Implementation in Mixed Nash Equilibrium

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Abstract

A mechanism implements a social choice correspondence f in mixed Nash equilibrium if, at any preference profile, the set of *all* (pure and mixed) Nash equilibrium outcomes coincides with the set of f-optimal alternatives at that preference profile. This definition generalizes Maskin's definition of Nash implementation in that it does not require each optimal alternative to be the outcome of a *pure* Nash equilibrium. We show that the condition of weak set-monotonicity, a weakening of Maskin's monotonicity, is necessary for implementation. We provide sufficient conditions for implementation and show that important social choice correspondences that are not Maskin monotonic can be implemented in mixed Nash equilibrium.

Keywords: implementation, Maskin monotonicity, pure and mixed Nash equilibrium, weak set-monotonicity, social choice correspondence.

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1 Introduction

This paper studies the problem of implementation in mixed Nash equilibrium. According to our definition, a mechanism implements a social choice correspondence f in mixed Nash equilibrium if the set of all (pure and mixed) equilibrium outcomes corresponds to the set of f-optimal alternatives at each preference profile. Crucially, and unlike the classical definition of implementation, this definition of implementation does not give a predominant role to pure equilibria: an f-optimal alternative does not have to be the outcome of a pure Nash equilibrium. This sharply contrasts with most of the existing literature on Nash implementation, which does not consider equilibria in mixed strategies (see Jackson, 2001, and Maskin and Sjöström, 2002, for excellent surveys). Two notable exceptions are Maskin (1999) for Nash implementation and Serrano and Vohra (2009) for Bayesian implementation. These authors do consider mixed equilibria, but still require each f-optimal alternative to be the outcome of a *pure* equilibrium. Pure equilibria are yet again given a special status.

Perhaps, the emphasis on pure equilibria expresses a discomfort with the classical view of mixing as deliberate randomizations on the part of players. However, it is now accepted that even if players do not randomize but choose definite actions, a mixed strategy may be viewed as a representation of the other players' uncertainty about a player's choice (e.g., see Aumann and Brandenburger, 1995). Moreover, almost all mixed equilibria can be viewed as pure Bayesian equilibria of nearby games of incomplete information, in which players are uncertain about the exact profile of preferences, as first suggested in the seminal work of Harsanyi (1973). This view acknowledges that games with commonly known preferences are an idealization, a limit of near-complete information games. This interpretation is particularly important for the theory of implementation in Nash equilibrium, whereby the assumption of common knowledge of preferences, especially on large domains, is at best a simplifying assumption.[1] Furthermore, recent evidence in the

[1] The point that the assumption of common knowledge of preferences might be problematic is not new. For instance, Chung and Ely (2003) study the problem of *full* implementation of social choice functions under "near-complete information" and show that Maskin monotonicity is a necessary condition for implementation in undominated (pure) Nash equilibria. Their result sharply contrasts with Palfrey and Srivastava (1991), who have shown that almost all social choice functions are implementable in undominated (pure) Nash equilibria. Oury and Tercieux (2010) consider the problem of *partial* implementation of social choice functions under "almost complete" information and show that Maskin

experimental literature suggests that equilibria in mixed strategies are good predictors of behavior in some classes of games e.g., coordination games and chicken games (see chapters 3 and 7 of Camerer, 2003). Since, for some preference profiles, a mechanism can induce one of those games, paying attention to mixed equilibria is important if we want to describe or predict players' behavior. In sum, we believe that there are no compelling reasons to give pure Nash equilibria a special status and modify the definition of implementation accordingly.

It is important to stress that while we consider mixed strategies, we maintain an entirely ordinal approach. To be specific, we assume that a social choice correspondence f maps profiles of preference orderings over alternatives into subsets of alternatives (not lotteries) and we require that a given mechanism implements f irrespective of which cardinal representation of players' preferences is chosen.

Our definition of mixed Nash implementation yields novel insights. We demonstrate that the condition of Maskin monotonicity is not necessary for full implementation in mixed Nash equilibrium. Intuitively, consider a profile of preferences and an alternative, say a, that is f-optimal at that profile of preferences. According to Maskin's definition of implementation, there must exist a pure Nash equilibrium with equilibrium outcome a. Thus, any alternative a player can obtain by unilateral deviations must be less preferred than a. Now, if we move to another profile of preferences where a does not fall down in the players' ranking, then a remains an equilibrium outcome and must be f-optimal at that new profile of preferences. This is the intuition behind the necessary condition of Maskin monotonicity for full implementation. Unlike Maskin's definition of implementation, our definition does not require a to be a pure equilibrium outcome. So, suppose that there exists a mixed equilibrium with a as an equilibrium outcome.[2] The key observation to make is that the mixed equilibrium induces a *lottery* over optimal alternatives. Thus, when we move to another profile of preferences where a does not fall down in the players' ranking, the original profile of strategies does not have to be an equilibrium at the new state. In fact, we show that a much weaker condition, *weak set-monotonicity*, is necessary for implementation in mixed Nash equilibrium. Weak set-monotonicity states that the set $f(\theta)$ of optimal alternatives at state θ is included

monotonicity is necessary for implementation in pure Nash equilibrium.

[2]More precisely, let σ^* be the mixed Nash equilibrium and $\mathbb{P}_{\sigma^*,g}$ the distribution over alternatives induced by the strategy profile σ^* and the allocation rule g. Then a belongs to the support of $\mathbb{P}_{\sigma^*,g}$.

in the set $f(\theta')$ of optimal alternatives at state θ', whenever the weak *and* strict lower contour sets at state θ of *all* alternatives in $f(\theta)$ are included in their respective weak and strict lower contour sets at state θ', for all players.

To substantiate our claim that weak set-monotonicity is a substantially weaker requirement than Maskin monotonicity, we show that the strong Pareto and the strong core correspondences are weak set-monotonic on the unrestricted domain of preferences, while they are not Maskin monotonic. Similarly, on the domain of strict preferences, the top-cycle correspondence is weak set-monotonic, but not Maskin monotonic.

We also show that a mild strengthening of weak set-monotonicity, that we call weak* set-monotonicity, is necessary for mixed Nash implementation of social choice *functions*. Furthermore, weak* set-monotonicity and no-veto power are sufficient for implementation of social choice correspondences in general settings, while on the domain of strict preferences weak* set-monotonicity is equivalent to weak set-monotonicity. We also provide an additional condition, called top-D-inclusiveness, which together with weak* set-monotonicity, guarantees the implementation by *finite* mechanisms in separable environments. Lastly, since no-veto power is not satisfied by important social choice correspondences like the strong Pareto and the strong core, we also present sufficient conditions that dispense with the no-veto power condition. (See Benoît and Ok, 2008, and Bochet, 2007.)

An important feature of our sufficiency proofs is the use of randomized mechanisms. This is a natural assumption given that players can use mixed strategies. Indeed, although a randomized mechanism introduces some uncertainty about the alternative to be chosen, the concept of a mixed Nash equilibrium already encapsulates the idea that players are uncertain about the messages sent to the designer and, consequently, about the alternative to be chosen. We also stress that the randomization can only be among optimal alternatives in equilibrium. In the literature on (exact) Nash implementation, randomized mechanisms have been studied by Benoît and Ok (2008) and Bochet (2007). These authors restrict attention to mechanisms in which randomization by the designer can only occur out of equilibrium, and do not attempt to rule out mixed strategy equilibria with undesirable outcomes. On the contrary, we allow randomization among f-optimal alternatives at equilibrium and rule out mixed equilibria with outcomes that are not f-optimal. Our approach also differs from the use of randomized mechanisms in the literature on virtual implementation (e.g., see Matsushima, 1998, and Abreu and Sen, 1991), which heavily exploits the possibility of selecting undesirable alternatives with positive probability in equilibrium.

The paper is organized as follows. Section 2 presents a simple example illustrating our ideas. Section 3 contains preliminaries and introduces the definition of mixed Nash implementation. Section 4 presents the necessary conditions of weak and weak* set-monotonicity, while sections 5, 6 and 7 provides several sets of sufficient conditions. Section 8 applies our results to some well-known social choice correspondences and section 9 concludes.

2 A Simple Example

This section illustrates our notion of mixed Nash implementation with the help of a simple example.

Example 1 There are two players, 1 and 2, two states of the world, θ and θ', and four alternatives, a, b, c, and d. Players have state-dependent preferences represented in the table below. For instance, player 1 ranks b first and a second in state θ, while a is ranked first and b last in state θ'. Preferences are strict.

θ		θ'	
1	2	1	2
b	c	a	c
a	a	d	d
c	b	c	a
d	d	b	b

The designer aims to implement the social choice correspondence f, with $f(\theta) = \{a\}$ and $f(\theta') = \{a, b, c, d\}$. We say that alternative x is f-optimal at state θ if $x \in f(\theta)$.

We first argue that the social choice correspondence f is not implementable in the sense of Maskin (1999). Maskin's definition of Nash implementation requires that for each f-optimal alternative at a given state, there exists a *pure* Nash equilibrium (of the game induced by the mechanism) corresponding to that alternative. So, for instance, at state θ', there must exist a pure Nash equilibrium with b as equilibrium outcome.

Maskin requires, furthermore, that no such equilibrium must exist at state θ. However, if there exists a pure equilibrium with b as equilibrium outcome at state θ', then b will also be an equilibrium outcome at state θ, since b moves up in every players' ranking when going from state θ' to state θ. Thus, the correspondence f is not implementable in the sense of Maskin. In other words, the social choice correspondence f violates Maskin monotonicity, a necessary condition for implementation in the sense of Maskin.

In contrast with Maskin, we do *not* require that for each f-optimal alternative at a given state, there exists a pure Nash equilibrium corresponding to that alternative. We require instead that the set of f-optimal alternatives coincides with the set of *mixed* Nash equilibrium outcomes. So, at state θ', there must exist a mixed Nash equilibrium with b corresponding to an action profile in the support of the equilibrium.

We now argue that with our definition of implementation, the correspondence f is implementable. To see this, consider the mechanism where each player has two messages m_1 and m_2, and the allocation rule is represented in the table below. (Player 1 is the row player.) For example, if both players announce m_1, the chosen alternative is a.

	m_1	m_2
m_1	a	b
m_2	d	c

Now, at state θ, (m_1, m_1) is the unique Nash equilibrium, with outcome a. At state θ', both (m_1, m_1) and (m_2, m_2) are pure Nash equilibria, with outcomes a and c. Moreover, there exists a mixed Nash equilibrium that puts strictly positive probability on each action profile (since preferences are strict), hence on each outcome.[3] Therefore, f is implementable in mixed Nash equilibrium, although it is not implementable in the sense of Maskin.

We conclude this section with two important observations. First, our notion of implementation in mixed Nash equilibrium is *ordinal*: the social choice correspondence f is implementable regardless of the cardinal representation chosen for the two players. Second, alternative d is f-optimal at state θ', and it moves down in player 1's ranking when moving from θ' to θ. This preference reversal guarantees the weak set-monotonicity of

[3]Note that at state θ, a is the unique rationalizable outcome, while all outcomes are rationalizable at state θ', so that f is implementable in rationalizable outcomes. See Bergemann, Morris and Tercieux (2010) for the study of social choice *functions* implementable in rationalizable outcomes.

the correspondence f, which, as we shall see, is a necessary condition for implementation in mixed Nash equilibrium.

3 Preliminaries

An environment is a triplet $\langle N, X, \Theta\rangle$ where $N := \{1, \ldots, n\}$ is a set of n players, X a finite set of alternatives, and Θ a finite set of states of the world. Associated with each state θ is a preference profile $\succcurlyeq^\theta := (\succcurlyeq_1^\theta, \ldots, \succcurlyeq_n^\theta)$, where $\succcurlyeq_i^\theta$ is player i's preference relation over X at state θ. The asymmetric and symmetric parts of $\succcurlyeq_i^\theta$ are denoted $\succ_i^\theta$ and $\sim_i^\theta$, respectively.

We denote with $L_i(x, \theta) := \{y \in X : x \succcurlyeq_i^\theta y\}$ player i's lower contour set of x at state θ, and $SL_i(x, \theta) := \{y \in X : x \succ_i^\theta y\}$ the strict lower contour set. For any (i, θ) in $N \times \Theta$ and $Y \subseteq X$, define $\max_i^\theta Y$ as $\{x \in Y : x \succcurlyeq_i^\theta y \text{ for all } y \in Y\}$.

We assume that any preference relation $\succcurlyeq_i^\theta$ is representable by a utility function $u_i(\cdot, \theta) : X \to \mathbb{R}$, and that each player is an expected utility maximizer. We denote with $\mathcal{U}_i^\theta$ the set of all possible cardinal representations $u_i(\cdot, \theta)$ of $\succcurlyeq_i^\theta$ at state θ, and let $\mathcal{U}^\theta := \times_{i \in N} \mathcal{U}_i^\theta$.

A social choice correspondence $f : \Theta \to 2^X \setminus \{\emptyset\}$ associates with each state of the world θ, a non-empty subset of alternatives $f(\theta) \subseteq X$. Two classic conditions for Nash implementation are Maskin monotonicity and no-veto power. A social choice correspondence f is *Maskin monotonic* if for all (x, θ, θ') in $X \times \Theta \times \Theta$ with $x \in f(\theta)$, we have $x \in f(\theta')$ whenever $L_i(x, \theta) \subseteq L_i(x, \theta')$ for all $i \in N$. Maskin monotonicity is a necessary condition for Nash implementation (à la Maskin). A social choice correspondence f satisfies *no-veto power* if for all $\theta \in \Theta$, we have $x \in f(\theta)$ whenever $x \in \max_i^\theta X$ for all but at most one player $j \in N$. Maskin monotonicity and no-veto power are sufficient conditions for Nash implementation (à la Maskin) when there are at least three players.

Let $\Delta(X)$ be the set of all probability measures over X. A mechanism (or game form) is a pair $\langle (M_i)_{i \in N}, g\rangle$ with M_i the set of messages of player i, and $g : \times_{i \in N} M_i \to \Delta(X)$ the allocation rule. Let $M := \times_{j \in N} M_j$ and $M_{-i} := \times_{j \in N \setminus \{i\}} M_j$, with m and m_{-i} generic elements.

A mechanism $\langle (M_i)_{i \in N}, g\rangle$, a state θ and a profile of cardinal representations $(u_i(\cdot, \theta))_{i \in N}$ of $(\succcurlyeq_i^\theta)_{i \in N}$ induce a strategic-form game as follows. There is a set N of n players. The set of pure actions of player i is M_i, and player i's expected payoff when he plays m_i

and his opponents play m_{-i} is

$$U_i(g(m_i, m_{-i}), \theta) := \sum_{x \in X} g(m_i, m_{-i})(x) u_i(x, \theta),$$

where $g(m_i, m_{-i})(x)$ is the probability that x is chosen by the mechanism when the profile of messages (m_i, m_{-i}) is announced. The induced strategic-form game is thus $G(\theta, u) := \langle N, (M_i, U_i(g(\cdot), \theta))_{i \in N} \rangle$. Let σ be a profile of mixed strategies. We denote with $\mathbb{P}_{\sigma,g}$ the probability distribution over alternatives in X induced by the allocation rule g and the profile of mixed strategies σ.[4]

Definition 1 *The mechanism $\langle (M_i)_{i \in N}, g \rangle$ implements the social choice correspondence f in mixed Nash equilibrium if for all $\theta \in \Theta$, for all cardinal representations $u(\cdot, \theta) \in \mathcal{U}^\theta$ of $\succcurlyeq^\theta$, the following two conditions hold:*

(i) For each $x \in f(\theta)$, there exists a Nash equilibrium σ^* of $G(\theta, u)$ such that x is in the support of $\mathbb{P}_{\sigma^*,g}$, and

(ii) if σ is a Nash equilibrium of $G(\theta, u)$, then the support of $\mathbb{P}_{\sigma,g}$ is included in $f(\theta)$.

Before proceeding, it is important to contrast our definition of implementation in mixed Nash equilibrium with Maskin (1999) definition of Nash implementation.

First, part (i) of Maskin's definition requires that for each $x \in f(\theta)$, there exists a pure Nash equilibrium m^* of $G(\theta, u)$ with equilibrium outcome x, while part (ii) of his definition is identical to ours. In contrast with Maskin, we allow for mixed strategy Nash equilibria in part (i) and, thus, restore a natural symmetry between parts (i) and (ii). Yet, our definition respects the spirit of full implementation in that only optimal outcomes can be observed by the designer as equilibrium outcomes.

Second, as in Maskin, our concept of implementation is ordinal as all equilibrium outcomes have to be optimal, regardless of the cardinal representation chosen. Also, our approach parallels the approach of Gibbard (1977). Gibbard considers probabilistic social choice functions, i.e., mapping from profiles of preferences to lotteries over outcomes, and characterizes the set of strategy-proof probabilistic social choice functions. Importantly to us, Gibbard requires each player to have an incentive to truthfully reveal

[4]Formally, the probability $\mathbb{P}_{\sigma,g}(x)$ of $x \in X$ is $\sum_{m \in M} \sigma(m) g(m)(x)$ if M is countable. If M is uncountable, a similar expression applies.

his preference, regardless of the cardinal representation chosen to evaluate lotteries (and announcements of others).[5]

Third, we allow the designer to use randomized mechanisms. This is a natural assumption given that players can use mixed strategies. Indeed, although a randomized mechanism introduces some uncertainty about the alternative to be chosen, the concept of mixed Nash equilibrium already encapsulates the idea that players are uncertain about the messages sent to the designer and, consequently, about the alternative to be chosen. We also stress that the randomization can only be among optimal alternatives in equilibrium. In the context of (exact) Nash implementation, Benoît and Ok (2008) and Bochet (2007) have already considered randomized mechanisms.[6] There are two important differences with our work, however. First, these authors restrict attention to mechanisms in which randomization only occurs out of equilibrium, while randomization can occur in equilibrium in our work, albeit only among optimal alternatives. Second, unlike us, they do not attempt to rule out mixed strategy equilibria with undesirable outcomes. Also, our work contrasts with the literature on virtual implementation (e.g., Abreu and Sen, 1991 and Matsushima, 1998). In that literature, randomization can also occur in equilibrium. Moreover, even non-optimal alternatives can occur with positive probability in equilibrium. Unlike this literature, we focus on exact implementation: only f-optimal alternatives can be equilibrium outcomes.

Finally, from our definition of mixed Nash implementation, it is immediate to see that if a social choice correspondence is Nash implementable (i.e., à la Maskin), then it is implementable in mixed Nash equilibrium. The converse is false, as shown by Example 1 in Section 2. The goal of this paper is to characterize the social choice correspondences implementable in mixed Nash equilibrium. The next section provides a necessary condition.

[5]See also Barberà et al. (1998) and Abreu and Sen (1991) for further discussions of the ordinal approach.

[6]Vartiainen (2007) also considers randomized mechanisms, but for the implementation of social choice correspondences in (pure) subgame perfect equilibrium on the domain of strict preferences.

4 Necessary Conditions

We begin by introducing a new condition, called *weak set-monotonicity*, which we show to be necessary for the implementation of social choice correspondences in mixed Nash equilibrium.

Definition 2 *A social choice correspondence f is weak set-monotonic if for all pairs $(\theta, \theta') \in \Theta \times \Theta$, we have $f(\theta) \subseteq f(\theta')$ whenever for all $x \in f(\theta)$, for all $i \in N$, the two following conditions hold: (i) $L_i(x, \theta) \subseteq L_i(x, \theta')$ and (ii) $SL_i(x, \theta) \subseteq SL_i(x, \theta')$.*[7]

Weak set-monotonicity is a weakening of Maskin monotonicity. It restricts f only when *all* alternatives in the set $f(\theta)$ move up in the weak and strict rankings of all players. More precisely, it requires that if for *all* players, the lower *and* strict lower contour sets of *all* alternatives in $f(\theta)$ do not shrink in moving from θ to θ', then the set $f(\theta')$ of optimal alternatives at θ' must be a superset of the set $f(\theta)$ of optimal alternatives at θ. Maskin monotonicity, on the contrary, restricts f whenever a *single* alternative in $f(\theta)$ moves up in the weak rankings of all players.[8] As we shall see in Section 8, important correspondences, like the strong Pareto correspondence, the strong core correspondence and the top-cycle correspondence are weak-set monotonic, while they fail to be Maskin monotonic.

Theorem 1 *If the social choice correspondence f is implementable in mixed Nash equilibrium, then it satisfies the weak set-monotonicity condition.*

Proof The proof is by contradiction on the contrapositive. Assume that the social choice correspondence f does not satisfy weak set-monotonicity and yet is implementable in mixed Nash equilibrium by the mechanism $\langle M, g \rangle$.

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[7] Alternatively, a social choice correspondence f is weak set-monotonic if $x^* \in f(\theta) \setminus f(\theta')$ implies that there exists a pair (x, y) in $f(\theta) \times X$ and a player $i \in N$ such that either (1) $x \succcurlyeq_i^{\theta} y$ and $y \succ_i^{\theta'} x$, or (2) $x \succ_i^{\theta} y$ and $y \succcurlyeq_i^{\theta'} x$.

[8] Weak set-monotonicity is also weaker than Sanver's (2006) almost monotonicity, and Cabrales and Serrano's (2009) quasimonotonicity. Quasimonotonicity and almost monotonicity restrict f when a *single* alternative in $f(\theta)$ moves up in the rankings of all players. The correspondence f is quasimonotonic if for all pairs $(\theta, \theta') \in \Theta \times \Theta$, and $x \in f(\theta)$, we have $x \in f(\theta')$ whenever for all $i \in N$ it is $SL_i(x, \theta) \subseteq SL_i(x, \theta')$. The correspondence f is almost monotonic if for all pairs $(\theta, \theta') \in \Theta \times \Theta$, and $x \in f(\theta)$, we have $x \in f(\theta')$ whenever for all $i \in N$ it is: (i) $L_i(x, \theta) \subseteq L_i(x, \theta')$ and (ii) $SL_i(x, \theta) \subseteq SL_i(x, \theta')$.

Since f does not satisfy weak set-monotonicity, there exist x^*, θ, and θ' such that $x^* \in f(\theta) \setminus f(\theta')$, while $L_i(x,\theta) \subseteq L_i(x,\theta')$ and $SL_i(x,\theta) \subseteq SL_i(x,\theta')$ for all $x \in f(\theta)$, for all $i \in N$.

Claim. For each player $i \in N$, fix any cardinal representation $u_i(\cdot,\theta)$ of $\succsim_i^\theta$. We claim that there exists a cardinal representation $u_i(\cdot,\theta')$ of $\succsim_i^{\theta'}$ such that $u_i(x,\theta') \leq u_i(x,\theta)$ for all $x \in X$, $u_i(x,\theta') = u_i(x,\theta)$ for all $x \in f(\theta)$.

Proof of claim. To prove our claim, consider any pair $(x,x') \in f(\theta) \times f(\theta)$ with $x \succsim_i^\theta x'$. Since $L_i(\hat{x},\theta) \subseteq L_i(\hat{x},\theta')$ for all $\hat{x} \in f(\theta)$, we have that $x \succ_i^\theta x'$ implies $x \succ_i^{\theta'} x'$ and $x \sim_i^\theta x'$ implies $x \sim_i^{\theta'} x'$. Hence, we can associate with each alternative in $f(\theta)$ the same utility at θ' as at θ. Now, fix an alternative $x \in f(\theta)$ and consider $y \in L_i(x,\theta)$. Since $L_i(x,\theta) \subseteq L_i(x,\theta')$, we must have $u_i(y,\theta') \leq u_i(x,\theta') = u_i(x,\theta)$. If $x \sim_i^\theta y$, then we can choose $u_i(y,\theta') \leq u_i(y,\theta) = u_i(x,\theta)$. If $x \succ_i^\theta y$, then we must have $x \succ_i^{\theta'} y$ since $SL_i(x,\theta) \subseteq SL_i(x,\theta')$; we can therefore choose $u_i(y,\theta')$ in the open set $(-\infty, u_i(y,\theta))$ and still represent $\succsim_i^{\theta'}$ by $u_i(\cdot,\theta')$. Finally, if $y \notin \cup_{x \in f(\theta)} L_i(x,\theta)$, we have that $u_i(y,\theta) > u_i(x,\theta)$ for all $x \in f(\theta)$. If $y \in L_i(x,\theta')$ for some $x \in f(\theta)$, then we can set $u_i(y,\theta') \leq u_i(x,\theta') = u_i(x,\theta) \leq \max_{x' \in f(\theta)} u_i(x',\theta) < u_i(y,\theta)$. If $y \notin \cup_{x \in f(\theta)} L_i(x,\theta')$, then we can choose $u_i(y,\theta')$ in the open set $(\max_{x' \in f(\theta)} u_i(x',\theta), u_i(y,\theta))$. This concludes the proof of our claim.

Before proceeding, we should stress the importance of the nestedness of the strict lower-contour sets in part (ii) of the definition of weak set-monotonicity. Let $x \in f(\theta)$ and assume that $x \sim_i^\theta y \succ_i^\theta z$ at state θ and $x \sim_i^{\theta'} z \succ_i^{\theta'} y$ at state θ'. Both alternatives z and y are in the lower contour set of x at θ and θ', but the strict lower-contour sets are not nested. Clearly, we cannot assign the same utility to x at θ and θ' and weakly decrease the utility of both y and z when moving from θ to θ'; the claim does not hold.

Since f is implementable and $x^* \in f(\theta)$, for any cardinal representation $u(\cdot,\theta)$ of $\succsim^\theta$, there exists an equilibrium σ^* of the game $G(\theta,u)$ with x^* in the support of $\mathbb{P}_{\sigma^*,g}$. Furthermore, since $x^* \notin f(\theta')$, for all cardinal representations $u(\cdot,\theta')$ of $\succsim^{\theta'}$, for all equilibria σ of $G(\theta',u)$, x^* does not belong to the support of $\mathbb{P}_{\sigma,g}$. In particular, this implies that σ^* is not an equilibrium at θ' for all cardinal representations $u(\cdot,\theta')$. Thus, assuming that M is countable, there exist a player i, a message m_i^* in the support of σ_i^*, and a message m_i' such that:[9]

$$\sum_{m_{-i}} \left[U_i(g(m_i^*, m_{-i}), \theta) - U_i(g(m_i', m_{-i}), \theta)\right] \sigma_{-i}^*(m_{-i}) \geq 0$$

and

$$0 > \sum_{m_{-i}} \left[U_i(g(m_i^*, m_{-i}), \theta') - U_i(g(m_i', m_{-i}), \theta')\right] \sigma_{-i}^*(m_{-i}).$$

It follows that

$$\begin{aligned} & \sum_{m_{-i}} \left[U_i(g(m_i^*, m_{-i}), \theta) - U_i(g(m_i^*, m_{-i}), \theta')\right] \sigma_{-i}^*(m_{-i}) \qquad (1) \\ > & \sum_{m_{-i}} \left[U_i(g(m_i', m_{-i}), \theta) - U_i(g(m_i', m_{-i}), \theta')\right] \sigma_{-i}^*(m_{-i}) \end{aligned}$$

Let us now consider the cardinal representations constructed in the claim above; that is, $u_i(x, \theta') \leq u_i(x, \theta)$ for all $x \in X$ and $u_i(x, \theta) = u_i(x, \theta')$ for all $x \in f(\theta)$. Since f is implementable, we have that the support of $\mathbb{P}_{\sigma^*, g}$ is included in $f(\theta)$. Therefore, $U_i(g(m_i^*, m_{-i}), \theta) = U_i(g(m_i^*, m_{-i}), \theta')$ for all m_{-i} in the support of σ_{-i}^*. Hence, the left-hand side of the inequality (1) is zero. Furthermore, we have that $U_i(g(m_i', m_{-i}), \theta) \geq U_i(g(m_i', m_{-i}), \theta')$ for all m_{-i}. Hence, the right-hand side of (1) is non-negative, a contradiction. This completes the proof. □

Several remarks are worth making. First, Theorem 1 remains valid if we restrict ourself to deterministic mechanisms, so that weak-set monotonicity is a necessary condition for implementation in mixed Nash equilibrium, regardless of whether we consider deterministic or randomized mechanisms. Second, it is easy to verify that weak-set monotonicity is also a necessary condition for implementation if we require the Nash equilibria to be in pure strategies, but allow randomized mechanisms. Third, while we have restricted attention to von Neumann-Morgenstern preferences, the condition of weak set-monotonicity remains necessary if we consider larger classes of preferences that include the von Neumann-Morgenstern preferences. This is because we follow an ordinal approach and require that f be implemented by all admissible preference representations.

[9] If the mechanism is uncountable, a similar argument holds with appropriate measurability conditions and integrals rather than sums.

We now introduce a strengthening of the notion of weak set-monotonicity, which turns out to be necessary for mixed Nash implementation when f is a social choice *function*, that is $f(\theta)$ is a singleton for all $\theta \in \Theta$.

Definition 3 *A social choice correspondence f is* weak* set-monotonic *if for all pairs $(\theta, \theta') \in \Theta \times \Theta$, we have $f(\theta) \subseteq f(\theta')$ whenever for all $x \in f(\theta)$, for all $i \in N$, the two following conditions hold: (i) $L_i(x, \theta) \subseteq L_i(x, \theta')$ and (ii) either $SL_i(x, \theta) \subseteq SL_i(x, \theta')$ or $x \in \max_i^{\theta'} X$.*

Clearly, if a social choice correspondence is Maskin monotonic, then it is weak* set-monotonic, and if it is weak* set-monotonic, then it is weak set-monotonic. Moreover, weak* set-monotonicity coincides with weak set-monotonicity if $\max_i^{\theta} X$ is a singleton for each $i \in N$, for each $\theta \in \Theta$. We refer to this domain of preferences as the *single-top preferences*. This mild domain restriction will prove useful in applications (see Section 8). Furthermore, on the domain of strict preferences (a subset of single-top preferences) $SL_i(x, \theta) = L_i(x, \theta) \setminus \{x\}$ for all x and θ, and weak* set-monotonicity is equivalent to weak set-monotonicity.

Theorem 2 *If the social choice function f is implementable in mixed Nash equilibrium, then it satisfies the weak* set-monotonicity condition.*

Proof The proof is by contradiction. Assume the function f violates weak* set-monotonicity, but is implementable in mixed Nash equilibrium by the mechanism $\langle M, g \rangle$. In light of Theorem 1, f must satisfy weak set-monotonicity. Hence, there exist x^*, θ, and θ' such that $x^* = f(\theta) \neq f(\theta')$, $L_i(x^*, \theta) \subseteq L_i(x^*, \theta')$ for all $i \in N$, for at least one $i \in N$ we have $SL_i(x^*, \theta) \subsetneq SL_i(x^*, \theta')$ and $x^* \in \max_i^{\theta'} X$, while for all other $i \in N$ we have $SL_i(x^*, \theta) \subseteq SL_i(x^*, \theta')$.

Since f is implementable and $x^* = f(\theta)$, for any cardinal representation $u(\cdot, \theta)$ of $\succsim^{\theta}$, there exists an equilibrium σ^* of the game $G(\theta, u)$ with the unique element in the support of $\mathbb{P}_{\sigma^*, g}$ being x^*. Consider first all players i such that $SL_i(x^*, \theta) \subseteq SL_i(x^*, \theta')$. The proof of Theorem 1 shows that for some cardinal representations of these players' utilities at θ', it cannot be the case that they have a profitable deviation from σ_i^*. Now, consider the players $i \in N$ with $SL_i(x^*, \theta) \subsetneq SL_i(x^*, \theta')$ and $x^* \in \max_i^{\theta'} X$. Since $x^* \in \max_i^{\theta'} X$, for all possible cardinal representations of player i's preferences at θ' there cannot be a profitable deviation from strategy σ_i^*. This shows that σ^* is an equilibrium of $G(\theta', u')$ for some cardinal representation u' of preferences at state θ', and contradicts the assumption that f is implementable. $\square$

An alternative definition of implementation suggested to us would replace part (i) of our definition with the requirement that, for each $x \in f(\theta)$, there exists a Nash equilibrium σ^* such that x coincides with the support of $\mathbb{P}_{\sigma^*,g}$. It follows from the proof of Theorem 2 that weak* set-monotonicity is necessary for implementation of social choice correspondences with this alternative definition.[10]

5 Sufficient Conditions

Before stating the main result of this section, we need to introduce a natural restriction on the set of cardinal representations at each state, which guarantees that the set of admissible cardinal representations is closed. Example 5 below illustrates the difficulties arising when the set of cardinal representations is open.

We assume that for each player $i \in N$, for each state $\theta \in \Theta$, the set of admissible cardinal representations is a compact subset $\overline{\mathcal{U}}_i^\theta$ of $\mathcal{U}_i^\theta$. It follows that there exists $\varepsilon > 0$ such that for all $i \in N$, for all θ, for all pairs $(x, y) \in X \times X$ with $x \succ_i^\theta y$, and for all $u_i(\cdot, \theta) \in \overline{\mathcal{U}}_i^\theta : u_i(x, \theta) \geq (1 - \varepsilon) u_i(y, \theta) + \varepsilon \max_{w \in X} u_i(w, \theta)$.[11] Accordingly, we modify Definition 1 of mixed Nash implementation so as to include this restriction on the set of cardinal representations. We call this weaker notion of implementation, *mixed Nash C-implementation.*

We are now ready to present the main result of this section, which states that in any environment with at least three players, weak* set-monotonicity and no veto-power are sufficient conditions for implementation in mixed Nash equilibrium.

Theorem 3 *Let $\langle N, X, \Theta \rangle$ be an environment with $n \geq 3$. If the social choice correspondence f is weak* set-monotonic and satisfies no-veto power, then it is C-implementable in mixed Nash equilibrium.*

[10] We did not adopt this definition because we see no compelling reasons to require that for each f-optimal alternative, there exists a mixed equilibrium attaching probability one to that alternative.

[11] Rule 2 of the mechanism used in the proof of Theorem 3 exploits the existence of such an ε.

Proof Consider the following mechanism $\langle M, g\rangle$. For each player $i \in N$, the message space M_i is $\Theta \times \{\alpha^i : \alpha^i : X \times \Theta^2 \to X\} \times X \times \mathbb{Z}_{++}$. In words, each player announces a state of the world, a function from alternatives and pairs of states into alternatives, an alternative, and a strictly positive integer. A typical message m_i for player i is $(\theta^i, \alpha^i, x^i, \mathbf{z}^i)$. (Note that we denote any integer $\mathbf{z}$ in bold.) Let $M := \times_{i\in N} M_i$ with typical element m.

Let $\{f_1(\theta), \ldots, f_{K^\theta}(\theta)\} = f(\theta)$ be the set of f-optimal alternatives at state θ ; note that $K^\theta = |f(\theta)|$. Let $1 > \varepsilon > 0$ be such that for all $i \in N$, for all $\theta \in \Theta$, for all pairs $(x, y) \in X \times X$ with $x \succ_i^\theta y$, and for all $u_i(\cdot, \theta) \in \overline{\mathcal{U}}_i^\theta$, we have $u_i(x, \theta) \geq (1-\varepsilon)u_i(y, \theta) + \varepsilon \max_{w\in X} u_i(w, \theta)$. Since $\overline{\mathcal{U}}_i^\theta$ is a compact subset of $\mathcal{U}_i^\theta$, such an ε exists. Let $\mathbf{1}[x] \in \Delta(X)$ be the lottery that puts probability one on outcome $x \in X$. The allocation rule g is defined as follows:

Rule 1: If $m_i = (\theta, \alpha, x, 1)$ for all $i \in N$ (i.e., all agents make the same announcement m_i) and $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)$, then $g(m)$ is the "uniform" lottery over alternatives in $f(\theta)$; that is,

$$g(m) = \frac{1}{K^\theta} \sum_{k=1}^{K^\theta} \mathbf{1}\left[f_k(\theta)\right].$$

Rule 2: If there exists $j \in N$ such that $m_i = (\theta, \alpha, x, 1)$ for all $i \in N \setminus \{j\}$, with $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)$, and $m_j = (\theta^j, \alpha^j, x^j, \mathbf{z}^j) \neq m_i$, then $g(m)$ is the lottery:

$$\frac{1}{K^\theta} \sum_{k=1}^{K^\theta} \left\{\delta_k(m)(1-\varepsilon_k(m))\mathbf{1}\left[\alpha^j(f_k(\theta), \theta, \theta^j)\right] + \delta_k(m)\varepsilon_k(m)\mathbf{1}\left[x^j\right] + (1-\delta_k(m))\,\mathbf{1}\left[f_k(\theta)\right]\right\},$$

with

$$\delta_k(m) = \begin{cases} \delta & \text{if } \alpha^j(f_k(\theta), \theta, \theta^j) \in L_j(f_k(\theta), \theta) \\ 0 & \text{if } \alpha^j(f_k(\theta), \theta, \theta^j) \notin L_j(f_k(\theta), \theta) \end{cases}$$

for $1 > \delta > 0$, and

$$\varepsilon_k(m) = \begin{cases} \varepsilon & \text{if } \alpha^j(f_k(\theta), \theta, \theta^j) \in SL_j(f_k(\theta), \theta) \\ 0 & \text{if } \alpha^j(f_k(\theta), \theta, \theta^j) \notin SL_j(f_k(\theta), \theta) \end{cases}.$$

That is, suppose all players but player j send the same message $(\theta, \alpha, x, 1)$ with $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $k \in \{1, \ldots, K^\theta\}$. Let $m_j = (\theta^j, \alpha^j, x^j, \mathbf{z}^j)$ be the message sent by player j.

If $\alpha^j(f_k(\theta), \theta, \theta^j)$ selects an alternative x in player j' strict lower-contour set $SL_j(f_k(\theta), \theta)$ of $f_k(\theta)$ at state θ, then the designer replaces the outcome $f_k(\theta)$ from the uniform lottery with the lottery that attaches probability $\delta(1-\varepsilon)$ to x, probability $\delta\varepsilon$ to x^j, and probability $(1-\delta)$ to $f_k(\theta)$. If $\alpha^j(f_k(\theta), \theta, \theta^j)$ selects an alternative x in $L_j(f_k(\theta), \theta) \setminus SL_j(f_k(\theta), \theta)$ (i.e., player j is indifferent between x and $f_k(\theta)$ at state θ), then the designer replaces the outcome $f_k(\theta)$ from the uniform lottery with the lottery that attaches probability δ to x and probability $(1-\delta)$ to $f_k(\theta)$. Otherwise, the designer does not replace the outcome $f_k(\theta)$ from the uniform lottery.

Rule 3: If neither rule 1 nor rule 2 applies, then $g\left((\theta^i, \alpha^i, x^i, \mathbf{z}^i)_{i\in N}\right) = x^{i^*}$, with i^* a player announcing the highest integer $\mathbf{z}^{i^*}$. (If more than one player i selects the highest integer, then g randomizes uniformly among their selected x^i.)

Fix a state θ^* and a cardinal representation $u_i \in \overline{\mathcal{U}}_i^{\theta^*}$ of $\succsim_i^{\theta^*}$ for each player i. Let u be the vector of cardinal representations. For future reference, we divide the rest of the proof in several steps.

Step 1. We first show that for any $x \in f(\theta^*)$, there exists a Nash equilibrium σ^* of $G(\theta^*, u)$ such that x belongs to the support of $\mathbb{P}_{\sigma^*,g}$. Consider a profile of strategies σ^* such that $\sigma_i^* = (\theta^*, \alpha, x, 1)$ for all $i \in N$, so that rule 1 applies. The (pure strategy) profile σ^* is clearly a Nash equilibrium at state θ^*. By deviating, each player i can trigger rule 2, but none of these possible deviations are profitable. Any deviation can either induce a probability shift in the uniform lottery from $f_k(\theta^*)$ to a lottery with mass $(1-\varepsilon)$ on an alternative in $SL_i(f_k(\theta^*), \theta^*)$ and mass ε on x^j, or shift δ probability mass from $f_k(\theta^*)$ to an alternative indifferent to $f_k(\theta^*)$ (i.e., an alternative in $L_i(f_k(\theta^*), \theta^*) \setminus SL_i(f_k(\theta^*), \theta^*)$). By definition of ε, the former type of deviation is not profitable and the latter type of deviation is clearly not profitable. Moreover, under σ^*, the support of $\mathbb{P}_{\sigma^*,g}$ is $f(\theta^*)$. Hence, for any $x \in f(\theta^*)$, there exists an equilibrium that implements x.

Step 2. Conversely, we need to show that if σ^* is a mixed Nash equilibrium of $G(\theta^*, u)$, then the support of $\mathbb{P}_{\sigma^*,g}$ is included in $f(\theta^*)$. Let m be a message profile and denote with $g^O(m)$ the set of alternatives that occur with strictly positive probability when m is played: $g^O(m) = \{x \in X : g(m)(x) > 0\}$. Let us partition the set of messages M into three subsets corresponding to the three allocation rules. First, let R_1 be the set of message profiles such that rule 1 applies, i.e., $R_1 = \{m : m_j = (\theta, \alpha, x, 1)$ for all $j \in N$, with $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)\}$. Second, if all agents $j \neq i$ send some message $m_j = (\theta, \alpha, x, 1)$ with $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)$, while agent i sends a different message $m_i = (\theta^i, \alpha^i, x^i, \mathbf{z}^i)$, then rule 2 applies and agent i is the only agent differentiating his message. Let R_2^i be the set of these message profiles and define $R_2 = \cup_{i \in N} R_2^i$. Third, let R_3 be the set of message profiles such that rule 3 applies (i.e., R_3 is the complement of $R_1 \cup R_2$ in M).

Consider an equilibrium σ^* of $G(\theta^*, u)$ and let M_i^* be the set of message profiles that occur with positive probability under σ_i^*. (M_i^* is the support of σ_i^*.) We need to show that $g^O(m^*) \subseteq f(\theta^*)$ for all $m^* \in M^* := \times_{i \in N} M_i^*$.

Step 3. For any player $i \in N$, for all $m_i^* = (\theta^i, \alpha^i, x^i, \mathbf{z}^i) \in M_i^*$, define the (deviation) message $m_i^D(m_i^*) = (\theta^i, \alpha^D, x^D, \mathbf{z}^D)$, where: 1) α^D differs from α^i in at most the alternatives associated with elements $(f_k(\theta), \theta, \theta)$ for all $\theta \in \Theta$, for all $k \in \{1, \ldots, K^\theta\}$; that is, we can only have $\alpha^D(f_k(\theta), \theta, \theta) \neq \alpha^i(f_k(\theta), \theta, \theta)$ for some $k \in \{1, \ldots, K^\theta\}$ and some $\theta \in \Theta$, while we have $\alpha^D(x, \theta', \theta'') = \alpha^i(x, \theta', \theta'')$, otherwise, 2) $x^D \in \max_i^{\theta^*} X$, and 3) $\mathbf{z}^D > \mathbf{z}^i$ and for $1 > \mu \geq 0$, the integer $\mathbf{z}^D$ is chosen strictly larger than the integers $\mathbf{z}^j$ selected by all the other players $j \neq i$ in all messages $m_{-i}^* \in M_{-i}^*$, except possibly a set of message profiles $M_{-i}^\mu \subseteq M_{-i}^*$ having probability of being sent less than μ. (Note that μ can be chosen arbitrarily small, but not necessarily zero because other players may randomize over an infinite number of messages.) Consider the following deviation σ_i^D for player i from the equilibrium strategy σ_i^*:

$$\sigma_i^D(m_i) = \begin{cases} \sigma_i^*(m_i^*) & \text{if } m_i = m_i^D(m_i^*) \text{ for some } \ m_i^* \in M_i^* \\ 0 & \text{otherwise} \end{cases}.$$

Step 4. First, note that under $(\sigma_i^D, \sigma_{-i}^*)$, the set of messages sent is a subset of $R_2^i \cup R_3$: either rule 2 applies and all players but player i send the same message or rule 3 applies. Second, whenever rule 3 applies, player i gets his preferred alternative at state θ^* with arbitrarily high probability $(1 - \mu)$. Third, suppose that under σ^*, there exists $m^* \in R_2^j$ with $j \neq i$. Under $(\sigma_i^D, \sigma_{-i}^*)$, with the same probability that m^* is played, $(m_i^D(m_i^*), m_{-i}^*) \in R_3$ is played (rule 3 applies) and with probability at most μ, the lottery $g((m_i^D(m_i^*), m_{-i}^*))$ under $(m_i^D(m_i^*), m_{-i}^*)$ might be less preferred by player i than the lottery $g(m^*)$. (With probability $1 - \mu$, $g((m_i^D(m_i^*), m_{-i}^*)) = \max_i^{\theta^*} X$.) Yet, since μ can be made arbitrarily small and utilities are bounded, the loss can be made arbitrarily small. Consequently, by setting $\alpha^D(f_k(\theta), \theta, \theta^i) \succcurlyeq_i^{\theta^*} \alpha^i(f_k(\theta), \theta, \theta^i)$ for all θ

and all $k \in \{1, \ldots, K^\theta\}$, player i can guarantee himself an arbitrarily small, worst-case loss of $\bar{u}$, in the event that $m^* \in \cup_{j \neq i} R_2^j$ under σ^*.

Step 5. Let us now suppose that there exists $(m_i^*, m_{-i}^*) \in R_1$; that is, for all $j \neq i$, $m_j^* = m_i^* = (\theta, \alpha, x, 1)$. In the event the message sent by all others is $m_j^* = m_i^*$, player i strictly gains from the deviation if $\alpha^D(f_k(\theta), \theta, \theta) \in L_i(f_k(\theta), \theta)$ and either (1) $\alpha^D(f_k(\theta), \theta, \theta) \succ_i^{\theta^*} f_k(\theta)$ or (2) $\alpha^D(f_k(\theta), \theta, \theta) \in SL_i(f_k(\theta), \theta)$, $\alpha^D(f_k(\theta), \theta, \theta) \succcurlyeq_i^{\theta^*} f_k(\theta)$ and $f_k(\theta) \notin \max_i^{\theta^*} X$. Since the expected gain in this event can be made greater than $\bar{u}$ by appropriately choosing μ, (1) and (2) cannot hold for any player i. It follows that for σ^* to be an equilibrium, for all i and all k, we must have (1) $L_i(f_k(\theta), \theta) \subseteq L_i(f_k(\theta), \theta^*)$ and (2) either $SL_i(f_k(\theta), \theta) \subseteq SL_i(f_k(\theta), \theta^*)$ or $f_k(\theta) \in \max_i^{\theta^*} X$. Therefore, by the weak* set-monotonicity of f, we must have $f(\theta) \subseteq f(\theta^*)$. This shows that $g^O(m_i^*, m_{-i}^*) \subseteq f(\theta^*)$ for all $(m_i^*, m_{-i}^*) \in R_1$.

Step 6. Let us now suppose that there exists $(m_i^*, m_{-i}^*) \in R_2^i$; that is, for all $j \neq i$, $m_j^* = (\theta, \alpha, x, 1) \neq m_i^*$. In this case, any player $j \neq i$ strictly gains from the deviation σ_j^D whenever $\mathbf{z}^D$ is the largest integer, which occurs with a probability of at least $1 - \mu$, unless $g^O(m_i^*, m_{-i}^*) \subseteq \max_j^{\theta^*} X$. Since μ can be made arbitrarily small, it must be $g^O(m_i^*, m_{-i}^*) \subseteq \max_j^{\theta^*} X$ for all $j \neq i$. Therefore, by no-veto power, it must be $g^O(m_i^*, m_{-i}^*) \subseteq f(\theta^*)$ for all $(m_i^*, m_{-i}^*) \in R_2^i$.

Step 7. It only remains to consider messages $(m_i^*, m_{-i}^*) \in R_3$. For such messages the argument is analogous to messages in R_2^i. For no player i to be able to profit from the deviation σ_i^D, it must be $g^O(m_i^*, m_{-i}^*) \subseteq \max_i^{\theta^*} X$ for all $i \in N$. Therefore, the condition of no-veto power implies $g^O(m_i^*, m_{-i}^*) \subseteq f(\theta^*)$ for all $(m_i^*, m_{-i}^*) \in R_3$. □

Several remarks are in order. First, the mechanism constructed in the proof is inspired by the mechanism in the appendix of Maskin (1999), but ours is a randomized mechanism.[12] As we have already explained, we believe this is natural given that we consider the problem of implementation in mixed Nash equilibrium.

Second, our construction uses integer games. While we agree that integer games are not entirely satisfactory (e.g., see Jackson, 2001), Theorem 3 is no different from the large literature on implementation in having to resort to integer games in order to

[12]The mechanism in the main body of Maskin (1999) does not deal with the issue of ruling out unwanted mixed Nash equilibria.

rule out unwanted (not f-optimal) outcomes. In section 6, we will provide sufficient conditions for mixed Nash implementation by finite mechanisms, mechanisms in which (unlike in an integer game) each player only has a finite number of strategies.

Third, Theorem 3 strongly relies on the condition of weak* set-monotonicity, a weakening of Maskin monotonicity, which is relatively easy to check in applications. We have not tried to look for necessary and sufficient conditions for mixed Nash implementation. We suspect that such a characterization will involve conditions that are hard to check in practice, as it is the case for Nash implementation à la Maskin (e.g., condition μ of Moore and Repullo, 1990, condition M of Sjöström, 1991, condition β of Dutta and Sen, or strong monotonicity of Danilov, 1992). We do know, however, as example 4 below shows, that weak* set-monotonicity is not necessary for Nash implementation.

Fourth, we have not attempted to find sufficient conditions for the case of two players. Such a case requires a special treatment and is better left to another paper.

Fifth, as the following example shows, with C-implementation the conditions of weak and weak* set-monotonicity might fail to be necessary (for social choice correspondences and functions, respectively). However, they remain necessary for "large" enough compact sets of cardinal representations.

Example 2 There are two players, 1 and 2, two states of the world, θ and θ', and a unique cardinal representation at each state, indicated below:

θ		θ'	
1	2	1	2
d : 2	d : 5	d : 5	$c \sim d$: 2
c : 1.5	c : 2	a : 1	a : 1
a : 1	a : 1	c : 0.5	b : 0
b : -1	b : 0	b : -1	

For instance, at state θ, player 1's utility of d is 2, while player 2's utility is 5. The social choice function is $f(\theta) = \{a\}$ and $f(\theta') = \{c\}$; it is not weak* set-monotonic. Yet, it is easy to verify that f is implementable by the mechanism:

	m_1	m_2
m_1	a	$(1/2)\mathbf{1}[a] + (1/2)\mathbf{1}[b]$
m_2	$(1/2)\mathbf{1}[d] + (1/2)\mathbf{1}[b]$	c

,

where $(1/2)\mathbf{1}[x] + (1/2)\mathbf{1}[y]$ denotes a 50-50 lottery on x and y. To see that for "large" enough compact sets of cardinal representations the conditions of weak and weak* set-monotonicity remains necessary, fix $\delta > 0$, and let $\overline{\mathcal{U}}_i^\theta := \{u_i(\cdot, \theta) : X \to [-K, K] : x \succ_i^\theta y \Leftrightarrow u_i(x, \theta) \geq u_i(y, \theta) + \delta\}$ with K large enough but finite. The set $\overline{\mathcal{U}}_i^\theta$ of cardinal representations is clearly compact, and the proofs of Theorems 1 and 2 carry over if $\delta < 2K/(|X|^2)$.[13]

Sixth, the next example shows that the necessary condition of weak set-monotonicity together with no-veto power are not sufficient for mixed Nash implementation. This is analogous to the literature on implementation with incomplete information where no-veto power together with the necessary conditions of incentive compatibility and Bayesian monotonicity are not sufficient for Bayesian implementation (see Jackson, 1991).

Example 3 There are three players, 1, 2 and 3, two states of the world, θ and θ', and three alternatives a, b and c. Preferences are represented in the table below.

θ			θ'		
1	2	3	1	2	3
b	$a \sim d$	c	b	$a \sim b \sim d$	c
c		a	c		a
a	b	b	a		b
d	c	d	d	c	d

The social choice correspondence is $f(\theta) = \{a, d\}$ and $f(\theta') = \{b, d\}$. It is weak set-monotonic since $SL_2(a, \theta) \not\subseteq SL_2(a, \theta')$ and $L_2(b, \theta') \not\subseteq L_2(b, \theta)$. It also satisfies no-veto power. However, it is not implementable in mixed Nash equilibrium. If it were implementable, then for any cardinal representation of $\succsim^\theta$, there would exist an equilibrium σ^* at θ such that the support of $\mathbb{P}_{\sigma^*,g}$ is either $\{a\}$ or $\{a, d\}$. Then, σ^* would

[13] To see this, fix the following cardinal representation of $\succsim_i^\theta$: $u_i(x, \theta) = K$ for all $x \in \max_i^\theta X$, $u_i(x, \theta) = K - \delta|X|$ for all $x \in \max_i^\theta(X \setminus \max_i^\theta X)$, etc. This cardinal representation is clearly admissible and the difference in utilities between any two alternatives that are not indifferent is at least $\delta|X|$. It is then easy to see that, as required by the claim in the proof of Theorem 1, if we move to a state θ' where $L_i(x, \theta) \subseteq L_i(x, \theta')$ and $SL_i(x, \theta) \subseteq SL_i(x, \theta')$ for all $x \in f(\theta)$, for all $i \in N$, then there exists a cardinal representation $u_i(\cdot, \theta')$ with $u_i(x, \theta') \leq u_i(x, \theta)$ for all $x \in X$, and $u_i(x, \theta') = u_i(x, \theta)$ for all $x \in f(\theta)$. That is all we need for the proofs of Theorems 1 and 2 to go through.

also be an equilibrium at θ' for some cardinal representation of $\succsim^{\theta'}$, a contradiction. For instance, fix a cardinal representation $u(\cdot,\theta)$ at θ. Since players 1 and 3's preferences do not change from θ to θ', we can use the same cardinal representations at θ'. As for player 2, we can use $u_2(a,\theta') = u_2(b,\theta') = u_2(d,\theta') = u_2(a,\theta) = u_2(d,\theta) > u_2(b,\theta) > u_2(c,\theta) = u_2(c,\theta')$. The intuition is clear. Since players 1 and 3's preferences do not change from θ to θ' and a and d are top-ranked for player 2 at both states, any equilibrium at θ with outcome a in the support remains an equilibrium at θ' (with the above cardinal representation). At state θ', there is no alternative that can be used to generate a profitable deviation for player 2. Note that, as implied by Theorem 3, f is not weak* set-monotonic since $L_2(a,\theta) \subseteq L_2(a,\theta')$, $a \in \max_2^{\theta'}\{a,b,c\}$, and yet $a \notin f(\theta')$.

Seventh, as the next example shows, weak* set-monotonicity is not necessary for the implementation of social choice *correspondences*.

Example 4 There are three players, 1, 2 and 3, three alternatives a, b and c, and two admissible profiles of preferences θ and θ'. Preferences are given in the table below.

θ			θ'		
1	2	3	1	2	3
a	b	b	$a \sim c$	$b \sim c$	b
c	c	a			a
b	a	c	b	a	c

The social choice correspondence is $f(\theta) = \{a,b,c\}$ and $f(\theta') = \{b,c\}$. It is not weak* set-monotonic, but it is implementable in mixed Nash equilibrium. To see that f is not weak* set-monotonic, note that $a \notin f(\theta')$ and yet: $L_i(x,\theta) \subseteq L_i(x,\theta')$ for all $x \in \{a,b,c\}$, for all $i \in \{1,2,3\}$; $SL_3(x,\theta) \subseteq SL_3(x,\theta')$ for all $x \in \{a,b,c\}$; $SL_1(c,\theta) \subseteq SL_1(c,\theta')$, $SL_1(b,\theta) \subseteq SL_1(b,\theta')$, and $SL_1(a,\theta) \nsubseteq SL_1(a,\theta')$, but $a \in \max_1^{\theta'} X$; $SL_2(a,\theta) \subseteq SL_2(a,\theta')$, $SL_2(c,\theta) \subseteq SL_2(c,\theta')$, and $SL_2(b,\theta) \nsubseteq SL_1(b,\theta')$, but $b \in \max_2^{\theta'} X$. To show that f is implementable in mixed Nash equilibrium, consider the mechanism in which players 1 and 2 have two messages each, m_1 and m_2, player 3 has no message, and the allocation rule is represented below (player 1 is the row player):

	m_1	m_2
m_1	b	c
m_2	a	b

If the profile of preferences is θ', (m_1, m_2) is the unique pure Nash equilibrium of the game, with outcome c. There is also an equilibrium in which player 1 chooses m_1 and player 2 (appropriately) mixes over m_1 and m_2. There is no equilibrium in which both players mix. (Note that m_2 is weakly dominant for player 2 at state θ'.) On the other hand, it is clear that if the profile of preferences is θ, then there is an equilibrium in which both players totally mix between m_1 and m_2. Therefore, f is implementable in mixed Nash equilibrium, although it is not weak* set-monotonic.

We now claim that the restriction to compact sets of cardinal representations in Theorem 3 can be relaxed if we strengthen the condition of weak* set-monotonicity to *strong set-monotonicity*. Theorem 4 formally states this result without proof.[14]

Definition 4 *A social choice correspondence f is strong set-monotonic if for all pairs $(\theta, \theta') \in \Theta \times \Theta$, we have $f(\theta) \subseteq f(\theta')$ whenever for all $x \in f(\theta)$, for all $i \in N$: $L_i(x, \theta) \subseteq L_i(x, \theta')$.*

Note that on the domain of strict preferences, strong set-monotonicity coincides with weak and weak* set-monotonicity. Moreover, if a social choice correspondence is Maskin monotonic, then it is strong set-monotonic, and if it is strong set-monotonic, then it is weak set-monotonic.

Theorem 4 *Let $\langle N, X, \Theta \rangle$ be an environment with $n \geq 3$. If the social choice correspondence f is strong set-monotonic and satisfies no-veto power, then it is implementable in mixed Nash equilibrium.*

We conclude this section with an example showing that weak* set-monotonicity and no-veto power are not sufficient for (non compact) mixed Nash implementation; either a restriction to a compact set of cardinal representations is needed or the condition of weak* set-monotonicity needs to be strengthened to strong set-monotonicity.

Example 5 There are three players, 1, 2 and 3, four alternatives, a, b, c and d, and two

[14]The proof is obtained from the proof of Theorem 3 by setting $\varepsilon_k(m) = 0$ for all m.

admissible profiles of preferences θ and θ'. Preferences are given in the table below.

θ			θ'		
1	2	3	1	2	3
b	d	c	b	d	c
c	a	a	c	$a \sim b$	a
a	b	b	a	c	b
d	c	d	d		d

The social choice correspondence is $f(\theta) = \{a\}$ and $f(\theta') = \{b\}$. It satisfies weak* set-monotonicity and no-veto power; hence it is C-implementable in mixed Nash equilibrium. Let η be any real number with $\eta \geq 2$. Consider the following (non-compact) family of preference cardinalizations: $u_1(x, \theta) = u_1(x, \theta')$ for all $x \in X$; $u_3(x, \theta) = u_3(x, \theta')$ for all $x \in X$; $u_2(d, \theta) = u_2(d, \theta') = 2$, $u_2(a, \theta) = u_2(a, \theta') = 1$, $u_2(b, \theta) = 1 - \frac{1}{\eta}$, $u_2(b, \theta') = 1$, $u_2(c, \theta) = u_1(c, \theta') = 0$. Note that there is a unique utility representation for player 1 and 3 and the representation for player 2 only varies in the utility assigned to alternative b in state θ. We will show that with such a set of cardinal representations, f cannot be implemented in mixed Nash equilibrium. Suppose, to the contrary, that it can. Then, when θ is the true state, there is an equilibrium σ^* with full support on outcome a. Consider a possible deviation σ_2^D by player 2 and let p_a, p_b, p_c, p_d be the probability on each of the four outcomes induced by such a deviation. It must be the case that the deviation is not profitable when the state is θ; that is,

$$1 = u_2(a, \theta) \geq p_a u_2(a, \theta) + p_b u_2(b, \theta) + p_c u_2(c, \theta) + p_d u_2(d, \theta) = p_a + p_b \left(1 - \frac{1}{\eta}\right) + 2p_d. \quad (2)$$

Since σ^* cannot be an equilibrium when the true state is θ' (otherwise f would not be implemented) and the preferences of players 1 and 3 do not change with the state, there must be a deviation σ_2^D by player 2 that is profitable when the state is θ' and player 1 and 3 play σ_1^* and σ_3^*; that is,

$$1 = u_2(a, \theta') < p_a u_2(a, \theta') + p_b u_2(b, \theta') + p_c u_2(c, \theta') + p_d u_2(d, \theta') = p_a + p_b + 2p_d. \quad (3)$$

Equation (2) implies that for all $\eta \geq 2$, $1 - p_a - p_b \geq -\frac{1}{\eta} p_b + 2p_d$ and, hence, $1 - p_a - p_b \geq 2p_d$, while equation (3) requires that $1 - p_a - p_b < 2p_d$, a contradiction.

6 Finite Mechanisms

Theorem 3 relies on integer games to provide sufficient conditions for mixed Nash implementation. As pointed out by Jackson (2001, p. 684), this is not totally satisfactory: *"A player's best response correspondence is not well-defined when that player faces a mixed strategy of the others that places weight on an infinite set of integers."* In this section, we look at *finite* mechanisms; that is, for each player $i \in N$, we impose that the set of messages M_i is finite.[15] Since this paper considers finite environments (finite sets of alternatives and preference profiles), the restriction to finite mechanisms is natural. We use two additional conditions. First, following Jackson, Palfrey and Srivastava (1994), we restrict attention to separable environments. An environment is separable if the following two properties hold:[16]

A1 A *worst outcome* relative to f: there exists $w \in X$ such that $x \succ_i^\theta w$ for all $i \in N$, all $(\theta, \theta') \in \Theta \times \Theta$, and all $x \in f(\theta')$.

A2 Separability: For all $x \in X$, all $\theta' \in \Theta$, and $i \in N$, there exists $y^i(x) \in X$ such that $y^i(x) \sim_i^{\theta'} x$, while $y^i(x) \sim_j^\theta w$ for all $\theta \in \Theta$ and all $j \in N \setminus \{i\}$.

There are several examples of separable environments, e.g., pure exchange economies with strictly monotone preferences or environments with transferable utilities. We refer the reader to Jackson et al. (1994) for more examples.[17]

Let D be a subset of N containing at least one player. The second condition we require is that for each player i in D and for each state of the world θ, $f(\theta)$ contains the top alternatives for player i in the range $X^f := f(\Theta)$ of f. A formal definition is as follows.

Definition 5 *A social choice correspondence f is* top-D-inclusive *(relative to f) if there exists a non-empty subset D of the set of players N such that $\cup_{i \in D} \max_i^\theta X^f \subseteq f(\theta)$ for all $\theta \in \Theta$.*

[15] Best responses are well-defined in games induced by finite mechanisms.

[16] The definition by Jackson, Palfrey and Srivastava (1994) includes a third property, strict value distinction, which we do not need. Also, our property A2 is weaker then their corresponding property.

[17] Sjöström (1994) provides another example of a separable environment: a production economy with public goods.

Top-D-inclusiveness is an efficiency condition. For instance, the weak Pareto correspondence is top-D-inclusive. Note also that if f is top-D-inclusive with D containing at least two elements and $X^f = X$, then it satisfies no-veto power. We have the following theorem:

Theorem 5 *Let $\langle N, X, \Theta \rangle$ be a separable environment with $n \geq 3$. If the social choice correspondence f is weak* set-monotonic and top-D-inclusive, then it is C-implementable in mixed Nash equilibrium by a finite mechanism.*

The intuition for Theorem 5 is simple. Consider a profile of messages m^* such that all but player i announces the same message (i.e., rule 2 of the proof of Theorem 3 applies: $m^* \in R_2^i$). An essential role of the integer game in Theorem 3 is to guarantee that any player $j \neq i$ can trigger the integer game, gets his favorite alternative (with arbitrary high probability) and, thus, cannot be worse off. Without the integer game, this is not always possible. However, with separable environments, we can guarantee player i the same expected payoff under m^*, while giving to all the other players the payoff corresponding to the worst outcome. In turn, this implies that any player $j \neq i$ can trigger a finite "game" between players in D where only alternatives in X^f can be implemented and, thus, not be worse off. Lastly, top-D-inclusiveness guarantees that the outcomes of the finite "game" are f-optimal. The formal proof of Theorem 5 is in the Appendix.

We end this section with three remarks. First, we do not know how tight our sufficient conditions for finite mechanisms are. Clearly, dictatorial and constant social choice correspondences are implementable in mixed Nash equilibrium by finite mechanisms in general environments.[18] Furthermore, in Example 1, we implement a social choice correspondence by a finite mechanism and yet it is neither top-D-inclusive nor dictatorial nor constant.[19] Second, as in Theorem 4, if we replace weak* set-monotonicity with strong set-monotonicity, then we obtain a sufficiency theorem for implementation in mixed Nash equilibrium by a finite mechanism, as opposed to C-implementation. Third, it is worth emphasizing the difference between Theorem 5 and the results in three related

[18] In separable environments, a constant social choice function is weak* set-monotonic and top-D-inclusive. Moreover, setting $|D| = 1$ and $X^f \subseteq \cup_{\theta \in \Theta} \max_i^\theta X$ allows to define weak* set-monotonic and top-D-inclusive social choice correspondences that are essentially dictatorial.

[19] Example 1 is with two players, but it is easy to modify it so as to have three players: add a third player with the same preferences as player 1.

papers. Jackson, Palfrey and Srivastava (1994) and Sjöström (1994) use bounded mechanisms that make no use of integer games, but their solution concept is different from ours; they consider implementation in undominated Nash equilibrium, rather than Nash implementation. Abreu and Matsushima (1992) use finite mechanisms, but their results are for virtual implementation, and their mechanism requires, among other things, that agents transmit cardinal information about their preferences to the center. We focus on exact implementation in mixed Nash equilibrium and on ordinal mechanisms.[20]

7 Dispensing with No-Veto Power

As pointed out by Benoît and Ok (2008) and Bochet (2007), the appeal of the no-veto power condition may be questioned in settings with a small number of agents. In the context of pure Nash implementation, and allowing for out-of-equilibrium randomness in the mechanism, they showed that no-veto power can be dispensed with, provided that some mild domain restrictions are imposed.[21] We now show that similar results can be obtained in the context of mixed Nash implementation.

Definition 6 (Bochet, 2007) *An environment* $\langle N, X, \Theta \rangle$ *satisfies* top-strict difference *if for any* $\theta \in \Theta$ *and* $x \in X$ *such that* $x \in \cap_{i \in I} \max_i^\theta X$ *for* $I \subseteq N$ *with* $|I| = n-1$, *there exist* $(j,k) \in N \times N$ *such that* $\max_j^\theta X = \max_k^\theta X = \{x\}$.

Top strict difference requires that if $n-1$ agents rank x at the top, then at least two agents must rank x strictly at the top.

[20] Almost any social choice correspondence is implementable in undominated Nash equilibrium (Palfrey and Srivastava, 1991) or virtually implementable in Nash equilibrium (Abreu and Sen, 1991). But, as we pointed out in fn.1, Chung and Ely (2003) show that if the assumption of complete information is relaxed to "near-complete information," then Maskin monotonicity is restored as a necessary condition for implementation in undominated, pure, Nash equilibrium.

[21] Bochet (2007) showed that, with $n \geq 3$, Maskin monotonicity is sufficient for Nash implementation if preferences satisfy top-strict difference. Benoît and Ok (2008), again with $n \geq 3$, showed that Maskin monotonicity and weak unanimity of f are sufficient if preferences satisfy the top-coincidence condition. Both papers use out-of-equilibrium randomness in the mechanism, but limit themselves to rule out unwanted *pure* equilibria.

Definition 7 (Benoît and Ok, 2008) *An environment $\langle N, X, \Theta \rangle$ satisfies the* top-coincidence *condition if for any $\theta \in \Theta$ and any $I \subseteq N$ with $|I| = n - 1$, the set $\cap_{i \in I} \max_i^\theta X$ is either empty or a singleton.*

Clearly, the top-coincidence condition and the top-strict difference condition are satisfied on the domain of single-top preferences. (Remember that on the domain of single-top preferences, each player has a single most preferred alternative at each state.)

Definition 8 *A social choice correspondence f is* weakly unanimous *if for all $\theta \in \Theta$, we have $x \in f(\theta)$ whenever $\{x\} = \cap_{i \in N} \max_i^\theta X$.*

As argued by Benoît and Ok (2008), the top-coincidence condition is a fairly mild domain restriction, while weak unanimity is a much weaker condition than no-veto power. Clearly, if f satisfies no-veto power, then it is weakly unanimous, but the converse does not hold. We have the following theorem.[22]

Theorem 6 *Let $\langle N, X, \Theta \rangle$ be an environment with $n \geq 3$. If the social choice correspondence f is weak* set-monotonic and either (a) the environment satisfies the top-coincidence condition and f is weakly unanimous, or (b) the environment satisfies the top-strict-difference condition, then f is C-implementable in mixed Nash equilibrium.*[23]

8 Applications

This section contains a series of remarks in which we provide applications of our results to some important social choice rules.

Remark 1 On the unrestricted domain of preferences (i.e., when all possible ordinal rankings are admissible), the strong Pareto correspondence f^{PO} is weak set-monotonic, while it fails to be Maskin monotonic. Therefore, on the domain of single-top preferences, f^{PO} satisfies weak* set-monotonicity and no veto power.[24] Hence, if we restrict attention

[22]The proof is relegated to the Appendix.

[23]Theorem 6 is stated for weak* set-monotonic correspondences and C-implementation, but also holds for strong set-monotonic correspondences and mixed Nash implementation. Only a modification like the one needed to prove Theorem 4 is required.

[24]Recall that on the domain of single-top preferences, $\max_i^\theta X$ is a singleton for each $i \in N$, for each $\theta \in \Theta$, and thus weak* set-monotonicity coincides with weak set-monotonicity.

to the domain of single-top preferences, Theorem 3 applies and f^{PO} is C-implementable in mixed Nash equilibrium.

The strong Pareto correspondence is defined as follows:

$$f^{PO}(\theta) := \{x \in X : \text{ there is no } y \in X \text{ such that } x \in L_i(y,\theta) \text{ for all } i \in N, \\ \text{and } x \in SL_i(y,\theta) \text{ for at least one } i \in N\}.$$

To see that f^{PO} is weak set-monotonic, consider two states θ and θ' such that for all $i \in N$, for all $x \in f^{PO}(\theta)$, (i) $L_i(x,\theta) \subseteq L_i(x,\theta')$ and (ii) $SL_i(x,\theta) \subseteq SL_i(x,\theta')$. Suppose that $x^* \in f^{PO}(\theta)$, but $x^* \notin f^{PO}(\theta')$. At state θ', there must then exists $y \in X$ such that $y \notin SL_i(x^*,\theta')$ for all $i \in N$ and $y \notin L_i(x^*,\theta')$ for at least one $i \in N$. It follows that $y \notin SL_i(x^*,\theta)$ for all $i \in N$ and $y \notin L_i(x^*,\theta)$ for at least one $i \in N$, a contradiction with $x^* \in f^{PO}(\theta)$. Consequently, $f^{PO}(\theta) \subseteq f^{PO}(\theta')$ and f^{PO} is weak set-monotonic.

To see that the strong Pareto correspondence is not Maskin monotonic on the domain of single-top preferences (and, therefore, on the unrestricted domain), consider the following example. There are three players, 1, 2 and 3, and two states of the world θ and θ'. Preferences are given in the table below.

θ			θ'		
1	2	3	1	2	3
d	d	b	d	d	b
b	a	a	b	$a \sim b$	a
c	b	c	c		c
a	c	d	a	c	d

The strong Pareto correspondence is: $f^{PO}(\theta) = \{a,b,d\}$ and $f^{PO}(\theta') = \{b,d\}$. Maskin monotonicity does not hold since $L_2(a,\theta) \subseteq L_2(a,\theta')$ and yet $a \notin f^{PO}(\theta')$.[25]

Remark 2 Using arguments that parallel the ones used for the strong Pareto correspondence, it can be verified that on the unrestricted domain of preferences the strong

[25] On the unrestricted domain of preferences, f^{PO} is not weak* set-monotonic. To see this, suppose alternative d is not available in the example. The strong Pareto correspondence is then $f(\theta) = \{a,b\}$ and $f(\theta') = \{b\}$. Weak* set-monotonicity fails, since it is $L_2(a,\theta) \subseteq L_2(a,\theta')$, $a \in \max_2^{\theta'}\{a,b,c\}$ and yet $a \notin f^{PO}(\theta')$. In fact, following the reasoning in Example 3, we can see that f^{PO} is not C-implementable in mixed Nash equilibrium in this modified example.

core correspondence f^{SC} is weak set-monotonic. If we restrict attention to the domain of single-top preferences f^{SC} also satisfies weak* monotonicity, while it is not Maskin monotonic on either domain of preferences. Since f^{SC} also satisfies weak unanimity, by Theorem 6, it is C-implementable in mixed Nash equilibrium in the domain of single-top preferences.

A coalitional game is a quadruple $\langle N, X, \theta, v\rangle$, where N is the set of players, X is the finite set of alternatives, θ is a profile of preference relations, and $v : 2^N \setminus \{\emptyset\} \rightarrow 2^X$. An alternative x is weakly blocked by the coalition $S \subseteq N \setminus \{\emptyset\}$ if there is a $y \in v(S)$ such that $x \in L_i(y, \theta)$ for all $i \in S$ and $x \in SL_i(y, \theta)$ for at least one $i \in S$. If there is an alternative that is not weakly blocked by any coalition in $2^N \setminus \{\emptyset\}$, then $\langle N, X, \theta, v\rangle$ is a game with a non-empty strong core. A coalitional environment with non-empty strong core is a quadruple $\langle N, X, \Theta, v\rangle$, where Θ is a set of preference relations such that $\langle N, X, \theta, v\rangle$ has a non-empty strong core for all $\theta \in \Theta$.

The strong core correspondence f^{SC} is defined for all coalitional environments with non-empty strong core as follows:

$$f^{SC}(\theta) := \{x \in v(N) : \ x \text{ is not weakly blocked by any } \emptyset \neq S \subseteq N\}.$$

Remark 3 On the unrestricted domain of preferences, a Maskin monotonic social choice *function* must be constant (Saijo, 1988). It is simple to see that this is also true for a weak* set-monotonic social choice function. Suppose, to the contrary, that $f(\theta) = x \neq y = f(\theta')$. Let θ'' be such that $x, y \in \max_i^{\theta''} X$ for all $i \in N$. Then weak* set-monotonicity implies $\{x, y\} \subseteq f(\theta'')$, contrary to the assumption that $f(\theta'')$ is a singleton.

Remark 4 On the domain of strict preferences, the top-cycle correspondence, an important voting rule, is weak* (and hence weak) set-monotonic, while it is not Maskin monotonic. Since it also satisfies no-veto power, it follows that Theorem 4 applies: on the domain of strict preferences, the top-cycle correspondence is implementable in mixed Nash equilibrium.

We say that alternative x defeats alternative y at state θ, written $x \gg^{\theta} y$, if the number of players who prefer x to y is strictly greater than the number of players who prefer y to x. At each state θ, the top-cycle correspondence selects the smallest subset of X such that any alternative in it defeats all alternatives outside it.

$$f^{TC}(\theta) := \cap\{X' \subseteq X : x' \in X', x \in X \setminus X' \text{ implies } x' \gg^{\theta} x\}.$$

To prove that the top-cycle correspondence is weak set-monotonic, assume to the contrary that there is at least an alternative x^* such that $x^* \in f^{TC}(\theta)$, $x^* \notin f^{TC}(\theta')$, and $L_i(x,\theta) \subseteq L_i(x,\theta')$ for all $x \in f^{TC}(\theta)$, for all $i \in N$. (When preferences are strict $L_i(x,\theta) \setminus \{x\} = SL_i(x,\theta)$ and strong set-monotonicity coincides with weak set-monotonicity and weak* set-monotonicity.) Clearly, if x^* is a Condorcet winner at θ so that $f^{TC}(\theta) = \{x^*\}$, then x^* is also a Condorcet winner at θ'; hence $x^* \in f^{TC}(\theta')$, a contradiction. Assume that x^* is not a Condorcet winner; that is, the set $f^{TC}(\theta)$ is not a singleton. Take any alternative $x \in f^{TC}(\theta)$ and any $y \notin f^{TC}(\theta)$. By definition of f^{TC}, it must be that $x \gg^{\theta} y$. Since $L_i(x,\theta) \subseteq L_i(x,\theta')$ for all $x \in f^{TC}(\theta)$, it must also be that $x \gg^{\theta'} y$. Hence, it must be $f^{TC}(\theta') \subseteq f^{TC}(\theta)$. For all $x^* \in f^{TC}(\theta) \setminus f^{TC}(\theta')$, it must be the case that $x \gg^{\theta'} x^*$ for all $x \in f^{TC}(\theta')$. Furthermore, since the lower contour sets of x^* satisfy $L_i(x^*,\theta) \subseteq L_i(x^*,\theta')$, the strict upper contour set of x^* at θ' is a subset of the strict upper contour set at θ for all $i \in N$, and hence it must be $x \gg^{\theta} x^*$ for all $x \in f^{TC}(\theta')$. This contradicts the assumption that $x^* \in f^{TC}(\theta)$ and $f^{TC}(\theta)$ is the smallest subset of X such that any alternative in it defeats all alternatives outside it at θ.

To see that f^{TC} is not Maskin monotonic, consider the following example with two states, three alternatives and three players.

θ			θ'		
1	2	3	1	2	3
a	c	b	a	c	c
b	a	c	b	a	b
c	b	a	c	b	a

We have that $f^{TC}(\theta) = \{a,b,c\}$ and $f^{TC}(\theta') = \{c\}$. Since $L_i(a,\theta) \subseteq L_i(a,\theta')$ for all $i \in N$, Maskin monotonicity and $f^{TC}(\theta) = \{a,b,c\}$ would require $a \in f^{TC}(\theta')$.

Remark 5 The Borda, Kramer and plurality voting rules fail to satisfy weak set-monotonicity.[26]

[26]The Kramer score of alternative x at state θ is $\max_{y \neq x} |\{i \in N : x \succ_i^{\theta} y\}|$; the Kramer rule selects

It is simple to see that the example in Maskin (1999, page 30) shows that the Borda rule fails to satisfy not only Maskin monotonicity, but also weak set-monotonicity. For the Kramer rule, consider the example in the table below with five players, three alternatives and two states θ and θ'.

θ					θ'				
1	2	3	4	5	1	2	3	4	5
a	a	a	a	c	a	a	a	a	b
b	b	c	c	b	b	b	b	b	c
c	c	b	b	a	c	c	c	c	a

The Kramer rule selects a at state θ and b at state θ', a violation of weak set-monotonicity. (Remember that weak* set-monotonicity coincides with weak set-monotonicity when preferences are strict, as in the above example.) For the plurality rule, consider the table below with seven players, three alternatives and two states θ and θ'.

θ							θ'						
1	2	3	4	5	6	7	1	2	3	4	5	6	7
a	a	a	b	b	c	c	a	a	a	b	b	b	b
b	b	b	c	c	b	b	b	b	b	c	c	c	c
c	c	c	a	a	a	a	c	c	c	a	a	a	a

In this example, the plurality rule selects a at θ and b at θ', which also violates weak set-monotonicity. So, our results are not so permissive so as to imply that all "reasonable" social choice correspondences are implementable in mixed Nash equilibrium.

9 Conclusions

In this paper, we introduce the concept of *mixed Nash implementation.* According to our definition, a mechanism implements a social choice correspondence f in mixed Nash equilibrium if the set of all pure *and* mixed Nash equilibrium outcomes corresponds to the set of f-optimal alternatives at each preference profile. Crucially, and unlike Maskin, our definition does not give a predominant role to pure equilibria. While we allow players

the alternatives with the highest Kramer score at each state. The plurality rule selects the alternatives ranked top by the highest number of players.

and the planner to randomize, we maintain an entirely ordinal approach. In our model, a social choice correspondence f maps profiles of preference orderings over alternatives into subsets of alternatives and we require that the chosen game form implements f for all possible cardinal representation of players' preferences. Also, we require exact, as opposed to virtual, implementation.

We show that weak set-monotonicity, a substantial weakening of Maskin's monotonicity, is necessary for implementation of social choice correspondences in mixed Nash equilibrium. Weak* set-monotonicity, a mild strengthening of weak set-monotonicity, is necessary for implementing social choice functions, and together with no veto power is sufficient for the implementation of social choice correspondences if there are at least three players. We also provide sufficient conditions which include weak* set-monotonicity and mild domain restrictions, but do not include no-veto power. Restricting attention to finite mechanisms, we show that an efficiency condition, called top-D-inclusiveness, together with weak* set-monotonicity, is sufficient for implementation in separable environments. Important social choice correspondences that are not Maskin monotonic, like the strong Pareto, the strong core and the top-cycle may be implemented in mixed Nash equilibrium.

Appendix

Proof of Theorem 5. We need to make a few changes to the mechanism considered in the proof of Theorem 3. Let D be a subset of players, with cardinality $|D| \geq 1$ that satisfies the definition of top-D-inclusiveness. For each player $i \in N$, the message space M_i is $\Theta \times \{\alpha^i : \alpha^i : X \times \Theta^2 \to X\} \times X \times X^f \times \{1, 2\}$. Thus, each i must also announce an element s of X^f and may announce only two integers, 1 or 2. Rule 1 is the same as in the proof of Theorem 3, except that that the common message sent by all players $m_i = (\theta, \alpha, x, s, 1)$ includes two alternatives. We replace rules 2 and 3 of the allocation rule g with the following:

Rule $2^{\mathbf{F}}$: If there exists $j \in N$ such that $m_i = (\theta, \alpha, x, s, 1)$ for all $i \in N \setminus \{j\}$, with $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)$, and $m_j = (\theta^j, \alpha^j, x^j, s^j, \mathbf{z}^j) \neq m_i$, then $g(m)$ is the lottery:

$$\frac{1}{K^\theta} \sum_{k=1}^{K^\theta} \Big\{ \delta_k(m)(1 - \varepsilon_k(m)) \mathbf{1} \left[y^j(\alpha^j(f_k(\theta), \theta, \theta^j)) \right] + \delta_k(m) \varepsilon_k(m) \mathbf{1} \left[y^j(x^j) \right] + (1 - \delta_k(m)) \, \mathbf{1} \left[y^j(f_k(\theta)) \right] \Big\}$$

with

$$\delta_k(m) = \begin{cases} \delta & \text{if } \alpha^j(f_k(\theta), \theta, \theta^j) \in L_j(f_k(\theta), \theta) \\ 0 & \text{if } \alpha^j(f_k(\theta), \theta, \theta^j) \notin L_j(f_k(\theta), \theta) \end{cases}$$

for $1 > \delta > 0$, and

$$\varepsilon_k(m) = \begin{cases} \varepsilon & \text{if } \alpha^j(f_k(\theta), \theta, \theta^j) \in SL_j(f_k(\theta), \theta) \\ 0 & \text{if } \alpha^j(f_k(\theta), \theta, \theta^j) \notin SL_j(f_k(\theta), \theta) \end{cases}.$$

It is important to note that the assumption of a separable environment and the definition of $y^j(x)$ in A2 imply that under this rule any player $i \neq j$ gets the same payoff as under the worst outcome w.

Rule $3^{\mathbf{F}}$: If neither rule 1 nor rule 2 applies, then $g\left((\theta^i, \alpha^i, x^i, s^i, \mathbf{z}^i)_{i \in N}\right)$ is the uniform lottery over the alternatives s^i selected by the players in D:

$$\frac{1}{|D|} \sum_{j \in D} \mathbf{1} \left[s^i \right].$$

Fix a state θ^*, and a cardinal representation $u_i \in \overline{\mathcal{U}}_i^{\theta^*}$ of $\succsim_i^{\theta^*}$ for each player i. Let u be the vector of cardinal representations.

Step 1. Step 1 of the proof is the same as Step 1 in the proof of Theorem 3. It shows that for any $x \in f(\theta^*)$, there exists a Nash equilibrium σ^* of $G(\theta^*, u)$ such that x belongs to the support of $\mathbb{P}_{\sigma^*,g}$.

Step 2. Step 2, defining the sets R_1, R_2^j and R_3, in which the different rules apply, is also the same as Step 2 in the proof of Theorem 3.

Consider an equilibrium σ^* of $G(\theta^*, u)$, and let M_i^* be the set of message profiles that occur with positive probability under σ_i^*. We need to show that $g^O(m^*) \subseteq f(\theta^*)$ for all $m^* \in M^* := \times_{i \in N} M_i^*$.

Step 3. Suppose there exists $(m_i^*, m_{-i}^*) \in R_2^i$; that is, for all $j \neq i$, $m_j^* = (\theta, \alpha, x, , s, 1) \neq m_i^*$. In this case, for all $m_j^* = (\theta^j, \alpha^j, x^j, s^j, \mathbf{z}^j)$ define the deviation message function $m_j^D\left(m_j^*\right) = (\theta^j, \alpha^j, x^j, s^j, 2)$; that is, $m_j^D\left(m_j^*\right)$ replaces $\mathbf{z}^j$ with the integer 2 in all messages sent by player j. Note that: (i) when $(m_j^*, m_{-j}^*) \in R_1 \cup R_2^j$, then $\left(m_j^D\left(m_j^*\right), m_{-j}^*\right) \in R_2^j$ and player j obtains the same payoff under g from the two message profiles; (ii) when $(m_j^*, m_{-j}^*) \in R_2^i$, then $\left(m_j^D\left(m_j^*\right), m_{-j}^*\right) \in R_3$ and player j obtains a strictly higher payoff from the deviation message, since he obtains the same payoff as with the worst outcome in R_2^i, while any outcome under R_3 belongs to X^f and hence is preferred to w; (iii) when $(m_j^*, m_{-j}^*) \in R_3$, then $\left(m_j^D\left(m_j^*\right), m_{-j}^*\right) \in R_3$ and player j obtains the same payoff from the two message profiles. It follows that if a message profile $(m_i^*, m_{-i}^*) \in R_2^i$ occurs with positive probability, then player j has a strictly profitable deviation. This implies that there are only two possibilities. The first is that the equilibrium σ^* of $G(\theta^*, u)$ is in pure strategies and all players send the same message $m_i^* = (\theta, \alpha, x, s, 1)$ and hence $(m_i^*, m_{-i}^*) \in R_1$. The second possibility is that the equilibrium σ^* of $G(\theta^*, u)$ is in mixed (or pure) strategies and all message profiles sent with positive probability by players belong to the set R_3. Step 4 deals with the first possibility, while Step 5 deals with the second.

Step 4. Suppose all players send the same message $m_i^* = (\theta, \alpha, x, s, 1)$ and hence $(m_i^*, m_{-i}^*) \in R_1$ with probability one. For any player $i \in N$, define the (deviation) message $m_i^D = (\theta^i, \alpha^D, x^D, s^D, 2)$, where: 1) α^D differs from α^i in at most the alternatives associated with elements $(f_k(\theta), \theta, \theta)$ for all $\theta \in \Theta$, for all $k \in \{1, \ldots, K^\theta\}$;

2) $x^D \in \max_i^{\theta^*} X$, and 3) $s^D \in \max_i^{\theta^*} X^f$. Player i strictly gains from the deviation if $\alpha^D(f_k(\theta),\theta,\theta) \in L_i(f_k(\theta),\theta)$ and either (1) $\alpha^D(f_k(\theta),\theta,\theta) \succ_i^{\theta^*} f_k(\theta)$ or (2) $\alpha^D(f_k(\theta),\theta,\theta) \in SL_i(f_k(\theta),\theta)$, $\alpha^D(f_k(\theta),\theta,\theta) \succcurlyeq_i^{\theta^*} f_k(\theta)$ and $f_k(\theta) \notin \max_i^{\theta^*} X$. Hence, (1) and (2) cannot hold for any player i. It follows that for σ^* to be an equilibrium, for all i and all k it must be (1) $L_i(f_k(\theta),\theta) \subseteq L_i(f_k(\theta),\theta^*)$ and (2) either $SL_i(f_k(\theta),\theta) \subseteq SL_i(f_k(\theta),\theta^*)$ or $f_k(\theta) \in \max_i^{\theta^*} X$. Therefore, by the weak* set-monotonicity of f, we must have $f(\theta) \subseteq f(\theta^*)$. This shows that $g^O(m_i^*, m_{-i}^*) \subseteq f(\theta^*)$.

Step 5. Suppose the equilibrium σ^* of $G(\theta^*, u)$ is in mixed (or pure) strategies and all message profiles sent with positive probability by players belong to the set R_3. For all $j \in D$ and all $m_j^* = (\theta^j, \alpha^j, x^j, s^j, \mathbf{z}^j)$ sent with positive probability by player j, define the deviation message function $m_j^D\left(m_j^*\right) = \left(\theta^j, \alpha^j, x^j, s^D, 2\right)$, where $s^D \in \max_i^{\theta^*} X^f$. Note that if player $j \in D$ follows the deviation strategy of replacing each message m_j^* with the deviation message defined by the function $m_j^D(\cdot)$, then he strictly profits unless $s^j \in \max_i^{\theta^*} X^f$. Since this is true for all $j \in D$, and Rule 3^{F} applies to all equilibrium message profiles, it must be $g^O(m_i^*, m_{-i}^*) \subseteq \cup_{i\in D} \max_i^{\theta} X^f$. Then, by top-D-inclusiveness, it is $g^O(m_i^*, m_{-i}^*) \subseteq f(\theta^*)$. This completes the proof. □

Proof of Theorem 6. For part (a) we use the same mechanism as in the proof of Theorem 3. For part (b) we slightly modify rule 3, replacing it with the following:

Rule $3^{\mathbf{R}}$: Let i^* be a player announcing the highest integer $\mathbf{z}^{i^*}$. If neither rule 1 nor rule 2 applies, then $g\left((\theta^i, \alpha^i, x^i, \mathbf{z}^i)_{i\in N}\right)$ is the random lottery that assigns probability $\left(1 - \frac{1}{\mathbf{z}^{i^*}}\right)$ to x^{i^*} and probability $\frac{1}{\mathbf{z}^{i^*}}$ to the uniform lottery over all alternatives in X.

The proof of both parts is very similar to the proof of Theorem 3; only two changes are needed.

The first, more substantial, change is for the case of a message realization $m^* \in R_2^i$. Replace Step 6 with the following.

Step 6^R. As in the proof of Theorem 3, we may conclude that all the alternatives in the support of $\mathbb{P}_{m^*,g}$ must belong to $\max_j^{\theta^*} X$ for all $j \in N \setminus \{i\}$. (Recall that $m_j^* = (\theta, \alpha, x, 1)$ for all $j \in N \setminus \{i\}$, with $\alpha(f_k(\theta),\theta,\theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)$, and $m_i^* = (\theta^i, \alpha^i, x^i, \mathbf{z}^i) \neq m_j^*$.)

(a) By the top-coincidence condition, the support of $\mathbb{P}_{m^*,g}$ must then consist of a single alternative x^*. Hence it must be $x^* = f(\theta)$. Player i may deviate and send the message $m_i^D = (\theta^i, \alpha^D, x^D, \mathbf{z}^i)$, with $x^D \in \max_i^{\theta^*} X$ and with α^D differing from α^i only in the component $\alpha^D(x^*, \theta, \theta^i) \in L_i(x^*, \theta)$. For such a deviation not to be profitable it must be: (i) $x^* = f(\theta) \succcurlyeq_i^{\theta^*} \alpha^D(x^*, \theta, \theta^i)$ and (ii) if $\alpha^D(x^*, \theta, \theta^i) \in SL_i(x^*, \theta)$ then $x^* \succ_i^{\theta^*} \alpha^D(x^*, \theta, \theta^i)$ or $x^* \in \max_i^{\theta^*} X$. We can conclude that: (1) $L_i(x^*, \theta) \subseteq L_i(x^*, \theta^*)$ and (2) either $SL_i(x^*, \theta) \subseteq SL_i(x^*, \theta^*)$ or $x^* \in \max_i^{\theta *} X$. Furthermore, since $x^* \in \max_j^{\theta^*} X$ for all $j \in N \backslash \{i\}$, (1) and (2) hold for all j. Consequently, $f(\theta) = x^* \subseteq f(\theta^*)$ by the weak* set-monotonicity of f.

(b) By the top-strict-difference condition, there must be at least a $j \in N \backslash \{i\}$ such that $\max_j^{\theta^*} X$ is a singleton. Hence the support of $\mathbb{P}_{m^*,g}$ must consist of a single alternative x^* and it must be $x^* = f(\theta)$. The rest of the proof of this case is as the proof of part (a).

The second, minor, change in the proof is for the case of a message realization m^* such that rule 3 (rule 3^{R}) applies. Replace Step 7 with the following.

Step 7^R. Let i^* be a player announcing the highest integer $\mathbf{z}^{i^*}$. Since no player must be able to profitably gain from a deviation, it must be the case that $x^{i^*} \in \max_i^{\theta^*} X$ for all $i \in N$. (a) It follows from the top-coincidence condition and weak unanimity that $x^{i^*} \in f(\theta^*)$. (b) It follows from the top-strict-difference condition that for at least one agent j, we have $\max_j^{\theta^*} X = \{x^{i^*}\}$, then, by rule 3^{R}, setting $\mathbf{z}^j > \mathbf{z}^{i^*}$ and $x^j = x^{i^*}$ is a profitable message deviation for agent j. $\square$

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