always accepts an offer x_B if and only if $x_B \le x_B^*$,*
• *player B always offers x_B^* and always accepts an offer x_A if and only if $x_A \le x_A^*$, where*

$$x_A^* = b_A + \frac{1}{2-p}\Big(\pi - b_A - b_B\Big) \quad \textit{and}$$
$$x_B^* = b_B + \frac{1}{2-p}\Big(\pi - b_A - b_B\Big).$$

Proof. The proof involves a straightforward adaptation of the arguments in sections 3.2. In particular, in any SPE that satisfies Properties 1 and 2 player i is indifferent between accepting and rejecting player j's ($j \neq i$) equilibrium offer. That is

$$\pi - x_A^* = pb_B + (1-p)x_B^* \quad \text{and}$$
$$\pi - x_B^* = pb_A + (1-p)x_A^*.$$

The unique solution to these two equations is stated in the proposition. □

In the unique SPE, agreement is reached at time 0, and the bargaining outcome is Pareto efficient: in equilibrium, the negotiations do not break down in disagreement. As I argued in Section 3.3.2, attention should in general be focused upon arbitrarily small values of Δ. It is assumed that as the time interval Δ between two consecutive offers decreases, the probability of breakdown p between two consecutive offers decreases, and that $p \to 0$ as $\Delta \to 0$.

Corollary 2 (Split-The-Difference Rule)**.** *In the limit, as $\Delta \to 0$, the unique SPE shares of the cake to players A and B respectively converge to*

$$b_A + \frac{1}{2}\Big(\pi - b_A - b_B\Big) \quad \textit{and} \quad b_B + \frac{1}{2}\Big(\pi - b_A - b_B\Big).$$

The friction in the bargaining process underlying the above described game arises from the risk that negotiations break down between two consecutive offers, which is captured by the probability p. As the absolute magnitude of this friction becomes arbitrarily small — and thus the common cost of haggling to the players becomes arbitrarily small — the limiting equilibrium partition of the cake, which is independent of who makes the first offer, may be interpreted as follows. The players agree first of all to give player i ($i = A, B$) a share b_i of the cake (which gives her a payoff equal to the payoff she obtains when, and if, negotiations break down), and then they split equally the remaining cake $\pi - b_A - b_B$.

5. Outside Options

Consider a situation in which University A and academic economist B bargain over the wage. Moreover, suppose that the academic economist has been offered a job at some alternative (but similar) university at a fixed, non-negotiable, wage w_B. A main objective of this chapter is to investigate the impact of such an 'outside option' on the outcome of bargaining between A and B. Although the academic economist B has to receive at least a wage of w_B if she is to work at University A, it is far from clear, for example, whether the negotiated wage will equal w_B, or strictly exceed w_B. In this section I study the role of outside options through a simple extension of the basic alternating-offers mode.[8]

Two players, A and B, bargain over the partition of a cake of size π (where $\pi > 0$) according to the alternating-offers procedure, but with the following modification. Whenever any player has to respond to any offer, she has three choices, namely: (i) accept the offer, (ii) reject the

[8]In contrast to the model studied above, in the model studied here there is no exogenous risk of breakdown. A player's decision to take up her outside option (and thus, to terminate the negotiations in disagreement) is a *strategic* decision: no random event forces a player to opt out.

offer and make a counteroffer Δ time units later, and (iii) reject the offer and opt out, in which case negotiations terminate in disagreement.

The payoffs are as follows. If the players reach agreement at time $t\Delta$ ($t = 0, 1, 2, 3, \ldots$, and $\Delta > 0$) on a partition that gives player i ($i = A, B$) a share x_i ($0 \leq x_i \leq \pi$) of the cake, then her payoff is $x_i \exp(-r_i t\Delta)$. If, on the other hand, the players do not reach agreement because a player opts out at time $t\Delta$, then player i takes up her outside option, and obtains a payoff of $w_i \exp(-r_i t\Delta)$, where $w_A < \pi$, $w_B < \pi$ and $w_A + w_B < \pi$. The *outside option point* is the payoff pair (w_A, w_B), where w_i is player i's outside option. Finally, if the players perpetually disagree (i.e., each player always rejects any offer made to her, and never opts out), then each player's payoff is zero. The following proposition characterizes the unique subgame perfect equilibrium of this model in the limit as $\Delta \to 0$.

Proposition 3 (The Outside Option Principle)**.** *In the limit, as* $\Delta \to 0$*, the unique SPE share* x_A^* *obtained by player A is as follows:*

$$x_A^* = \begin{cases} \eta_A \pi & \text{if } w_A \leq \eta_A \pi \text{ and } w_B \leq \eta_B \pi \\ \pi - w_B & \text{if } w_A \leq \eta_A \pi \text{ and } w_B > \eta_B \pi \\ w_A & \text{if } w_A > \eta_A \pi \text{ and } w_B \leq \eta_B \pi, \end{cases}$$

where $\eta_A = r_B/(r_A + r_B)$ *and* $\eta_B = r_A/(r_A + r_B)$*, and the unique SPE share obtained by player B is to* $x_B^* = \pi - x_A^*$*.*

Proof. The proof involves a minor and straightforward adaptation of the arguments in Sections 3.2. In any SPE that satisfies Properties 1 and 2 player i is indifferent between accepting and not accepting player j's ($j \neq i$) equilibrium offer. That is

$$\pi - x_A^* = \max\{\delta_B x_B^*, w_B\} \quad \text{and}$$
$$\pi - x_B^* = \max\{\delta_A x_A^*, w_A\}.$$

These equations have a unique solution, which converge to those stated in the proposition, as $\Delta \to 0$. □

In the unique SPE, agreement is reached at time 0, and the bargaining outcome is Pareto efficient: in equilibrium, the players do not take up their respective outside options. However, the presence of the outside options do influence the equilibrium partition of the cake. If each player's outside option is less than or equal to the share she receives in the (limiting) SPE of Rubinstein's model (cf. Corollary 1), then the outside options have no influence on the (limiting) SPE partition. On the other hand, if one player's outside option strictly exceeds her (limiting) Rubinsteinian SPE share, then her (limiting) SPE share is equal to her outside option.

As I now illustrate, the results derived above capture, in particular, the notion that the players should not be influenced by threats which are not credible. Suppose that University A and academic economist B are bargaining over the wage w when neither player has any outside option. The (instantaneous) utilities to A and B if they reach agreement on wage w are $1 - w$ and w, respectively, and both players discount future utilities at a common rate $r > 0$. Applying Corollary 1, it follows that the (limiting) SPE wage $w^* = 0.5$. Now suppose that B has an outside option $w_B > 0$. It seems intuitive that if B's outside option w_B is less than or equal to 0.5, then her threat to opt out is not credible. If she is receiving a wage of 0.5, then getting an alternative job offer with a wage of 0.49, for example, should be useless: University A should ignore any threats made by B to quit. If, on the other hand, the alternative wage exceeds 0.5, then University A has only to exactly match the outside wage offer; there is no need to give more.

I now show how to apply the asymmetric Nash bargaining solution in bargaining situations with outside options. First, note that it follows from Proposition 3 that the unique limiting (as $\Delta \to 0$) SPE payoff pair $(x_A^*, \pi - x_A^*)$ is identical to the unique solution of the following maximization problem

$$\max_{u_A, u_B} (u_A)^{\eta_A} (u_B)^{\eta_B}$$

subject to $(u_A, u_B) \in \Omega$, $u_A \geq 0$ and $u_B \geq 0$, where

(7)
$$\Omega = \{(u_A, u_B) : 0 \leq u_A \leq \pi, u_B = \pi - u_A, u_A \geq w_A \text{ and } u_B \geq w_B\}.$$

Thus, in the limit, as $\Delta \to 0$, the unique SPE payoff pair of the model with outside options and discounting converges to the asymmetric Nash bargaining solution of the bargaining problem (Ω, d) with $\tau = r_B/(r_A + r_B)$, where Ω is defined in (7) and $d = (0, 0)$. Thus, the Nash's bargaining solution is applicable when the friction in the bargaining process is arbitrarily small (i.e., Δ is arbitrarily close to zero). The important point to note here concerns, in particular, how the outside option point should be mapped into the objects upon which Nash's bargaining solution is defined. It should be noted that in the bargaining situation considered above, there are two possible ways in which the players fail to reach agreement: through perpetual disagreement and when a player opts out. The disagreement point in Nash's set-up should be identified with the players' payoff from impasse; the outside option point, on the other hand, constrains the set Ω of possible utility pairs on which Nash's bargaining solution should be defined — by requiring that each utility pair $(u_A, u_B) \in \Omega$ be such that u_i is at least as high as player i's outside option. I thus emphasize that *the outside option point does not affect the disagreement point.*

6. Inside Options

Consider the basic exchange situation in which a seller and a buyer are bargaining over the price at which the seller sells an indivisible object (such as a house) to the buyer. If agreement is reached on price p, then the seller's payoff is p and the buyer's payoff is $\pi - p$. Furthermore, the seller obtains utility at rate g_S while the object is in her possession, where $g_S \geq 0$; thus, for $\Delta > 0$ but small, she obtains a payoff of $g_S\Delta$ if she owns the house for Δ units of time. Given her discount rate $r_S > 0$, this means that if she keeps possession of the house forever, then her payoff is g_S/r_S, which is assumed to be less than π — for otherwise gains from trade do not exist. The payoff that the seller obtains while the parties *temporarily* disagree is her *inside option* — which equals $g_S[1 - \exp(-r_S\Delta)]/r_S$ if they disagree for Δ units of time. In contrast, her *outside option* is the payoff she obtains if she chooses to *permanently* stop bargaining, and chooses not to reach agreement with the buyer; for example, this could be the price p^* (where $p^* > g_S/r_S$) that she obtains by selling the house to some other buyer.

Two players, A and B, bargain over the partition of a cake of size π ($\pi > 0$) according to the alternating-offers procedure. The payoffs are as follows. If the players reach agreement at time $t\Delta$ ($t = 0, 1, 2, 3, \ldots$ and $\Delta > 0$) on a partition that gives player i a share x_i ($0 \leq x_i \leq \pi$) of the cake, then her payoff is

$$\int_0^{t\Delta} g_i \exp(-r_i s) ds + x_i \exp(-r_i t\Delta),$$

where $r_i > 0$ and $g_i \geq 0$. The interpretation behind this payoff is as follows: the second term is her (discounted) utility from x_i units of the cake (where r_i is her discount rate), while the first term captures the notion that until agreement is struck player i obtains a flow of utility at rate g_i. After integrating the first term, it follows that this payoff equals

$$\frac{g_i[1 - \exp(-r_i t\Delta)]}{r_i} + x_i \exp(-r_i t\Delta).$$

If an offer is rejected at time $t\Delta$, then in the time interval Δ — before a counteroffer is made at time $(t+1)\Delta$ — player i obtains a utility of $g_i[1 - \exp(-r_i\Delta)]/r_i$, which is her *inside option.* Notice that for

Δ small, her inside option is approximately equal to $g_i\Delta$. The pair (g_A, g_B) is called the *inside option point*.

If the players perpetually disagree (i.e., each player always rejects any offer made to her), then player i's payoff is g_i/r_i. Assume that $g_A/r_A + g_B/r_B < \pi$; for otherwise, gains from co-operation do not exist. A straightforward observation is that player i can guarantee a payoff of g_i/r_i by always asking for the whole cake and always rejecting all offers. Thus, in any subgame perfect equilibrium of any subgame of the model with inside options and discounting, player i's payoff is greater than or equal to g_i/r_i. The following proposition characterizes the unique subgame perfect equilibrium (SPE) of this model in the limit as $\Delta \to 0$.

Proposition 4 (Split-The-Difference Rule)**.** *In the limit, as $\Delta \to 0$, the unique subgame perfect equilibrium shares of the cake to players A and B respectively are:*

$$Q_A = \frac{g_A}{r_A} + \eta_A\left(\pi - \frac{g_A}{r_A} - \frac{g_B}{r_B}\right) \quad \textit{and}$$

$$Q_B = \frac{g_B}{r_B} + \eta_B\left(\pi - \frac{g_A}{r_A} - \frac{g_B}{r_B}\right),$$

where $\eta_A = r_B/(r_A + r_B)$ and $\eta_B = r_A/(r_A + r_B)$.

Proof. The proof involves a straightforward adaptation of the arguments in Sections 3.2. In particular, in any SPE that satisfies Properties 1 and 2 player i is indifferent between accepting and rejecting player j's equilibrium offer. That is

$$\pi - x_A^* = \frac{g_B(1-\delta_B)}{r_B} + \delta_B x_B^* \quad \text{and}$$

$$\pi - x_B^* = \frac{g_A(1-\delta_A)}{r_A} + \delta_A x_A^*.$$

These equations have a unique solution, which converge to those stated in the proposition as $\Delta \to 0$. □

The limiting equilibrium partition of the cake may be interpreted as follows. The players agree first of all to give each player i a share g_i/r_i of the cake — which gives her a payoff equal to the payoff that she obtains from perpetual disagreement — and then they split the remaining cake.

The limiting SPE payoff pair (Q_A, Q_B) is the unique solution of the following maximization problem:

$$\max_{u_A, u_B} (u_A - d_A)^{\eta_A}(u_B - d_B)^{\eta_B}$$

subject to $(u_A, u_B) \in \Omega$, $u_A \geq d_A$ and $u_B \geq d_B$, where

$$\Omega = \{(u_A, u_B) : 0 \leq u_A \leq \pi \text{ and } u_B = \pi - u_A\} \tag{8}$$

$$d = (g_A/r_A, g_B/r_B). \tag{9}$$

This observation implies the result that the limiting SPE payoff pair is identical to the asymmetric Nash bargaining solution of the bargaining problem (Ω, d) with $\tau = \eta_A$, where Ω and d are respectively defined in (8) and (9). Thus, in the limit, as $\Delta \to 0$, the unique subgame perfect equilibrium payoff pair in the model with inside options and discounting converges to the asymmetric Nash bargaining solution of the bargaining problem (Ω, d) with $\tau = \eta_A$, where Ω and d are respectively defined in (8) and (9), and $\eta_A = r_B/(r_A + r_B)$. Thus, Proposition 4 shows how to incorporate the impact of the inside options — as captured by the flow rates g_A and g_B — in Nash's bargaining solution: they affect the disagreement point in Nash's bargaining solution. This is another illustration of the insight — obtained also in other contexts — that the disagreement point in Nash's framework should be identified with the players' payoffs from perpetual disagreement (impasse).

6.1. **An Application to Sovereign Debt Renegotiations.** Country B, who produces one unit of some domestic commodity β per unit time, owes a large amount of some foreign commodity α to a foreign bank A. By trading on international markets, B obtains P units of commodity α for one unit of commodity β, where $P > 1$. The utility per unit time to B is the sum of the quantities of commodities α and β that it consumes. In the absence of any outside interference, in each unit of time B would trade the unit of commodity β for P units of the foreign commodity α, and obtain a utility of P. However, if A and B fail to reach agreement on some debt repayment scheme, then the bank seizes a fraction ν of the country's traded output.

The players bargain over the payment per unit time x that B makes to A everafter. If agreement is reached on x at time $t\Delta$, then in each future unit of time B trades the unit of its domestic commodity β for P units of the foreign commodity α — and thus, the payoffs to A and B (from time $t\Delta$ onwards) are respectively x/r and $(P - x)/r$, where $r > 0$ is the players' common discount rate. It is assumed that the amount that B owes A exceeds P/r, and furthermore, the parties are committed to the agreed debt repayment scheme. In the framework of the model studied in above, the players are bargaining over the partition of a cake of size $\pi = P/r$; and if player A receives a share x_i of this cake, then this means that the per unit time repayment $x = rx_i$.

The inside options to the players are now derived. If any offer is rejected at any time $t\Delta$, then — before a counteroffer is made at time $(t+1)\Delta$ — Δ units of the domestic commodity is produced. Country B either consumes all of it or trades without agreement. In the former case the inside options of B and A are respectively Δ and zero, while in the latter case the inside options of B and A are respectively $(1-\nu)P\Delta$ and $\nu P\Delta - \epsilon$, where ϵ denotes an infinitesimal (small) cost of seizure. Hence, since B makes the decision on whether to consume or trade, it follows (in the notation of the previous section) that

$$(g_A, g_B) = \begin{cases} (0,1) & \text{if } 1 > (1-\nu)P \\ (\nu P - \epsilon, (1-\nu)P) & \text{if } 1 \leq (1-\nu)P. \end{cases}$$

Noting that $g_A/r + g_B/r < \pi$, one may apply Proposition 4 and obtain that the players reach agreement immediately (at time 0) with Country B agreeing to pay the foreign bank an amount x per unit time, where, in the limit as $\epsilon \to 0$, x converges to

$$x^* = \begin{cases} (P-1)/2 & \text{if } \nu > 1 - (1/P) \\ \nu P & \text{if } \nu \leq 1 - (1/P). \end{cases}$$

If international trade sanctions (as captured by the value of ν) are sufficiently harsh, then Country B's inside option is derived from consuming the domestic commodity — which implies that the equilibrium debt payment per unit time equals half the gains from trade. On the other hand, if international trade sanctions are not too harsh, then Country B's inside option is derived from trading the domestic commodity — which implies that the equilibrium debt payment per unit time equals the quantity of traded good seized.

A major insight obtained from this analysis is as follows. If ν is sufficiently high, then further international trade sanctions (as captured by an increase in ν) have no effect on debt payments. In contrast, an increase in the terms of Country B's international trade (as captured by an increase in P) always increases debt payments.

It seems reasonable to assume that the value of ν can be (strategically) chosen by the foreign bank. Assuming that ν is chosen before the renegotiations begin, and that the bank is committed to its choice, the bank will set ν to maximize the equilibrium debt payment per unit time x^*. For any P, x^* is strictly increasing in ν over the closed interval $[0, 1-(1/P)]$, and equals $(P-1)/2$ for any ν in the interval $(1-(1/P), 1]$. Hence, since at $\nu = 1-(1/P)$, $x^* = P-1$, the optimal value of ν is $\nu^* = 1-(1/P)$. Thus, since $1-(1/P) < 1$, the optimal level of ν is strictly less than one; that is, it is not optimal for the foreign bank to seize all the traded output. The optimal level of debt payment per unit time equals $P-1$, the gains from international trade per unit time. Thus, at the optimum, the bank captures all of the gains from international trade — it cannot, however, extract any greater amount of payment per unit time from Country B.

7. Asymmetric Information

In some bargaining situations at least one of the players knows something of relevance that the other player does not. For example, when bargaining over the price of her second-hand car the seller knows its quality but the buyer does not. In such a bargaining situation, the seller has private information; and there exists an asymmetry in information between the players. In this section I study the role of asymmetric information on the bargaining outcome.

A player may in general have private information about a variety of things that may be relevant for the bargaining outcome, such as her preferences, outside option and inside option. However, in order to develop the main fundamental insights in a simple manner attention is focused on the following archetypal bargaining situation. A seller and a buyer are bargaining over the price at which to trade an indivisible object (such as a second-hand car, or a unit of labour). The payoff to each player (from trading) depends on the agreed price and on her reservation value. A key assumption is that at least one player's reservation value is her private information.

The following argument illustrates the possibility that the bargaining outcome cannot be efficient. A buyer and a seller are bargaining over the price of a second-hand car, whose quality is the seller's private information. If she owns a low quality car, then she has an incentive to pretend to own a high quality car in order to obtain a relatively high price. Since the buyer is aware of this 'incentive to lie', the maximum price that she might be willing to pay may be strictly less than the high reservation value of a seller owning a high quality car. Thus, if the seller actually owns a high quality car, then mutually beneficial trade between the two parties may fail to occur.

Consider a bargaining situation in which player S owns (or, can produce) an indivisible object that player B wants to buy. If agreement is reached to trade at price p ($p \geq 0$), then the payoffs to the seller (player S) and the buyer (player B) are respectively $p-c$ and $v-p$, where c denotes the seller's reservation value (or, cost of production) and v denotes the buyer's reservation value (or, the maximum price at which she is willing to buy). If the players do not reach an agreement to trade, then each player's payoff is zero. The outcome of this bargaining situation is *ex-post* efficient if and only if when $v \geq c$ the players reach an agreement to trade, and when $v < c$ the players do not reach an agreement to trade.

A key assumption is that exactly one player's reservation value is her private information. Section 7.1 studies the case in which the players' reservation values are independent of each other, while section 7.2 studies the case in which the players' reservation values are correlated.

7.1. **The Case of Private Values.** In this section it is assumed that the players' reservation values are independent of each other, and exactly one player, say the buyer, has private information about her reservation value. The seller's reservation value is known to both players. This asymmetry in information is modelled as follows. The buyer's reservation value is a random draw from the following (binary) probability distribution: with probability α (where $0 < \alpha < 1$) the buyer's reservation value is H, and with probability $1 - \alpha$ the buyer's reservation value is L, where $H > L$. The buyer knows the realization of the random draw, but the seller does not. The seller only knows that the buyer's reservation value is a random draw from this probability distribution. The following proposition establishes that the bargaining outcome can be *ex-post* efficient.

Proposition 5. *There exists a bargaining procedure such that the induced bargaining game has an ex-post efficient Bayesian Nash Equilibrium (BNE).*

Proof. Consider the following bargaining procedure. The buyer makes an offer to the seller. If she accepts the offer, then agreement is struck and the game ends. But if she rejects the offer, then the game ends with no agreement. Letting p^*_H and p^*_L respectively denote the buyer's price offers when $v = H$ and $v = L$, the following pair of strategies is a BNE: $p^*_H = \min\{H, c\}$, $p^*_L = \min\{L, c\}$, and the seller accepts a price offer p if and only if $p \geq c$. The proposition follows immediately, because this BNE is *ex-post* efficient. □

It is straightforward to generalise this argument and show that if the players' reservation values are independent of each other, and exactly one player's reservation value is her private information, then the bargaining outcome can be *ex-post* efficient.

7.2. **The Case of Correlated Values.** I now assume that the player's reservation values are correlated, and exactly one player has private information about her reservation value. This assumption is modelled as follows. There is a parameter θ — which is a real number — that determines both players' reservation values, and furthermore, the value of θ is the private information of exactly one player. It is assumed that each player's reservation value is strictly increasing in θ. Furthermore, for any θ, the buyer's reservation value — which is denoted by $v(\theta)$ — is greater than or equal to the seller's reservation value — which is denoted by $c(\theta)$.

Assume that it is the seller who has private information about θ. This asymmetry in information is modelled as follows. The value of θ is a random draw from the following (binary) probability distribution: with probability α (where $0 < \alpha < 1$) the value of θ is H, and with probability $1 - \alpha$ the value of θ is L, where $H > L$. The seller knows the realization of the random draw, but the buyer does not. The buyer only knows that the value of θ is a random draw from this probability distribution. The following proposition establishes that the bargaining outcome can be *ex-post* efficient if and only if $v^e \geq c(H)$, where $v^e = \alpha v(H) + (1 - \alpha)v(L)$ is the buyer's expected reservation value.

Proposition 6. *(a) If $v^e \geq c(H)$, where $v^e = \alpha v(H) + (1 - \alpha)v(L)$, then there exists a bargaining procedure such that the induced bargaining game has an ex-post efficient BNE.*
(b) If $v^e < c(H)$, then for any bargaining procedure the induced bargaining game does not have an ex-post efficient BNE.

Proof. I first establish Proposition 6(a). Consider the following bargaining procedure. The seller makes an offer to the buyer. If she accepts the offer, then agreement is struck and the game ends. But if she rejects the offer, then the game ends with no agreement. Since $v^e \geq c(H)$, the following pair of strategies is a BNE: $p^*(H) = p^*(L) = c(H)$ (where $p^*(H)$ and $p^*(L)$ are respectively the seller's price offers when $\theta = H$ and $\theta = L$), the buyer accepts the price $p = c(H)$ and rejects any price $p \neq c(H)$. The desired conclusion follows immediately, because this BNE is *ex-post* efficient. □

I now proceed to prove Proposition 6(b). In order to do so I need to consider the set of *all* possible bargaining procedures. However, I begin by considering a particular subset of the set of all bargaining procedures, which is called the set of all *direct revelation* procedures. In the context of the bargaining situation under consideration, a direct revelation procedure (DRP) is characterized by four numbers: λ_L, λ_H, p_L and p_H, where $\lambda_s \in [0, 1]$ and $p_s \geq 0$ ($s = L, H$). In a DRP the seller announces a possible value of θ. If s denotes the announced value (where $s \in \{L, H\}$), then with probability λ_s trade occurs at price p_s, and with probability $1 - \lambda_s$ trade does not occur.

Fix an arbitrary DRP, and consider the induced bargaining game (which is a single-person decision problem). Let $s(\theta) \in \{L, H\}$ denote the seller's announcement if the true (realized) value is θ ($\theta = L, H$). The DRP is *incentive-compatible* if and only if in the induced bargaining game the seller announces the truth — that is, $s^*(L) = L$ and $s^*(H) = H$. Thus, the DRP is incentive-compatible if and only if the following two inequalities are satisfied

$$\lambda_L(p_L - c(L)) \geq \lambda_H(p_H - c(L)) \tag{10}$$

$$\lambda_H(p_H - c(H)) \geq \lambda_L(p_L - c(H)). \tag{11}$$

Inequalities 10 and 11 are respectively known as the incentive-compatibility constraints for the *low-type* seller and *high-type* seller. Inequality 10 states that the expected payoff to the low-type seller by announcing the truth is greater than or equal to her expected payoff by telling a lie. Similarly, inequality 11 states that the expected payoff to the high-type seller by announcing the truth is greater than or equal to her expected payoff by telling a lie.

An incentive-compatible DRP is *individually-rational* if and only if in the incentive-compatible DRP each type of seller and the buyer obtain an expected payoff that is not less than their respective payoff from disagreement (which equals zero). That is, if and only if the following three inequalities are satisfied

$$\lambda_L(p_L - c(L)) \geq 0 \tag{12}$$

$$\lambda_H(p_H - c(H)) \geq 0 \tag{13}$$

$$\alpha\lambda_H(v(H) - p_H) + (1 - \alpha)\lambda_L(v(L) - p_L) \geq 0. \tag{14}$$

Since (by assumption) $v(H) \geq c(H)$ and $v(L) \geq c(L)$, a DRP is *ex-post efficient* if and only if the buyer trades with the seller of either type with probability one. That is, if and only if

$$\lambda_L = \lambda_H = 1. \tag{15}$$

The Revelation Principle allows me to consider only the set of all incentive-compatible and individually-rational direct revelation procedures.[9] It follows from the Revelation Principle that if there does not

[9]The Revelation Principle is as follows. Fix an arbitrary bargaining situation with asymmetric information and an arbitrary bargaining procedure. For any BNE outcome of the induced bargaining game there exists an incentive-compatible and individually-rational DRP that implements the BNE outcome.

exist an incentive-compatible and individually-rational DRP that is *ex-post* efficient, then there does not exist a bargaining procedure whose induced bargaining game has an *ex-post* efficient BNE. Proposition 6(b) is therefore an immediate consequence of the following claim.

Claim 1. *If $v^e < c(H)$, where v^e is defined in Proposition 6, then there does not exist an incentive-compatible and individually-rational DRP that is ex-post efficient.*

Proof. Suppose, to the contrary, that there exists a DRP that satisfies (10)–(15). Substituting (15) into (10) and (11), it follows that $p_L = p_H$. Hence, after substituting (15) into (13)–(15), it follows from (13)–(15) that $v^e \geq c(H)$, thus contradicting the hypothesis. □

8. Repeated Bargaining Situations

In this section I study a model of a situation in which two players have the opportunity to be involved in a sequence of bargaining situations. Such a situation will be called a 'repeated' bargaining situation (RBS). Examples of repeated bargaining situations abound. For instance: (i) in any marriage the wife and the husband are in a RBS, and (ii) in most bilateral monopoly markets the seller and the buyer are in a RBS.

Two players, A and B, bargain over the partition of a cake of size π ($\pi > 0$) according to the alternating-offers procedure. If agreement is reached at time t_1, where $t_1 = 0, \Delta, 2\Delta, \ldots$, and Δ ($\Delta > 0$) is the time interval between two consecutive offers, then immediately the players consume their respective (agreed) shares. Then τ ($\tau > 0$) time units later, at time $t_1 + \tau$, the players bargain over the partition of a second cake of size π according to the alternating-offers procedure. Agreement at time t_2, where $t_2 = t_1 + \tau, t_1 + \tau + \Delta, t_1 + \tau + 2\Delta, \ldots$, is followed immediately with the players consuming their respective (agreed) shares. Then τ time units later, at time $t_2 + \tau$, the players bargain over the partition of a third cake of size π according to the alternating-offers procedure. This process continues indefinitely, provided that the players always reach agreement. However, if the players perpetually disagree over the partition of some cake, then there is no further bargaining over new cakes; the players have terminated their relationship. Without loss of generality, I assume that player i makes the first offer when bargaining begins over the partition of the $(n+1)$th cake ($n = 1, 2, \ldots$) if it was player j ($j \neq i$) whose offer over the partition of the nth cake was accepted by player i. Furthermore, player A makes the offer at time 0.

The payoffs to the players depend on the number N (where $N = 0, 1, 2, \ldots$) of cakes that they partition. If $N = 0$ — that is, they perpetually disagree over the partition of the first cake — then each player's payoff is zero. If $1 \leq N < \infty$ — that is, they partition N cakes and perpetually disagree over the partition of the $(N+1)$th cake — then player i's $(i = A, B)$ payoff is

$$\sum_{n=1}^{N} x_i^n \exp(-r_i t_n),$$

where x_i^n $(0 \leq x_i^n \leq \pi)$ is player i's share of the nth cake, t_n is the time at which agreement over the partition of the nth cake is struck, and r_i $(r_i > 0)$ is player i's discount rate. Finally, if $N = \infty$ — that is, they partition all the cakes — then player i's payoff is

$$\sum_{n=1}^{\infty} x_i^n \exp(-r_i t_n).$$

Define for each $i = A, B$, $\delta_i \equiv \exp(-r_i \Delta)$ and $\alpha_i \equiv \exp(-r_i \tau)$. The parameters δ_A and δ_B capture the bargaining frictions: they respectively represent the costs to players A and B of haggling over the partition of a cake. In contrast, the parameters α_A and α_B respectively represent the values to players A and B of future bargaining situations.

One interpretation of the repeated bargaining model described above is as follows. Two players have the opportunity to engage in an infinite sequence of 'one-shot' transactions, where π denotes the size of the *surplus* generated from each one-shot transaction and τ the *frequency* of such one-shot transactions. The outcomes of any pair of one-shot transactions are negotiated separately, and, moreover, the outcome of each one-shot transaction is negotiated when (and if) it materializes. Letting a *one-shot* contract denote a contract that specifies the outcome of a single one-shot transaction, the model embodies the notion that the long-term relationship is governed by a sequence of one-shot contracts. An alternative interpretation of the model is as follows. Two players have the opportunity to generate (through some form of co-operation) a *flow* of money at rate $\widehat{\pi}$. They bargain over a contract that specifies the partition of the money over the duration of the contract, where τ is the *duration* of the contract. Thus, the present discounted value of the total amount of money generated over the duration of any single contract is $\pi = \widehat{\pi}[1 - \exp(-r_m \tau)]/r_m$, where r_m $(r_m > 0)$ is the market interest rate. The model embodies the notion that the long-term relationship is governed by a sequence of *limited-term* contracts (of duration τ).

8.1. The Unique Stationary Subgame Perfect Equilibrium.

Fix an arbitrary subgame perfect equilibrium (SPE) that satisfies Properties 1 and 2 (which are stated in section 3). Given Property 2, let x_i^* $(i = A, B)$ denote the equilibrium offer that player i makes whenever she has to make an offer. I adopt the convention that an offer is the share to the proposer. Furthermore, letting V_i^* denote player i's equilibrium payoff in any subgame beginning with her offer, it follows from Properties 1 and 2 that $V_i^* = x_i^* + \alpha_i(\pi - x_j^*) + \alpha_i^2 V_i^*$ $(j \neq i)$. Hence, it follows that

$$V_A^* = \frac{x_A^* + \alpha_A(\pi - x_B^*)}{1 - \alpha_A^2} \quad \text{and} \quad V_B^* = \frac{x_B^* + \alpha_B(\pi - x_A^*)}{1 - \alpha_B^2}. \tag{16}$$

Consider an arbitrary point in time at which player i has to make an offer to player j. By definition, player j's equilibrium payoff from rejecting any offer is $\delta_j V_j^*$. Therefore, perfection requires that player j accept any offer x_i (where $0 \leq x_i \leq \pi$) such that $\pi - x_i + \alpha_j V_j^* > \delta_j V_j^*$, and reject any offer x_i such that $\pi - x_i + \alpha_j V_j^* < \delta_j V_j^*$. Hence, if $\Delta \geq \tau$, then optimality implies that $x_A^* = x_B^* = \pi$.

On the other hand, if $\Delta < \tau$ then player i is indifferent between accepting and rejecting player j's equilibrium offer. That is

$$\pi - x_B^* + \alpha_A V_A^* = \delta_A V_A^* \quad \text{and} \quad \pi - x_A^* + \alpha_B V_B^* = \delta_B V_B^*. \tag{17}$$

After substituting for V_A^* and V_B^* in (17) using (16), and then solving for x_A^* and x_B^*, it follows that

$$x_A^* = \frac{(1 - \delta_A\alpha_A)(1 - \delta_B)(1 + \alpha_B)\pi}{(1 - \delta_A\alpha_A)(1 - \delta_B\alpha_B) - (\delta_A - \alpha_A)(\delta_B - \alpha_B)} \quad \text{and} \tag{18}$$

$$x_B^* = \frac{(1 - \delta_B\alpha_B)(1 - \delta_A)(1 + \alpha_A)\pi}{(1 - \delta_A\alpha_A)(1 - \delta_B\alpha_B) - (\delta_A - \alpha_A)(\delta_B - \alpha_B)}. \tag{19}$$

Hence, we obtain the following result:

Proposition 7. *The unique stationary subgame perfect equilibrium of the repeated bargaining model is as follows:*

- *player A always offers x_A^* and always accepts an offer x_B if and only if $x_B \leq x_B^*$,*
- *player B always offers x_B^* and always accepts an offer x_A if and only if $x_A \leq x_A^*$,*

where if $\Delta \geq \tau$ then $x_A^* = x_B^* = \pi$, and if $\Delta < \tau$ then x_A^* and x_B^* are respectively stated in equations 18 and 19. In equilibrium agreement is reached immediately over the partition of each and every cake. The equilibrium partition of the nth cake is $(x_A^*, \pi - x_A^*)$ if n is odd (i.e., $n = 1, 3, 5, \ldots$), and $(\pi - x_B^*, x_B^*)$ if n is even (i.e., $n = 2, 4, 6, \ldots$).

Hence, if $\Delta \geq \tau$, then in the unique stationary subgame perfect equilibrium (SSPE) player A obtains the whole of the nth cake when n is odd and player B obtains the whole of the nth cake when n is even. The intuition behind this result is that if $\Delta \geq \tau$, then (in a SSPE) the proposer effectively makes a 'take-it-or-leave-it-offer'. Although this result is rather provocative, it is not plausible that $\Delta \geq \tau$. It is more likely that $\tau > \Delta$ — the time interval between two consecutive offers during bargaining over the partition of any cake is smaller than the time taken for the 'arrival' of a new cake.

Before proceeding further, however, I note that (in general) the unique SSPE partition of each and every cake is different from the unique SPE partition of the *single* available cake in Rubinstein's bargaining model. The intuition for this difference — which is further developed below — is based on the following observation. In Rubinstein's model the cost to player i of rejecting an offer is captured by δ_i; rejecting an offer shrinks, from player i's perspective, the single available cake by a factor of δ_i. In contrast, in the model studied here the rejection of an offer not only shrinks the current cake, but it also shrinks *all* the future cakes, thus inducing a relatively higher cost of rejecting an offer.

As is evident from the expressions for x_A^* and x_B^* in (18) and (19), when $\tau > \Delta$ the equilibrium partition of each and every cake depends on the parameters r_A, r_B, Δ and τ in a rather complex manner. However, one of the main insights of the model can be obtained in a simple manner by examining the impact of the *derived* parameters α_A, α_B, δ_A and δ_B on the unique SSPE partitions. Notice that if $\tau > \Delta$, then $\delta_i > \alpha_i$ $(i = A, B)$.

Corollary 3. *For each $i = A, B$, x_i^* is (i) strictly increasing in δ_i on the set Z, (ii) strictly decreasing in α_i on the set Z, (iii) strictly decreasing in δ_j $(j \neq i)$ on the set Z and (iv) strictly increasing in α_j on the set Z, where*

$$Z = \{(\alpha_A, \alpha_B, \delta_A, \delta_B) : \delta_A > \alpha_A \text{ and } \delta_B > \alpha_B\}.$$

Proof. The corollary follows in a straightforward manner from the four derivatives of x_i^* (which is stated in (18)–(19)) — with respect to δ_i, α_i, δ_j and α_j. □

It follows immediately from Corollary 3 and Proposition 7 that if $\tau > \Delta$, then player i's share of *each and every cake* in the unique SSPE is strictly increasing in δ_i, but strictly decreasing in α_i. The former effect is consistent with the insight obtained in the context of Rubinstein's bargaining model when the players bargain over the partition of the *single* available cake. The latter effect, however, is novel, but the intuition behind it is straightforward. As α_i increases, the value to player i of future bargaining situations increases. Thus, when bargaining over the partition of any cake, her desire to proceed to bargain over the partition of the next cake has increased, which works to player j's advantage.

Notice that Corollary 3 implies that a decrease in r_i has two opposite effects on player i's equilibrium share of each and every cake — because a decrease in r_i increases both δ_i and α_i. It will be shown below that if Δ is arbitrarily small and (for each $i = A, B$) $r_i\tau > 0$ but small, then the effect through α_i dominates that through δ_i, thus implying that as player i becomes more patient her equilibrium share of each and every cake decreases.

8.2. **Small Time Intervals Between Consecutive Offers.** In Corollary 4 below I characterize the unique SSPE in the limit, as $\Delta \to 0$. I focus attention on this limit because it is the most persuasive case. Besides, the expressions for x_A^* and x_B^* in this limit are relatively more transparent compared to those in (18) and (19).

Corollary 4. *Fix any $r_A > 0$, $r_B > 0$ and $\tau > 0$. In the limit, as $\Delta \to 0$, x_A^* and x_B^* (as defined in Proposition 7) respectively converge to*

$$z_A^* = \frac{r_B\pi}{r_B + \phi_A r_A} \quad \textit{and} \quad z_B^* = \frac{r_A\pi}{r_A + \phi_B r_B}, \quad \textit{where}$$

$$\phi_A = \frac{(1+\alpha_A)(1-\alpha_B)}{(1-\alpha_A)(1+\alpha_B)} \quad \textit{and} \quad \phi_B = \frac{(1+\alpha_B)(1-\alpha_A)}{(1-\alpha_B)(1+\alpha_A)}.$$

Furthermore, the payoffs to players A and B in the limiting (as $\Delta \to 0$) unique SSPE are respectively

$$V_A^{**} = \frac{z_A^*}{1-\alpha_A} \quad \textit{and} \quad V_B^{**} = \frac{z_B^*}{1-\alpha_B}.$$

Proof. Fix $\tau > \Delta$. When $\Delta > 0$ but small, $\delta_i = 1 - r_i\Delta$. Using this to substitute for δ_A and δ_B in (18) and (19), it follows (after simplifying) that when $\Delta > 0$ but small

$$x_A^* = \frac{(1-\alpha_A + r_A\alpha_A\Delta)(1+\alpha_B)r_B\Delta\pi}{(1-\alpha_A\alpha_B)(r_A + r_B - r_A r_B\Delta)\Delta + (\alpha_B - \alpha_A)(r_B - r_A)\Delta} \quad \text{and}$$

$$x_B^* = \frac{(1-\alpha_B + r_B\alpha_B\Delta)(1+\alpha_A)r_A\Delta\pi}{(1-\alpha_A\alpha_B)(r_A + r_B - r_A r_B\Delta)\Delta + (\alpha_A - \alpha_B)(r_A - r_B)\Delta}.$$

After dividing the numerator and the denominator of each of these expressions by Δ, and then letting Δ tend to zero, it follows that

$x_i^* \to z_i^*$. Since $z_A^* + z_B^* = \pi$, it follows from Proposition 7 that the equilibrium payoffs in this limit are as stated in the corollary. □

It follows from Corollaries 1 and 4 that unless $\phi_A = \phi_B = 1$, the limiting (as $\Delta \to 0$) unique SSPE partition of each and every cake in the repeated bargaining model is different from the limiting (as $\Delta \to 0$) unique SPE partition of the single available cake in Rubinstein's bargaining model. Notice that for any $r_A > 0$, $r_B > 0$ and $\tau > 0$, $\phi_i = 1$ if and only if $r_A = r_B$. Furthermore, since $\phi_i \to 1$ as $r_i\tau \to \infty$, it follows that when $r_A \neq r_B$, the limiting (as $\Delta \to 0$) unique SSPE partition of each and every cake will have similar properties to the limiting (as $\Delta \to 0$) unique SPE partition of the single available cake in Rubinstein's model if and only if the value to each player of future bargaining situations is arbitrarily small.

The following corollary characterizes the properties of the limiting (as $\Delta \to 0$) unique SSPE when $r_A \neq r_B$ and (for each $i = A, B$) $r_i\tau$ is small (which means that the value to player i of future bargaining situations is large).

Corollary 5. *Assume that $r_A \neq r_B$, $r_A\tau > 0$ but small and $r_B\tau > 0$ but small.*
(i) *z_i^* $(i = A, B)$ is strictly increasing in r_i, and strictly decreasing in r_j $(j \neq i)$.*
(ii) *If $r_i > r_j$ $(i \neq j)$, then $z_i^* > z_j^*$.*
(iii) *If $r_i > r_j$ $(i \neq j)$, then $V_i^{**} < V_j^{**}$.*
(iv) *V_i^{**} $(i = A, B)$ is strictly decreasing in r_i, and strictly decreasing in r_j $(j \neq i)$.*
(v) *If $r_i > r_j$ $(i \neq j)$ then z_i^* is strictly increasing in τ, and if $r_i < r_j$ then z_i^* is strictly decreasing in τ.*
(vi) *In the limit as $r_A\tau \to 0$ and $r_B\tau \to 0$, $z_A^* \to \pi/2$ and $z_B^* \to \pi/2$.*

Proof. When $r_i\tau > 0$ but small, $\alpha_i = 1 - r_i\tau$. Using this to substitute for α_A and α_B in the expressions for ϕ_A and ϕ_B (stated in Corollary 4), it follows that when (for each $i = A, B$) $r_i\tau > 0$ but small

$$\phi_A = \frac{(2 - r_A\tau)r_B}{r_A(2 - r_B\tau)} \quad \text{and} \quad \phi_B = \frac{(2 - r_B\tau)r_A}{r_B(2 - r_A\tau)}.$$

This implies that when (for each $i = A, B$) $r_i\tau > 0$ but small

$$z_A^* = \frac{(2 - r_B\tau)\pi}{4 - (r_A + r_B)\tau} \quad \text{and} \quad z_B^* = \frac{(2 - r_A\tau)\pi}{4 - (r_A + r_B)\tau}.$$

The results in the corollary are now straightforward to derive, given these simple expressions for z_A^* and z_B^*. □

Corollary 5(i) states that when $r_A \neq r_B$ and the value to each player of future bargaining situations is large, player i's ($i = A, B$) share of each and every cake in the limiting (as $\Delta \to 0$) unique SSPE decreases as she becomes more patient and/or her opponent becomes less patient. In contrast, in Rubinstein's bargaining model a player's share of the single available cake in the limiting (as $\Delta \to 0$) unique SPE increases as she becomes more patient and/or her opponent becomes less patient (cf. Corollary 1). Thus, this fundamental insight of Rubinstein's model does not carry over to long-term relationships when the players have the opportunity to bargain (sequentially) over the partition of an infinite number of cakes and the value to each player of future bargaining situations is large. The intuition behind this conclusion is as follows.

In the repeated bargaining model studied here a player's discount rate determines not only her cost of rejecting an offer, but also her value of future bargaining situations. Suppose that one of the players — say, player i — becomes more patient. This means that her cost of rejecting an offer decreases. However, it also means that her value of future bargaining situations increases. When bargaining over the partition of a cake, the former effect increases her bargaining power (as she is more willing to reject offers), but the latter effect decreases her bargaining power — because she is more willing to accept offers so that the players can proceed to bargain over the partition of the next cake (cf. Corollary 3). When Δ is arbitrarily small the former effect is negligible, and when $r_i\tau > 0$ (but small) the latter effect is non-negligible. Thus, the latter effect dominates the former effect.

The above argument also provides intuition for the result stated in Corollary 5(ii) that the less patient of the two players receives a greater share of each and every cake. Not surprisingly, however, the limiting (as $\Delta \to 0$) unique SSPE *payoff* of the less patient of the two players is smaller than her opponent's limiting unique SSPE payoff (Corollary 5(iii)). This means that the more patient player's *overall* bargaining power *in the long-term relationship* is relatively higher. And Corollary 5(iv) shows that overall bargaining power increases as she becomes even more patient, but decreases as her opponent becomes less patient.

Corollary 5(v) implies that the more patient player's limiting (as $\Delta \to 0$) SSPE share of each and every cake decreases as τ increases. The intuition behind this conclusion runs as follows. As τ increases, the decrease in the more patient player's value of future bargaining situations is smaller than the decrease in her opponent's value of future bargaining situations, which works to her opponent's advantage.

Notice that if $r_A\tau > 0$ and $r_B\tau > 0$, then the players' shares of each and every cake (and payoffs) in the limiting (as $\Delta \to 0$) unique SSPE depend on the *relative* magnitude of the players' discount rates. However, Corollary 5(vi) shows that in the limit as $r_A\tau \to 0$ and $r_B\tau \to 0$, this is no longer true. This implies that (under these limiting conditions) the players' bargaining powers are identical no matter how patient or impatient player A is *relative* to player B. Hence, the insight of Rubinstein's bargaining model — that the *relative* magnitude of the players' discount rates critically determine the players' bargaining powers even when the time interval between two consecutive offers is arbitrarily small — does not carry over to long-term relationships when the players have the opportunity to bargain (sequentially) over an infinite number of cakes and the value to each player of future bargaining situations is arbitrarily large.

9. Concluding Remarks

The theory developed in the preceding sections contains some fundamental results and insights concerning the role of some key factors on the bargaining outcome. It cannot be overemphasized that the focus of this theory is on the *fundamentals*. Indeed, in the first stage of the development of an understanding of any phenomenon that is precisely the kind of theory that is required — one that cuts across a wide and rich variety of real-life scenarios and focuses upon their common core elements.

This objective to uncover the fundamentals of bargaining has meant that so far attention has centred on some basic, elementary models. That is how it should be. However, it is important that we now move beyond the fundamentals, in order to develop a richer theory of bargaining. With the theory just developed providing appropriate guidance and a firm foundation, it should be possible to construct tractable and richer models that capture more aspects of real-life bargaining situations. At the same time, future research should continue to develop the fundamentals; not only should we further study the roles of the factors studied in this article — especially the study of models in which many such factors are present — but we should also study the role of other factors that have not been addressed in this article.

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