Information Aggregation and Coordination[1]

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Abstract

In many strategic settings, if two players act, they receive a common-value payoff about which they have private information; but if only one acts, she incurs miscoordination costs. Thus, unlike standard global games, in addition to coordination considerations, players have information aggregation concerns—equilibrium outcomes are informative about the other player's information. Best responses are non-monotone reflecting interactions between information aggregation (a force for strategic substitutes) and miscoordination (a force for strategic complements). We first characterize monotone equilibria and show how the design of rewards and punishments (e.g. by regimes fearing coups) affect coordination probabilities. Subtle welfare consequences arise; e.g., when actions are strategic substitutes, making miscoordination less costly can harm a player. We show that for most plausible payoff structures, there is no informative cheap talk in monotone strategies. Finally, we use Karlin's theorem for TP_3 functions to partially characterize non-monotone equilibria in bounded interval strategies.

1 Introduction

Consider two interest groups deciding whether to support a known incumbent or an untried challenger. The challenger wins if both groups support him. However, the incumbent wins if only one group supports the challenger, and it then punishes the disloyal group, possibly rewarding the loyal one. The groups are interested in the same policies, and they know what the incumbent would do in office, but they are uncertain about the challenger and have noisy private information about him. We ask: From a welfare perspective, does the challenger receive sufficient support? Does an interest group gain if an incumbent increases its loyalty reward? Do the interest groups share their information? To maximize his chances of winning, should an incumbent punish the disloyal more harshly or reward the loyal more generously? Similar considerations arise in other settings: firms deciding whether to adopt a new technology standard or to make infrastructure investments in a developing country; countries deciding whether to remove import tariffs; officers deciding whether to mount a coup; or individuals contemplating breaking up existing relationships to start new ones.

We analyze behavior and welfare in a class of games that captures such strategic interactions. The games have three key features: (1) players receive private noisy signals about a common stochastic payoff that is received if they coordinate on a particular action; (2) extensive uncertainty remains even after players receive their private signals; and (3) if players miscoordinate, at least one is hurt. The first two features give rise to learning and hence information aggregation in equilibrium, and the third gives rise to coordination concerns.

		player j	
		0	*1*
player i	*0*	h, h	w, l
	1	l, w	θ, θ

Figure 1: Payoffs: h, l, and w are known, with $h > l$. θ is random with support $\mathbb{R}$.

The payoffs are depicted above.[1] The players receive private signals about a stochastic payoff θ that they receive if and only if they coordinate on action 1. The other expected

[1]The strategic considerations in this game is equivalent to the one in which l is replaced by $l + \alpha\theta$, with $\alpha \in [0, 1)$ and $l + \alpha E[\theta] < h$.

payoffs are common knowledge, with $h > l$. The key strategic considerations are that, by taking action 1, a player makes the other player pivotal in determining whether θ is received; but risks incurring the miscoordination cost $\mu = h - l$ if the other player takes action 0. Concretely, an interest group that supports the challenger (action 1) allows the other group's information to determine who wins, but risks punishment if that group sticks with the incumbent (action 0).

The presence of both coordination and information aggregation elements results in non-monotone best responses: a player's best response to a monotone strategy exhibits strategic substitutes whenever the other player is sufficiently willing to take action 1; and it exhibits strategic complements only if the other player is not too willing to do so. The force for strategic complements is that when one player is more likely to take action 1, the other player is less likely to incur miscoordination cost μ when he takes action 1. The force for strategic substitutes is that when a player is more willing to take action 1, then he does so following worse signals about θ. For example, if an officer is willing to attempt a coup even after a signal suggesting that a successful coup will lead to anarchy, others find mounting a coup less attractive. Most real world settings feature substantial uncertainty about payoffs even after private information is acquired. As a result, the value of information contained in the equilibrium actions of other players can be high, and hence the force for strategic substitutes can be strong. The global games literature focuses on the unique risk-dominant equilibrium that emerges when private signals are very accurate. Such a restriction not only does not fit many real-world situations where uncertainty is substantial, but it also precludes information aggregation, which can be a crucial strategic aspect in many settings.

We first characterize symmetric equilibria in cutoff strategies. We prove that if miscoordination costs μ are too high, no finite-cutoff equilibrium exists. Otherwise, there is at least one and at most two finite-cutoff equilibria. When there is a unique finite-cutoff equilibrium, it is stable and exhibits natural comparative statics—raising the miscoordination cost $\mu = h - l$ or predatory payoff w both reduce the likelihood that players take action 1. When there are two finite-cutoff equilirbia, the one in which players are most likely to take action 1 is stable and exhibits natural comparative statics. In contrast, the cutoff equilibrium in which players are less likely to take action 1 is unstable and has the opposite, perverse comparative statics, indicating that it is not a plausible candidate to describe the real world.[2]

We next establish that the likelihood of taking action 1 is more sensitive to increases in the punishment μ than to increases in the reward w if and only if the players are sufficiently unlikely to take action 1. This has important implications for a regime seeking to secure itself: to better discourage a coup, harsher punishment is more effective than better rewards if and only if the punishment μ *or* the reward w are already high.

We then prove that actions are always strategic complements in the (unstable) cutoff equilibrium where players take action 1 less. In contrast, in the equilibrium where players take action 1 more, actions are strategic substitutes if and only if the miscoordination cost μ is low. We show that whether equilibrium actions are strategic complements or substitutes has implications for how *asymmetric* changes in the environment affect player welfare. As long as $w > l$, slight increases in a player's predatory payoff w (e.g., a regime raises its reward for not supporting a coup) raise his welfare if actions are strategic substitutes, but can *harm* him if actions are strategic complements. Raising a player's predatory payoff has a positive direct welfare effect, but it also causes the player to take action 1 less, inducing the other player to adjust his strategy. When $w > l$, a player's *equilibrium* welfare is always *raised* when the other player takes action 1 slightly more. Thus, when increases in a player's predatory payoff cause him to take action 1 less and actions are strategic substitutes, the other player takes action 1 more, reinforcing the positive direct welfare effect. However, when actions are strategic complements, the other player takes action 1 less, and this welfare-damaging behavior can swamp the direct welfare effect. Similarly, the direct welfare effect of reducing a player's miscoordination punishment μ is also positive, but now, in contrast, the indirect welfare effects are positive if actions are strategic complements, and negative if actions are strategic substitutes.

A natural question that arises is: how do equilibrium cutoffs compare with the cutoff that a social planner would choose? We prove that as long as it is ever socially optimal for players to take action 1, then in equilibrium, players do not take action 1 as often as is socially optimal if and only if $w > l$. When deciding whether to take action 1, a player

[2]In addition, an equilibrium always exists in which both players take action 0 regardless of their signals. One may dismiss this latter equilibrium because information appears to matter in practice—interest groups sometimes support a challenger and officers sometimes mount coups.

internalizes the payoff l received when the other player takes action 0, but not the payoff w that the other player receives. Thus, when $l = w$, by maximizing his own expected payoffs, a player also maximizes the other player's payoffs. However, when $w > l$, a player who internalizes l does not take action 1 as often as the other player would like; and when $w < l$, the opposite obtains. As most economic and political settings feature $w > l$, in equilibrium, players do not take action 1 as often as is socially optimal.

The fact that when $w > l$, welfare would rise if players took action 1 slightly more does not imply that a player's welfare *always rises* when the other player takes action 1 more often: a player does *not* want to receive θ when he receives a slightly positive signal, but the other player receives a very negative one. When one player is too willing to take action 1, whether players receive θ depends little on that player's signal, preventing the players from properly using that signal to determine whether they should receive θ.

These results emphasize a disconnect between strategic complements and substitutes, and payoff complements and substitutes in equilibrium: when $w > l$, a player would always benefit if the other player took action 1 *slightly* more (actions are payoff complements), even though when μ is small, a player would respond by taking action 1 less (actions are strategic substitutes). Concretely, a military officer benefits if a fellow officer is more willing to mount a coup, even though this greater willingness to act may cause the officer, himself, to act less.

In the settings we analyze, cutoff strategies are natural—players take action 1 if and only if they receive sufficiently promising signals about its payoff. However, equilibrium strategies need not take a cutoff form. We use Karlin's theorem (1968) on variation diminishing properties of TP_3 functions to prove that equilibria can exist in which players take action 1 if and only if their signals are in a bounded interval, (k_L, k_R). That is, players take action 1 if and only if their signals about its common coordination payoff are high, but not *too* high. Paradoxically, endogenously generated fear of miscoordination can induce players not to take action 1 precisely when their signals suggest that the payoff from coordinated action is highest. We exploit the stability properties of different cutoff equilibria to prove that whenever a bounded interval equilibrium exists, (a) its lower bound k_L is between the cutoffs $\bar{k}$ and $\underline{k}$ of the low and high cutoff equilibria, and (b) k_L is less than $E[\theta]$. It follows directly that (a) players are less likely to take action 1 in a bounded interval equilibrium than in the low cutoff equilibrium, (b) bounded interval equilibria only exist when μ is sufficiently small, and (c) bounded interval equilibria are welfare dominated by cutoff equilibria.

We conclude by considering pre-play communication of private information. In some settings, pre-play communication is problematic: in revolutions and coups, communication risks conveying information to an undercover agent of the regime; and in other settings, players may not even know their strategic counterparts. Still, pre-play communication is plausible in some settings. We address this by allowing the players to simultaneously send unverifiable cheap talk messages prior to taking actions. We show that when $h > w$ so that there is no incentive to miscoordinate, there is a fully informative cheap talk equilibrium where players truthfully report their signals. However, in most real world settings, $h < w$, i.e., one player gains from the other's attempt to coordinate on receiving θ. When $h < w$, we show that not only is there no fully informative cheap talk equilibrium, but there is not even a partially informative cheap talk equilibrium in monotone strategies. This finding indicates that the impact of cheap talk is modest in most relevant settings.

We next discuss related literature. Section 2 presents the model and basic properties of cutoff equilibria. Section 3 compares equilibrium actions with the choices of a social planner. Section 4 studies the interactions between strategic complements/substitutes and welfare. Section 5 characterizes bounded interval equilibria. Section 6 investigates pre-play communication. Section 7 describes economic and political settings for which our analysis is germane. A conclusion follows. All proofs are in an appendix.

Related Literature. The closest paper is Shadmehr and Bernhardt (2011). Our Proposition 1 generalizes their characterization of cutoff equilibria. Beyond Proposition 1, *none* of our results has an analogue in that paper. Shadmehr and Bernhardt (2011) characterize cutoff equilibria under the assumption that $w = h$. This assumption means that when a player does not attempt to coordinate on θ, his payoff is unaffected by the other player's action choice. Relaxing this assumption is essential in two ways: (1) it allows us to extend the application to real world settings that mostly feature $w > l$ (see section 7); and (2) it allows for meaningful analyses of rewards and punishment, welfare, and pre-play communication, which are all *absent* in that paper.

We also contribute technically. We partially characterize *non-monotone* equilibria using Karlin's (1968) theorem on variation diminishing properties of totally positive functions of order *three* (TP_3). Total positivity of order two (TP_2) is equivalent to the monotone likelihood ratio property, and has been used to establish single-crossing properties under some integral transformations (Athey 2002; Friedman and Holden 2008). However, applications of higher orders are rare—Jewitt (1987) uses properties of TP_3 conditional distributions to show the preservation of quasi-concavity under integral transformations. We also relax Shadmehr and Bernhardt's (2011) additive signal structure. They assume that $s^i = \theta + \nu^i$ where θ and ν^i are independently-distributed normal random variables. Our analysis relies on affiliation and minimal structure on the tail properties of distributions. This generalization matters because Shadmehr and Bernhardt's (2011) analysis relies on special features of normal distributions that mask the driving forces.

More broadly, our paper relates to two literatures. There is a large literature exploring information aggregation in games where players internalize the information contained in equilibrium outcomes, for example, that a player is pivotal in determining electoral outcomes (Austen-Smith and Banks 1996; Duggan and Martinelli 2001; Feddersen and Pesendorfer 1996, 1998; McMurray 2013). However, this information aggregation literature focuses on settings where miscoordination is not costly.

There is also a vast global games literature, in which there are coordination concerns but no information aggregation issues (Angeletos et al. 2007; Carlsson and van Damme 1993; Frankel et al. 2003; Hellwig 2002; Morris and Shin 2003). This literature either assumes global strategic complementarity directly or analyzes games when the noise in signals is vanishingly small so the force for strategic substitutes due to learning about common value payoffs vanishes.[3] With global strategic complementarity, global games become a subset of super-modular games, which have largest and smallest equilibria, both in monotone strategies (Van Zandt and Vives 2007; see also Milgrom and Roberts 1990; Topkis 1998; Vives 1990). Uniqueness in global games settings is achieved by finding conditions (e.g, vanishing noise plus limit dominance) that guarantee the smallest and largest (monotone) equilibria coincide—and hence the unique equilibrium is in monotone strategies. Because this literature focuses on characterizing equilibrium when noise is vanishingly small, non-monotone equilibria that may exist when signals are less revealing have gone unstudied. To the best

[3] A few global games models feature strategic substitutes driven by congestion externalities (Clark and Polborn 2006; Goldstein and Pauzner 2005; Karp et al. 2007), rather than information aggregation. For example, Goldstein and Pauzner (2005) study bank runs and show how strategic substitutes emerge due to rationing when enough players "run on the bank".

of our knowledge, our analysis of non-monotone equilibria is the first in this class of games.

Our paper explores the middle ground between these two literatures where both information aggregation and coordination concerns are present, resulting in non-monotone best responses. To understand the strategic considerations that emerge, it is useful to consider a generalized version of the canonical investment game of Carlsson and van Damme (1993) and Morris and Shin (2003), whose payoffs are given in the left panel of Figure 2, where we add a parameter $\alpha \in [0, 1]$ that is 1 in those papers. When $\alpha = 1$, the game features

	no invest	*invest*
no invest	$0, 0$	$0, \alpha\theta - k$
invest	$\alpha\theta - k, 0$	θ, θ

	no invest	*invest*
no invest	$0, 0$	$0, \alpha\theta_j - k$
invest	$\alpha\theta_i - k, 0$	θ_i, θ_j

Figure 2: Generalized investment game. Left: *Common* values—as long as $\alpha < 1$, the informational forces for strategic substitutes are present. Right: *Private* values—there is no information aggregation, and thus no force for strategic substitutes.

global strategic complements because there is no way for a player to condition the receipt of θ on the other player's information—the force for strategic substitutes is completely absent. This game is a special case of ours when $\alpha = 0$. However, as long as $\alpha < 1$, the informational forces for strategic substitutes are present, as when the other player does not invest, this damps the consequences of θ for the player who invests.[4] As such, the other player's equilibrium action contains information about θ that can be exploited. Extending this logic, if the stochastic coordination payoffs are private values (Bueno de Mesquita 2010), as in the right panel of Figure 2, then once more the force for strategic substitutes disappears because player j's actions only contain information about θ_j, and not θ_i.

2 The Model

Two players A and B must choose between two actions 0 and 1. Payoffs are as in Figure 1 in the introduction; players only receive the stochastic payoff θ if they both take action 1,

[4]In fact, as far as $\alpha E[\theta] - k < 0$, this game is strategically equivalent to a special case of our game (where $h = w$) for all $\alpha \in [0, 1)$. That is, with normalization of payoffs, the net expected payoffs (see equation (1) below) for this game and our original game have similar properties. See section 7 for more details.

and payoffs h, l and w are common knowledge, with $h > l$.[5] Each player $i \in \{A, B\}$ receives a private signal s^i about θ. After receiving signals, players simultaneously take actions.

The signals and θ are jointly distributed according to a strictly positive, continuously differentiable density $f(\theta, s^A, s^B)$ on $\mathbb{R}^3$. Players are symmetrically situated in the sense that the signals are exchangeable, i.e., $f(\theta, s, s') = f(\theta, s', s)$, for all θ, s, s'. We assume that s^A, s^B and θ are strictly affiliated,[6] so that a higher signal s^i represents good news about s^j and θ. In addition, we impose modest structure on the tail properties of $f(\theta, s^A, s^B)$:

Assumption 1 *For every k,*

$$(a)\ \lim_{s^i \to \infty} E[\theta | s^j > k, s^i] = \infty, \quad \lim_{s^i \to -\infty} E[\theta | s^j > k, s^i] = -\infty.$$
$$(b)\ \lim_{s^i \to \infty} Pr(s^j > k | s^i) = 1, \quad \lim_{s^i \to -\infty} Pr(s^j > k | s^i) = 0.$$
$$(c)\ \lim_{k \to \infty} Pr(s^j > k | k)\ E[\theta | s^j > k, k] < \infty.$$

Assumption 1 holds with an additive noise signal structure, $s^i = \theta + \nu^i$, when θ and ν^is are (both) independent normal, logistic, or extreme value random variables—see below. However, if θ and ν^i are iid draws from a t distribution, the signals and θ are *not* affiliated[7], and $\lim_{k \to \infty} Pr(s^j > k | k)\ E[\theta | s^j > k, k] = \infty$.

Strategies, expected payoffs, and equilibrium. A pure strategy for player i is a function ρ_i mapping his signal s^i about θ into an action choice. That is, $\rho_i : \mathbb{R} \to \{0, 1\}$, where $\rho_i(s^i) = 1$ indicates that player i takes action 1, and $\rho_i(s^i) = 0$ indicates that i takes action 0. Our analysis first focuses on monotone strategies, where a player's strategy is (weakly) monotone in his signal. $\lim_{s^i \to -\infty} E[\theta | s^j > k, s^i] = -\infty$ ensures that always taking action 1 is never a best response, while the payoff structure of the game implies that never taking action 1 is always an equilibrium. All other monotone equilibria are in increasing cutoff

[5] Because optimal actions hinge on the net expected payoff from taking action 1 rather than action 0, strategic considerations and equilibrium behavior are identical when players receive signals about a "status quo" payoff that they receive *unless* they take action 1. That is, players must coordinate to avoid receiving θ. Collectively, these two scenarios encompass a larger set of phenomena: in some settings players may learn about status quo payoffs from private experience; and in other settings, players may understand the status quo payoffs, but be acquiring private information about the alternative.

[6] s^i, s^j and θ are strictly affiliated if, for all $z,\ z' \in \mathbb{R}^3$, with $z \neq z'$, $f(\min\{z, z'\}) f(\max\{z, z'\}) > f(z) f(z')$, where min and max are defined component-wise (Milgrom and Weber 1982; de Castro 2010).

[7] With additive noise, the signals and underlying parameter are affiliated if and only if the noise distribution is log-concave (de Castro 2010; Karlin 1968), and the t distribution is not log-concave (Bagnoli and Bergstrom 2006).

strategies, so that a player j's strategy can be summarized by a critical cutoff k^j: player j takes action 1 if and only if j receives a sufficiently promising signal s^j about θ, i.e.,

$$\rho_j(s^j) = 1 \text{ if } s^j > k^j \quad \text{and} \quad \rho_j(s^j) = 0 \text{ if } s^j \leq k^j.$$

Let $\Delta(s^i; k^j)$ be player i's expected net payoff from taking action 1 rather than 0, given his signal s^i and the other player's strategy k^j. Then,

$$\Delta(s^i; k^j) = Pr(s^j > k^j | s^i) \ (E[\theta | s^j > k^j, s^i] - w) - Pr(s^j \leq k^j | s^i) \ \mu, \tag{1}$$

where $\mu \equiv h - l > 0$ is the net miscoordination cost of taking action 1 when the other player takes action 0.[8] Player i takes action 1 if and only if $\Delta(s^i; k^j) > 0$. When player j takes action 1 more, player i is less likely to pay the miscoordination cost, i.e., $Pr(s^j \leq k^j | s^i) \ \mu$ falls as k^j is reduced, increasing i's incentive to take action 1. This is the force for strategic complements. However, the signal s^i conveys information about *both* (1) the likelihood that the other player takes action 1, and (2) the value of θ. Hence, when player j takes action 1 more, player i's expected payoff from successful coordination, $E[\theta | s^j > k^j, s^i]$, falls as k^j is reduced, reducing his incentive to take action 1. This is the force for strategic substitutes. Concretely, when another interest group is more willing to support an untried challenger, this reduces the likelihood of punishment by the incumbent for being the challenger's sole supporter (the force for strategic complements); but the other interest group is now supporting the challenger when he is less likely to be good (the force for strategic substitutes).

Characterizing the relative strength of strategic complements and substitutes—which determines the shape of the best response curve—is difficult. Shadmehr and Bernhardt (2011) show that one can tackle this problem by partially characterizing the non-linear differential equation describing the best response. To do this, they use the explicit functional form implications of the additive noise normal signal structure. We relax this structure, only requiring affiliation and minimal tail properties of signals. In the Appendix, we prove that the best response to a cutoff strategy is a unique cutoff strategy (Lemma 5). Let $k^i(k^j)$ be the player i's unique best response cutoff to player j's cutoff k^j, and define $\delta(k^i; k^j) = E[\theta | k^j, k^i(k^j)] - w + \mu$. To determine $sign\left(\frac{\partial k^i}{\partial k^j}\right)$, we first show that $sign\left(\frac{\partial k^i}{\partial k^j}\right) =$

[8] Player i's expected payoff from action 1 given signal s^i about θ is $Pr(s^j > k^j | s^i) \ E[\theta | s^j > k^j, s^i] + Pr(s^j \leq k^j | s^i) \ l$, and his expected payoff from action 0 is $Pr(s^j > k^j | s^i) \ w + Pr(s^j \leq k^j | s^i) \ h$.

$sign\,(\delta(k^i;k^j))$. Then, we sign δ by showing that it solves the following differential equation,

$$\frac{d\delta(k^i;k^j)}{dk^j} = \frac{\partial E[\theta|k^j,k^i(k^j)]}{\partial k^j} + \frac{\partial E[\theta|k^j,k^i(k^j)]}{\partial k^i}\;\frac{f(k^j|k^i)\;\delta(k^j,k^i)}{\frac{\partial\Delta(k^i;k^j)}{\partial k^i}}.$$

This differential equation is highly non-linear. However, what is crucial is that once δ is positive, its derivative remains positive: $\delta \geq 0$ implies $\frac{d\delta(k^i;k^j)}{dk^j} > 0$, and hence δ has a single-crossing property as k^j increases. This implies a unique k^* such that player i's best response features strategic substitutes if and only if $k^j < k^*$—see Lemma 6 in the Appendix.

A pair of finite cutoffs (k^i,k^j) is an equilibrium if and only if $\Delta(k^i;k^j) = \Delta(k^j;k^i) = 0$. We focus on symmetric equilibria where $k^j = k^i = k$. Thus, (k,k) is a symmetric cutoff equilibrium if and only if $\Delta_1(k) = 0$, where

$$\Delta_1(k) \equiv \Delta(k;k) = Pr(s^j > k|s^i = k)\;(E[\theta|s^j > k, s^i = k] - w + \mu) - \mu. \tag{2}$$

From Assumption 1, $\lim_{k\to-\infty}\Delta_1(k) < 0$, and

$$\lim_{k\to-\infty}\Delta_1(k) < \lim_{k\to\infty}\Delta_1(k) = \lim_{k\to\infty} Pr(s^j > k|s^i = k)\;E[\theta|s^j > k, s^i = k] - \mu.$$

If $\lim_{k\to\infty} Pr(s^j > k|s^i = k)\;E[\theta|s^j > k, s^i = k] - \mu > 0$, then at least one symmetric finite-cutoff equilibrium exists. When this condition does not hold, we establish conditions for the existence of symmetric finite-cutoff equilibria by partially characterizing the shape of $\Delta_1(k)$. Lemma 1 provides conditions that ensure $\Delta_1(k)$ is either single-peaked or monotone.

Lemma 1 *Suppose $Pr(s^j > k|s^i = k)$ is decreasing in k, and both $Pr(s^j > k|s^i = k)$ and $E[\theta|s^j > k, s^i = k]$ are log-concave functions of k. Then, $\Delta_1(k)$ is either single-peaked or strictly increasing in k.*

Examples of common signal structures for which $\Delta_1(k)$ is either single-peaked or strictly increasing include the following:

Additive Normal Distribution Signal Structure. Suppose that $s^i = \theta + \nu^i$, where θ and ν^i are independent normal random variables. In the Appendix, we establish

Result 1. With the classical additive normal noise signal structure, $\lim_{k\to-\infty}\Delta_1(k) = -\infty$, $\lim_{k\to\infty}\Delta_1(k) = -\mu$, and $\Delta_1(k)$ is single-peaked.

Truth or Noise Signal Structure. Suppose $\theta \sim f$, where f is a continuously differentiable density function with full support on $\mathbb{R}$. With an exogenous probability $p \in (0,1)$, player $i \in \{A, B\}$ observes θ, i.e., $s^i = \theta$. With residual probability $1 - p$, s^i is an independent draw from f. Papers using a variant of this signal structure in which all players receive the *same* signal draw include Lewis and Sappington (1994), Ottaviani (2000), Johnson and Myatt (2006), and Ganuza and Penalva (2010). In the Appendix, we show

Result 2. With a truth or noise signal structure, if f is log-concave, then $\Delta_1(k)$ is single-peaked, with $\lim_{k\to-\infty} \Delta_1(k) = -\infty$ and $\lim_{k\to\infty} \Delta_1(k) = -\mu$.

Additive Logistic Distribution Signal Structure. Suppose that $s^i = \theta + \nu^i$, where θ, ν^i and ν^j are iid according to the logistic distribution $\frac{e^{-x}}{(1+e^{-x})^2}$, where we have normalized the mean to zero without loss of generality and set the scale parameter to 1 to ease exposition. Because the logistic distribution is logconcave, θ and s^is are affiliated (de Castro 2010). In the Appendix, we establish

Result 3. With the logistic signal structure, $\Delta_1(k)$ is either single-peaked (when $w < 1+\mu$) or strictly increasing (when $w > 1 + \mu$).[9]

Lemma 1 and these examples lead us to make Assumption 2, which provides sufficient structure to characterize the number of symmetric finite cutoff strategy equilibria. When this property does not hold, there can be more than two equilibria in finite cutoff strategies due to strategic complementarities. Even then, our characterizations describe the equilibrium in which players take action 1 the most.

Assumption 2 $\Delta_1(x)$ *is either single-peaked or strictly increasing.*[10]

We next observe that if player j always takes action 1, then player i takes action 1 whenever his signal exceeds the $k^i(k^j = -\infty) \in \mathbb{R}$ that solves $E[\theta|k^i(k^j = -\infty)] = w$. Thus, as k^j traverses $\mathbb{R}$ from $-\infty$ to $+\infty$, the best response function, $k^i(k^j)$ crosses the 45° line at most twice, once from above and once from below. When a best response function crosses the 45° line from above, the equilibrium is (locally) stable—iterating on best responses converges to the equilibrium; otherwise it is (locally) unstable. Higher punishment costs μ make a player more hesitant to take action 1, and hence $k^i(k^j)$ shifts upward, i.e.,

[9] An analogous result holds with an extreme-value signal structure where θ and ν^is are iid according to the extreme value distribution $f(x) = e^{-x}\ e^{-e^{-x}}$.

[10] Recall also that a player's best response function has a unique minimum—it switches from strategic substitutes to complements once. Therefore, single-peakedness of $\Delta_1(k)$ follows whenever the strategic complements portion of best response curves is convex.

$\frac{dk^i(k^j;\mu)}{d\mu} > 0$. But whether $k^i(k^j) > k^j$ for all k^j depends on the slope of $k^i(k^j)$ in the right tail. If for sufficiently large μ, this slope remains less than 1 (e.g., with t distribution), then $k^i(k^j)$ always crosses the 45° line and an equilibium exists. Otherwise, once μ exceeds a threshold, $k^i(k^j) > k^j$ for all k^j, and no finite-cutoff equilibrium exists.

Proposition 1 ***Generalization of Shadmehr and Bernhardt (2011).*** *There exists a threshold $\mu^* > 0$ on miscoordination costs such that a finite-cutoff equilibrium exists if $\mu < \mu^*$, but not if $\mu > \mu^*$.*[11] *If $\mu < \mu^*$, then (a) a stable finite-cutoff equilibrium exists (with cutoff $\bar{k}$); (b) at most one other finite-cutoff equilibrium exists (with cutoff $\underline{k}$), and (c) when the $\underline{k}$ equilibrium exists, it is unstable and $\bar{k} < \underline{k}$. At $\mu = \mu^*$, if a finite-cutoff equilibrium exists, then it is unique and unstable. Further, for $\mu < \mu^*$, $\frac{\partial \bar{k}}{\partial \mu}, \frac{\partial \bar{k}}{\partial w} > 0$, but $\frac{\partial \underline{k}}{\partial \mu}, \frac{\partial \underline{k}}{\partial w} < 0$.*

In particular, when $\Delta_1(k)$ is single-peaked and $\lim_{k\to\infty} Pr(s^j > k|k)\ E[\theta|s^j > k, k] = 0$ (e.g., with an additive noise normal signal structure), then two finite cutoff equilibria exist whenever μ is small enough (fixing w), or conversely, whenever w is small enough (fixing μ).

Proposition 1 indicates that when multiple finite-cutoff equilibria exist, the $\bar{k}$ equilibrium in which players are most likely to take action 1 is a plausible candidate for describing real world outcomes, but the $\underline{k}$ equilibrium is not. That is, the $\bar{k}$ equilibrium is stable, and features "natural" comparative statics: raising the predatory payoff w or raising the miscoordination cost μ, which directly raise the attraction of action 0, in fact cause players to take action 0 more often in equilibrium. Concretely, if an incumbent raises the punishment for supporting a challenger or the reward for continuing to support the incumbent, then interest groups are less likely to fund the challenger. In contrast, the $\underline{k}$ equilibrium is unstable, and has perverse comparative static properties: raising w or μ, *raises* the equilibrium likelihood that players take action 1. In essence, at the $\underline{k}$ equilibrium, players take action 1 too little: the $\underline{k}$ equilibrium is supported by an "excessive" likelihood of miscoordination. Thus, as we raise w or μ, which directly reduce the attraction of action 1, the $\underline{k}$ equilibrium must adjust to maintain the attraction of action 1: player i must believe that player j is *more* likely to take action 1, thereby reducing the likelihood of miscoordination. In light of these observations, the bulk of our analysis focuses on the $\bar{k}$ equilibrium where players coordinate most.

[11] Note that μ^* will vary with the various primitives describing payoffs and signals of the model.

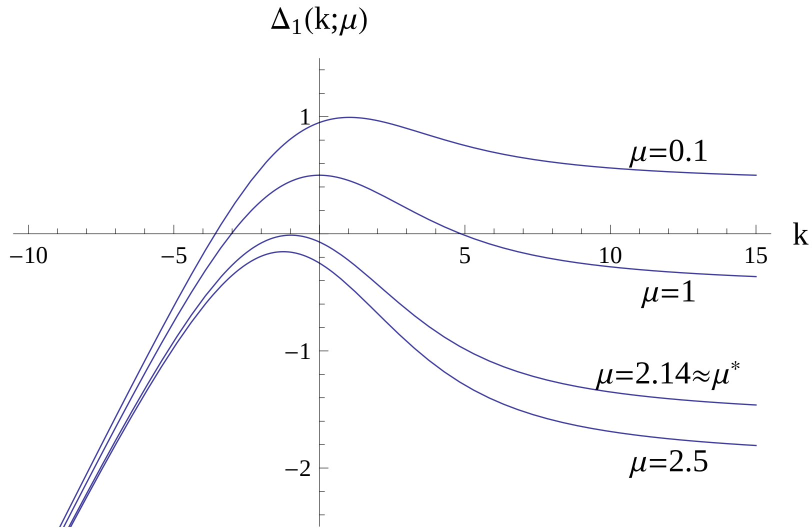

Figure 3: Symmetric net expected payoff $\Delta_1(k;\mu)$ as a function of k for different values of miscoordination costs μ. $s^i = \theta + \nu^i$, $i \in \{A, B\}$, with θ and ν^is are iid logistic $\frac{e^{-x}}{(1+e^{-x})^2}$. Parameters: $w = -1.5$.

2.1 Reward or Punishment?

Increases in the predatory payoff w or miscoordination punishment μ both reduce the likelihood that players take action 1. But which does it more? Consider an incumbent candidate who must decide whether to reward the interest group that supports him or to punish the interest group that supports his challenger. Which is more effective at keeping him in office?

Focusing on the largest finite-cutoff equilibrium $\bar{k}$, consider a player on the cusp of taking action 1 or action 0. With probability $Pr(s^j > \bar{k}|s^i = \bar{k})$, player i receives the reward w when he takes action 0, and with probability $1 - Pr(s^j > \bar{k}|s^i = \bar{k})$, he is punished when he takes action 1. Hence, his response is more affected by the more likely outcome—increasing the punishment μ has a greater effect on his willingness to take action 1 than increasing the reward w if and only if $Pr(s^j > \bar{k}|s^i = \bar{k}) < \frac{1}{2}$. Therefore, when players are unlikely to take action 1, harsher punishments are more effective at reducing the likelihood of coordination on action 1; but when players are likely to take action 1, higher rewards become more effective. Moreover, Proposition 1 shows that players are less likely to take action 1 when these punishments or rewards are higher. In standard signal settings, including the normal, logistic, extreme value and truth or noise settings that we discussed earlier, $Pr(s^j > k|s^i = k)$ is strictly decreasing in k, and $\lim_{k\to-\infty} Pr(s^j > k|s^i = k) > \frac{1}{2}$. With this structure, we have:

Proposition 2 *Suppose $Pr(s^j > k|s^i = k)$ is strictly decreasing in k, and $\lim_{k\to-\infty} Pr(s^j > k|s^i = k) > \frac{1}{2}$. To maintain the status quo, harsher punishment is more effective than better rewards if and only if punishments or rewards are sufficiently high.*

3 Welfare: Commitment

How would the likelihood of taking action 1 and player welfare change if they could commit ex ante to a common cutoff? Lemma 2 establishes the key welfare result that in the neighborhood of equilibrium cutoffs, player i's welfare would be raised were player j to take action 1 slightly more often (i.e., reduce k^j) if and only if $w > l$.

Lemma 2 *In equilibrium, actions are payoff complements if and only if $w > l$.*

The intuition for this result hinges on a comparison of how player i would evaluate player j's action choice were he in j's shoes (i.e., seeing signal s^j, but not s^i). Player i internalizes the expected coordination payoffs (θ or h) when they both take the same action in exactly the same way as player j. However, player j receives l when he takes action 1 and player i takes action 0, while player i's payoff in this case is w. When $w = l$, the players' miscoordination payoffs are the same, so i would make the same choices as j given j's information. Since player j is best responding in equilibrium, player i's payoff must also be maximized by player j's best response. However, players weigh miscoordination payoffs differently when $w \neq l$, so that j's best response is no longer what player i would have him select. In particular, when $w > l$, player j only internalizes the lower payoff l from taking action 1: from player i's perspective, this causes player j to take action 1 too infrequently. Conversely, when $w < l$, from player i's perspective, player j takes action 1 too frequently.

In most economic settings, the predatory payoff w exceeds l. For example, in a coup game, an officer who initiates an unsuccessful coup by taking action 1 expects to be punished by the regime; and in an investment game, with miscoordination, the firm taking action 1 incurs an additional investment cost. However, there are exceptions. For example, in a civil war, consider villages that must decide whether or not to join the rebels (take action 1). If only one group joins, it may be the unarmed non-combatants (who took action 0) who suffer the most (Kalyvas 2006).

We next characterize the symmetric cutoff strategy k^s to which players would like to jointly commit themselves, ex ante. We establish that Lemma 2 implies that if and only if $w > l$, players would be better off if they could jointly commit to taking action 1 for lower signals than they do in equilibrium.

Proposition 3 *Suppose* $\lim_{k \to -\infty} Pr(s^j > k | s^i = k) > 0$*, and that coordinating on action 0 is, on average, worse than coordinating on action 1, i.e.,* $h < E[\theta]$*. Then, it is socially optimal for players to take action 1 more often than they do in equilibrium if* $w > l$*, and it is optimal for them to take action 1 less if* $w < l$*.*

In the statement of the proposition, the condition $\lim_{k \to -\infty} Pr(s^j > k | s^i = k) > 0$ just guarantees that it is not socially optimal to revolt regardless of the signal received. The condition $h < E[\theta]$ is a simple sufficient condition for it to be socially optimal to take action 1 with positive probability. The welfare result only hinges on these limited properties.

We emphasize that in contrast to most global games, where actions exhibit global payoff complementarities due to the reduction in miscoordination, here a player's welfare is *always harmed* if the other player is too willing to take action 1: as k^j is reduced further below k^s, eventually player i is hurt because whether players receive θ depends less and less on the information contained in j's signal via j's action choice. Indeed, whenever μ is small, players *must* experience "interim regret" from taking action 1 conditional on the information contained in both players receiving signals k^i and k^j: given $s^i = k^i$, $s^j = k^j$, player i regrets taking action 1 when j does if and only if $E[\theta | k^i, k^j] < w$. When $\mu = 0$, $k^i(k^j)$ solves

$$E[\theta | s^j > k^j, k^i(k^j)] - w \quad = \quad 0,$$

and since $E[\theta | s^j > k^j, k^i(k^j)] > E[\theta | s^j = k^j, k^i(k^j)]$, we have

$$E[\theta | k^i(k^j), k^j] - w < 0.$$

By continuity, players experience interim regret as long as μ is small enough.

4 Welfare Implications of Strategic Structure

We nrcy characterize the nature of strategic interaction between players that can exist in the different equilibria. The $\underline{k}$ equilibrium is unstable—best responses cross the 45° line from below—and hence it always features strategic complements. In sharp contrast, in the $\bar{k}$ equilibrium in which players are most likely to take action 1, actions can be either strategic complements or strategic substitutes. Lemma 3 reveals that at the $\bar{k}$ equilibrium, actions are strategic substitutes if and only if the miscoordination cost μ is sufficiently low.

Lemma 3 *The $\bar{k}$ equilibrium features strategic substitutes if and only if μ is sufficiently low. Whenever the $\underline{k}$ equilibrium exists, it features strategic complements.*

The finding combines Proposition 1 and Lemma 6 in the Appendix. Lemma 6 shows that $k^i(k^j)$ exhibits strategic substitutes if and only if k^j is sufficiently low, and Proposition 1 shows that reducing μ lowers $\bar{k}$. Therefore, reductions in μ eventually shift $\bar{k}$ down to the range where best responses exhibit strategic substitutes. The logic reflects that reducing the miscoordination cost μ has two distinct effects: (1) In the $\bar{k}$ equilibrium, reducing μ causes players to take action 1 more. Thus, (a) players are less likely to miscoordinate, which reduces the force for strategic complements, and (b) $E[\theta|s^i, s^j > \bar{k}]$ falls, which raises the force for strategic substitutes. (2) Reducing μ directly lowers incentive to avoid miscoordination, which reduces the force for strategic complements. In particular, when $\mu = 0$, the force for strategic complements vanishes, ensuring that strategic substitutes obtain.

Comparing lemmas 2 and 3 reveals a very limited link between whether equilibrium actions are strategic substitutes or strategic complements (which hinges on the size of $\mu = h - l$), and whether actions are payoff substitutes or payoff complements (which hinges on the sign of $w - l$). In particular, as long as $w > l$, a player's welfare would rise if the other player took action 1 marginally more than he does in equilibrium (Lemma 2), even though when μ is small, his best response to this would be to take action 1 less (Lemma 3). This limited link underscores that strategic behavior reflects *marginal* considerations, while payoff substitutes/complements reflect *average* considerations. That is, the impact of a change in k^j on player i's best response, reflects a consideration of expected payoffs conditional on the marginal signals $s^j = k^j, s^i = k^i$, while the impact of a change in k^j on player i's expected utility reflects expectations over all signals that i could receive conditional on k^j.

Welfare Effects of Asymmetric Changes to Environment. Consider the game where two interest groups must decide whether to support an incumbent or an untried challenger. If the incumbent is elected, a loyal interest group that supported the incumbent is rewarded, and a disloyal group that supported the challenger is punished. Is an interest group better off if the incumbent promises it a slightly higher reward than the other group (see Figure 4)? We now derive the welfare effects of slight, asymmetric changes in the environment. We show that whether a player's welfare increases or decreases hinges in part on whether actions are strategic complements or substitutes in equilibrium.

	incumbent	*challenger*
incumbent	h, h	$w^i = w + \epsilon, l$
challenger	l, w	θ, θ

Figure 4: Perturbed Payoffs

The direct effects of raising w^i from w to $w + \epsilon$ are to raise player i's expected payoff and to cause him to take action 1 less. But what about the indirect strategic effects? When strategic substitutes obtain, player j takes action 1 more often when player i takes action 1 less. The equilibrium is (generically) continuous in ϵ, and Lemma 2 reveals that player i benefits when player j reduces his cutoff and takes action 1 more (as long as $w > l$). Thus, with strategic substitutes, raising w^i marginally benefits player i both directly and indirectly. But what happens when strategic complements obtain? To see that the negative indirect strategic effect can swamp the direct payoff gain, consider the special case where $\mu = \mu^*$ and there is a unique, unstable equlibrium. Then raising w^i causes player i to take action 1 less; player j reciprocates by doing the same; and, iterating on best responses, this spirals down to the resulting equilibrium in which players always take action 0, harming both players.

More generally, let $E[U^i|c^i, c^j, w^i]$ be player i's ex-ante expected utility when he adopts cutoff c^i and player j adopts cutoff c^j. From the proof of Lemma 2, differentiating player i's expected utility with respect to w^i at $w^i = w^j = w$, at the symmetric $\bar{k}$ equilibrium, yields

$$\begin{aligned} \frac{dE[U^i|c^{i^*}(w^i), c^{j^*}(c^{i*}(w^i)), w^i]}{dw^i} &= \frac{\partial E}{\partial w^i} + \left(\frac{\partial E}{\partial c^{i*}} + \frac{\partial E}{\partial c^{j*}} \frac{\partial c^{j*}}{\partial c^{i*}} \right) \frac{\partial c^{i*}}{\partial w^i} \\ &= Pr(s^i \leq \bar{k}, s^j > \bar{k}) + g(\bar{k})\ (l - w)\ \frac{\partial c^{j*}}{\partial c^{i*}} \frac{\partial c^{i*}}{\partial w^i}, \end{aligned}$$

where c^{i*} is i's equilibrium cutoff and $g(\cdot)$ is the pdf of s^j. The first term is the positive direct welfare effect. The other terms comprise the indirect strategic effect. The second term is zero because c^{i*} is a best response. The third term follows from equation (19) in the proof of Lemma 2. We have $\frac{\partial c^{i*}}{\partial w^i} > 0$, so that when $w > l$, all terms are positive with strategic substitutes, as $\frac{\partial c^{j*}}{\partial c^{i*}} < 0$. However, with strategic complements, $0 < \frac{\partial c^{j*}}{\partial c^{i*}}$: the strategic term is negative. Next observe that equal reductions in h and l to $h' = h - d$ and $l' = l - d$ keep their difference constant, i.e., $h' - l' = \mu$, leaving best responses and hence $\bar{k}$ unchanged. However, increases in d reduce $l' - w$. Hence, the relative magnitudes of the direct and strategic effects can be adjusted so that either effect can dominate. That is, depending on d, increasing w^i can raise or lower player i's welfare:

Proposition 4 *Let $h' = h - d$ and $l' = l - d$. Suppose μ is large enough that strategic complements obtain. Then there exists a d^w such that if and only if $d > d^w$, a marginal increase in player i's predatory payoff from w to $w^i = w + \epsilon$ reduces his equilibrium welfare.*

In contrast, the strategic effects of a slight reduction in the miscoordination cost/punishment that a player incurs are reversed, and hence there are opposing implications for welfare. Once more, the direct welfare effect of reducing μ^i from μ to $\mu - \epsilon$ is positive, and player i takes action 1 more. Thus, when $w > l$, with strategic complements the indirect strategic effect is also positive, but with strategic substitutes, it becomes negative. As a result, we have:

Proposition 5 *Let $h' = h - d$ and $l' = l - d$. Suppose μ is small enough that strategic substitutes obtain. Then there exists a d^μ such that if and only if $d > d^\mu$, a marginal reduction in player i's miscoordination cost from μ to $\mu^i = \mu - \epsilon$ reduces his equilibrium welfare.*

5 Bounded Interval Strategies

In this section, we characterize equilibria for a broader class of equilibrium strategies in which the set of signals for which players take action 1 needs only be connected, i.e., a player takes action 1 if and only if $s^i \in (k_L^i, k_R^i)$; a cutoff strategy has $k_R^i = \infty$. We say that player j adopts the bounded interval strategy (k_L^j, k_R^j) when

$$\rho_j(s^j) = 1 \quad \text{if and only if} \quad k_L^j < s^j < k_R^j, \quad \text{with} \quad k_L^j, k_R^j \in \mathbb{R}. \tag{3}$$

Let $\Gamma[s^i; k_L^j, k_R^j]$ be player i's expected net payoff from taking action 1 when his signal is s^i and player j adopts interval strategy (k_L^j, k_R^j). Mirroring the derivation of equation (1),

$$\begin{aligned}\Gamma[s^i; k_L^j, k_R^j] &= Pr(\rho_j = 1|s^i)\ (E[\theta|\rho_j = 1, s^i] - w + \mu) - \mu \\ &= Pr(k_L^j < s^j < k_R^j|s^i)\ (E[\theta|k_L^j < s^j < k_R^j, s^i] - w + \mu) - \mu.\end{aligned} \tag{4}$$

It is helpful to link $\Gamma[s^i; k_L^j, k_R^j]$ to player i's net payoff if she knew θ. Let $\pi(\theta; k_L^j, k_R^j)$ be i's incremental return from taking action 1 given θ and $\rho^j = (k_L^j, k_R^j)$:

$$\begin{aligned}\pi(\theta; k_L^j, k_R^j) &= (\theta - w)\ Pr(k_L^j < s^j < k_R^j|\theta) + (l - h)\ (1 - Pr(k_L^j < s^j < k_R^j|\theta)) \\ &= (\theta - (w - \mu))\ Pr(k_L^j < s^j < k_R^j|\theta) - \mu.\end{aligned} \tag{5}$$

Then we can write player i's net expected payoff from taking action 1 given signal s^i as

$$\Gamma(s^i; k_L^j, k_R^j) = \int_{\theta=-\infty}^{\infty} \pi(\theta; k_L^j, k_R^j)\ f(\theta|s^i)\ d\theta, \tag{6}$$

where $f(\theta|s^i)$ is the pdf of θ given s^i.

To analyze the properties of $\Gamma(s^i; k_L^j, k_R^j)$, we make use of the Variation Diminishing Property of totally positive functions (Karlin 1968). A function K is Totally Positive of order n, TP_n whenever

$$\begin{vmatrix} K(x,y) & \frac{\partial}{\partial y}K(x,y) & \cdots & \frac{\partial^{m-1}}{\partial y^{m-1}}K(x,y) \\ \frac{\partial}{\partial x}K(x,y) & \frac{\partial^2}{\partial x \partial y}K(x,y) & \cdots & \frac{\partial^m}{\partial x \partial y^{m-1}}K(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{m-1}}{\partial x^{m-1}}K(x,y) & \frac{\partial^m}{\partial x^{m-1}\partial y}K(x,y) & \cdots & \frac{\partial^{2(m-1)}}{\partial y^{m-1}\partial x^{m-1}}K(x,y) \end{vmatrix} \geq 0, \text{ for } m = 1, \cdots, n,$$

where X and Y are open intervals on $\mathbb{R}$, $K : X \times Y \to \mathbb{R}$, and all indicated derivatives exist. From the Variation Diminishing Property of totally positive functions, if $f(\theta|s^i)$ is totally positive of order n and $\pi(\theta)$ has $r \leq n-1$ sign changes, then $\Gamma(s^i; k_L^j, k_R^j)$ has *at most* r sign changes. Moreover, if $\Gamma(s^i; k_L^j, k_R^j)$ has exactly r sign changes, then π and Γ exhibit the same pattern of sign changes. We now impose the following properties on our information structure:

Assumption 3 *$f(s^i|\theta)$ is totally positive of order three (TP_3), and $Pr(k_L < s^i < k_R|\theta)$ is strictly log-concave in θ. Moreover, $\lim_{s^i \to \infty} E[\theta|s^j, s^i] f(s^j|s^i) = 0$ for any s^j.*

$\lim_{s^i\to\infty} E[\theta|s^j,s^i]f(s^j|s^i) = 0$ implies $\lim_{s^i\to+\infty} Pr(k_L < s^j < k_R|s^i)\ E[\theta|k_L < s^j < k_R, s^i] = \lim_{s^i\to+\infty} \int_{k_L}^{k_R} E[\theta|s^j,s^i]f(s^j|s^i) = 0$ for any k_L and k_R with $k_L < k_R$.

Thus, $f(s^i|\theta)$ is TP_3 whenever s^i and θ are affiliated (TP_2), and the associated determinant is non-negative for $m = n = 3$. For example, $f(s^i|\theta)$ satisfies Assumption 3 whenever $s^i = \theta + \nu^i$, where θ and ν^is are iid normal, logistic or extreme value random variables. For example, if $\theta \sim N(0,\sigma^2)$ and $\nu^i \sim N(0,\sigma_\nu^2)$, then $\frac{dPr(k_L<s^i<k_R|\theta)}{d\theta} = -\frac{\phi(\frac{k_R-\theta}{\sigma_\nu})-\phi(\frac{k_L-\theta}{\sigma_\nu})}{\Phi(\frac{k_R-\theta}{\sigma_\nu})-\Phi(\frac{k_L-\theta}{\sigma_\nu})}$, which is decreasing in θ, implying the log-concavity of $Pr(k_L < s^i < k_R|\theta)$ in θ. So, too, Assumption 3 holds when θ and $\nu^i \sim \frac{e^{-x}}{(1+e^{-x})^2}$: $f(s^i|\theta)$ is TP_3—the above determinant for $m = 3$ is $24\frac{e^{6(s+\theta)}}{(e^s+e^\theta)^{12}} > 0$, and $\frac{dLn[Pr(k_L<s^i<k_R|\theta)]}{d\theta} = \frac{Exp(k_R+k_L)-Exp(2\theta)}{Exp(k_R+k_L)+Exp((k_R+k_L)\theta)+Exp(2\theta)}$, which is decreasing in θ, implying that $Pr(k_L < s^i < k_R|\theta)$ is log-concave in θ.

From Assumption 1, $\lim_{s^i\to-\infty}\Gamma[s^i;k_L^j,k_R^j] < 0$, and Assumption 3 implies $\lim_{s^i\to\infty}\Gamma[s^i;k_L^j,k_R^j] < 0$. The log-concavity of $Pr(k_L < s^i < k_R|\theta)$ in θ means that, when θ unboundedly increases, $Pr(k_L < s^i < k_R|\theta)$ approaches zero at a rate faster than exponential functions (An 1998), and hence $\lim_{\theta\to\infty} Pr(k_L < s^i < k_R|\theta)(\theta-(w-\mu)) = 0$.

Lemma 5 in the Appendix shows that the best response to a cutoff strategy is a cutoff strategy, ensuring that our game satisfies Athey's (2001) "single crossing property for games of incomplete information." Thus, a nondecreasing pure strategy equilibrium exists. There is no general existence result for non-monotone equilibria. Lemma 4 characterizes the best response to a bounded interval strategy:

Lemma 4 *Suppose player j adopts a bounded interval strategy. Then, either player i's best response is a bounded interval strategy, or his best response is to always take action 0. Further, player i's best response is a bounded interval strategy if w is sufficiently low.*

We use Lemma 4 to solve for symmetric bounded interval equilibria when they exist. To solve for the bounds characterizing a bounded interval equilibrium, one must solve the system of non-linear equations that describe a player's indifference at the bounds between taking action 0 and 1: $\Gamma[k_L;k_L,k_R] = 0$ and $\Gamma[k_R;k_L,k_R] = 0$. Corollary 1 shows that any solution to these equations is, indeed, a bounded interval equilibrium:

Corollary 1 *(k_L,k_R) with $k_L,k_R \in \mathbb{R}$ and $k_L < k_R$ is a symmetric bounded interval equilibrium strategy if and only if $\Gamma[k_L;k_L,k_R] = 0$ and $\Gamma[k_R;k_L,k_R] = 0$.*

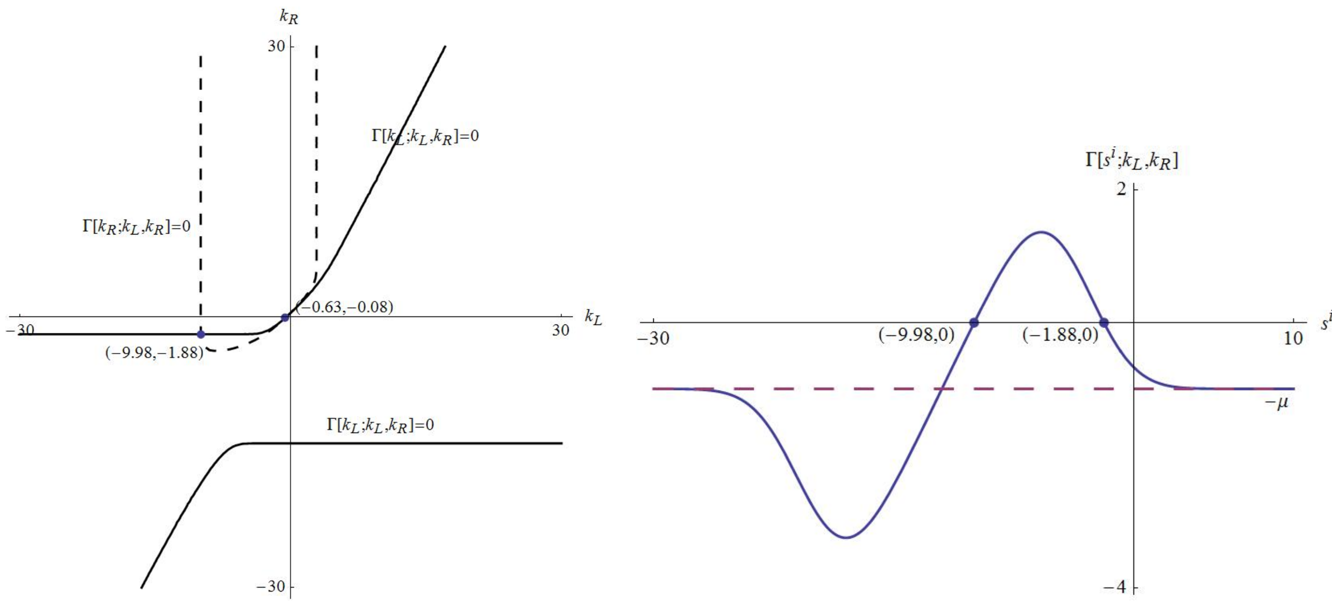

Figure 5: An interval equilibrium. The left panel depicts contour curves $\Gamma[k_L; k_L, k_R] = 0$, $\Gamma[k_R; k_L, k_R] = 0$, and their intersections. The right panel depicts $\Gamma[s^i; k_L = -9.98, k_R = -1.88]$. The figures together show that the interval equilibrium $(-9.98, -1.88)$ is a fixed point of the best response correspondence. The numbers are two-decimal approximations. Parameters: $w = -5$, $\mu = 1$, and $s^i = \theta + \nu^i$, with $\theta, \nu^i, \nu^j \sim iidN(0, 1)$.

Figure 5 illustrates an equilibrium in bounded interval strategies. We now relate the properties of bounded interval equilibria to those of cutoff equilibria. We use these results to derive upper bounds on the magnitude of w for which bounded interval equilibria exist.

Proposition 6 *If a symmetric bounded interval equilibrium exists, then finite-cutoff, symmetric, monotone equilibria exist. If (k_L, k_R) is a symmetric bounded interval equilibrium strategy, then $\bar{k} < k_L$. Moreover, if the $\underline{k}$ equilibrium exists, then $k_L < \underline{k}$.*[12]

This result immediately implies that players are strictly more likely to take action 1 in the $\bar{k}$ cutoff equilibrium than in any bounded interval equilibrium. We establish that $k_L > \bar{k}$ via contradiction, exploiting the facts that (a) $E[\theta|s^i, k_L < s^j < k_R]$ increases in k_R, so the lower bound $BR_L(k_L, k_R)$ characterizing the best response to (k_L, k_R) decreases in k_R, and (b) the stability of the $\bar{k}$ equilibrium then ensures that $BR_L(k_L, \infty) > k_L$ for $k_L < \bar{k}$. Analogously, when $\underline{k}$ exists, we establish that $k_L < \underline{k}$ exploiting the fact that the

[12]Recall that if the $\bar{k}$ equilibrium exists, then the $\underline{k}$ equilibrium exists when $\Delta_1(k)$ is single-peaked, and $lim_{k\to\infty}\Delta_1(k) < 0$, as in the additive normal or "truth or noise" signal structures.

$\underline{k}$ equilibrium is not stable. The intuition is that the players' failures to take action 1 when their signals about the common coordination payoffs are highest reflects their endogenously generated fear of miscoordination when they have high signals. But depriving players of the benefits of successfully coordinating on action 1 when one player's signal is high reduces the attraction to the other player of taking action 1. Hence, they take action 1 less.

The necessary and sufficient condition for a symmetric bounded interval equilibrium to exist is $\Gamma[s^i; k_L, k_R] > 0$ if and only if $s^i \in (k_L, k_R)$. It is difficult to determine sharp conditions on primitives that ensure the existence of a bounded interval equilibrium. However, the opposite type of existence characterization obtains: we can establish necessary and sufficient conditions for the symmetric bounded interval equilibrium associated with (k_L, k_R) to exist for *some* set of primitive parameters.

Proposition 7 *There exists a set of primitive parameters $\{\mu, w\}$ for which (k_L, k_R) is a symmetric bounded interval equilibrium strategy if and only if there exists $k_L < k_R$ such that $Pr(s^j \in (k_L, k_R)|s^i = k_L) > Pr(s^j \in (k_L, k_R)|s^i = k_R)$. With additive normal noise structure, this conditions becomes $k_L < E[\theta] = 0$ and $|k_L| > |k_R|$.*

The proof uses Corollary 1 to show that the necessary and sufficient condition for (k_L, k_R) to be a symmetric bounded interval equilibrium strategy is that $\Gamma[k_L; k_L, k_R] = \Gamma[k_R; k_L, k_R] = 0$. Since $E[\theta|s^i, k_L < s^j < k_R]$ increases in s^i,

$$E[\theta|s^i = k_L, k_L < s^j < k_R] < E[\theta|s^i = k_R, k_L < s^j < k_R].$$

Therefore, from equation (4), a player with signal k_L must believe he is more likely to receive θ than does a player with signal k_R, i.e., $Pr(k_L < s^j < k_R|k_L) > Pr(k_L < s^j < k_R|k_R)$. This requires that the peak of the conditional density of s^j given k_L, $f(s^j|k_L)$, be located closer to the middle of the (k_L, k_R) interval than the peak of $f(s^j|k_R)$, which happens if and only if $|k_L| > |k_R|$. Given $|k_L| > |k_R|$, we find $\mu > 0$ and w that simultaneously solve the two equations $\Gamma[k_L; k_L, k_R] = \Gamma[k_R; k_L, k_R] = 0$, where we note that $w << 0$ is required when $k_L << 0$. It is straightforward to compute numerically the fixed points of the associated best response correspondence, $\Gamma[k_L; k_L, k_R] = \Gamma[k_R; k_L, k_R] = 0$, for parameterizations in which such equilibria exist. Figure 5 depicts one such equilibrium.

The twin requirements from Propositions 6 and 7 that $k_L < E[\theta] = 0$ and $\bar{k} < k_L < \underline{k}$ imply that a bounded interval equilibrium only exists if $\bar{k} < 0$. This, in turn, imposes restrictions on primitives. From Propositions 1 and 6, when $\mu > \mu^*$, no bounded interval equilibria exist. Recall from Proposition 1 that $\bar{k}$ strictly increases in μ. Thus,

Corollary 2 *No bounded interval equilibrium exists for sufficiently large μ.*

6 Pre-play Communication

We have assumed that players cannot communicate their private information to each other prior to taking actions. This premise is natural in many settings. For example, potential coup leaders may not communicate due to the unmodeled consequences of suggesting coup to someone who may inform an autocratic leader. Or, a player may not even know his strategic counterpart—for example, when contemplating regentrification of the inner city, one may not know who else is considering such a move. Still, in some settings, pre-play communication may be feasible. We address whether and when players may communicate their unverifiable private information via cheap talk prior to taking actions.

Accordingly, we add a cheap talk stage prior to the stage where players take actions. Thus, a strategy for player i is a pair, $\sigma^i = (M^i, a^i)$, consisting of a cheap talk strategy and an action strategy. A cheap talk strategy, $M^i(s^i)$, is a function, mapping i's signals into a message $m^i \in M$, where M is the message space. An action strategy $a^i(s^i, m^i, m^j)$ is a function, mapping each tuple (s^i, m^i, m^j) into an action choice $a^i \in \{0, 1\}$.

First consider $h \geq w$. Then, given their joint information, players always want to coordinate: They want to take action 0 when $E[\theta|s^A, s^B] \leq h$, and take action 1 when $E[\theta|s^A, s^B] > h \geq w$. Consequently, if player j truthfully reveals his signal s^j, then player i wants to reveal his signal s^i. Thus, we have:

Result. If $h \geq w$, a completely informative cheap talk equilibrium exists in which players truthfully report their signals, and then take action 1 if and only if $E[\theta|s^A, s^B] > h$.

However, most real world settings feature $w > h$, where one party gains at the expense of the other's attempting to coordinate on θ. For example, in a trade game, a country gains from its positive tariff when the other country lowers its tariffs; in a coup game, the regime may reward a loyal general; and a firm may gain when its rival pursues a costly investment strategy that does not pay off. When $w > h$, a player who receives a very bad signal expects to take action 0 with very high probability, in which case he prefers miscoordination, wanting the other player to take action 1. But it is more attractive for a player to take action 1 when he believes the other player's signal is better. Therefore, a player receiving a sufficiently bad signal always wants to convince his rival that his signal is as high as possible. It follows that no completely informative cheap talk equilibrium exists.

In fact, when $w > h$, absent information aggregation incentives (e.g., in a private value game), a player always wants the other player to take action 1. If i takes action 0, she wants j to take action 1, in order to get w instead of h. If i takes action 1 (which implies $E[\theta^i|s^i] > h$), she wants j to take action 1 in order to get $E[\theta^i|s^i] > h > l$ instead of l. Thus, we have:

Result. If $w > h$, then in the private value analogue of our game, no informative cheap talk equilibrium exists.

In our common value setting, information aggregation incentives complicate players' incentives to communicate: if i takes action 1, she wants j to take action 0 whenever $E[\theta|s^i, s^j] < l$, i.e., she wants j to take action 0 whenever j's signal indicates that the coordination payoff θ is too low. Still, even with these incentives, informative cheap talk does not easily emerge in equilibrium. To see why, for any message m^k sent by player j, define $\underline{s}^k = \inf\{s^j : M^j(s^j) = m^k\}$ and $\bar{s}^k = \sup\{s^j : M^j(s^j) = m^k\}$. For any message m^k that j can send, if $E[\theta|s^i, \bar{s}^k] - w < 0$, then i wants to take action 0 even when j receives the maximal signal associated with m^k, and hence i wants j to take action 1. Similarly, if $E[\theta|s^i, \underline{s}^k] - l > 0$, then i wants to coordinate on action 1 even when j receives the least signal associated with m^k. In both cases, i wants to send a message that maximizes the probability j takes action 1. That is, unless there exists a m^k such that $E[\theta|s^i, \underline{s}^k] - l < 0 < E[\theta|s^i, \bar{s}^k] - w$, player i wants to send a message that maximizes the probability j takes action 1.[13]

Thus, a necessary condition for an equilibrium with informative cheap talk is that players' messages cannot be too informative: there must be a message that j can send that leaves player i uncertain about which action she wants j to take, i.e., $w - l > E[\theta|s^i, \bar{s}^k] - E[\theta|s^i, \underline{s}^k]$ for some m^k. In addition, message m^k must "matter enough" to swamp i's incentives to in-

[13]Indeed, if player j adopts a monotone strategy (see definition 2), then there is at most one message m^k that j can send such that this condition holds.

duce j to take action 1 following other messages. We now show that no partially informative equilibria exists in which players adopt monotone cheap talk strategies and take action 1.

Definition 1 *A cheap talk strategy M^i is informative whenever there exist signals s_1^i and s_2^i such that $M^i(s_1^i) \neq M^i(s_2^i)$. A (pure) cheap talk strategy M^i is monotone whenever the set of signals for which $M^i(s^i) = m$ is connected.*

A monotone cheap talk strategy is characterized by a disjoint collection of sets $\{O_k\}$ such that (1) $\{O_k\}$ cover $\mathbb{R}$, i.e., $\cup_{k \in \Lambda} O_k = \mathbb{R}$, where Λ is the index set, and (2) $m_k < m_{k'}$ if and only if $k < k'$, for all $k, k' \in \Lambda$. With monotone cheap talk strategies, we can extend the notion of monotone strategies to the game with cheap talk. Consider the usual ordering in $\mathbb{R}^2$ so that $(x_1', x_2') \geq (x_1, x_2)$ whenever $x_1' \geq x_1$ and $x_2' \geq x_2$.

Definition 2 *An action strategy a^i is monotone whenever $a(s^i, m^i, m^j) = 1$ implies that $a(s^{i'}, m^{i'}, m^j) = 1$ for $(s^{i'}, m^{i'}) \geq (s^i, m^i)$. A strategy $\sigma^i = (M^i, a^i)$ is monotone when both the cheap talk and action strategies are monotone.*

If player j believes that player i adopts a monotone strategy, then j's expected payoff from taking action 1 is higher if i's message is associated with a higher index k. Suppose there is a maximal index K on player j's equilibrium messages, i.e., j sends m^K for all s^j above some cutoff. Then, there exists s_K such that $E[\theta | s^i, m^K] < w$ for all signals $s^i < s_K$, and hence following sufficiently negative signals $s^i < s_K$, i wants to maximize the probability j takes action 1. Moreover, if i's monotone strategy is informative, there must be some $\hat{s}$ such that i sends some lower indexed message if $s^i < \hat{s}$ and some higher indexed message if $s^i > \hat{s}$. But for all sufficiently negative signals $s^i < \min\{\hat{s}, s_K\}$, i prefers to send a message with an index associated with a signal $s^i > \hat{s}$ rather than the message associated with signal $s^i < \hat{s}$. Thus, if there is a maximal index K on j's messages, and i's strategy is monotone, i must send the same message regardless of her signal. But then i's messages have a (trivial) maximal index, which in turn, implies that j must send the same message regardless of her signals. Therefore, the equilibrium cannot be informative.

When there is not a maximal message, we relax the equilibrium concept to allow for the possibility that a player may want to send an unboundedly high message to maximize the probability the other player takes action 1. Accordingly, we only require that monotone cheap talk strategies be part of an ϵ-equilibria ($\epsilon \to 0$). Then, even when no message maximizes the probability that the other player takes action 1, for a given ϵ, reporting any sufficiently high indexed message is a best response, and the reasoning above extends.

Proposition 8 *If $w > h$, then no informative ϵ-equilibrium exists in monotone strategies.*

7 Applications

We now expand on some of the strategic settings described by our games.

Investment Games. The payoffs in Figure 6 correspond to an investment game in which players receive signals about the uncertain payoffs that obtain if both invest. The players

	no invest	*invest*
no invest	h, h	$w', l + a\theta$
invest	$l + a\theta, w$	$\theta - c, \theta - c$

Figure 6: Investment game.

could be multinationals receiving signals about the payoffs from joint infrastructure investments in a developing country, where uncertainty about θ could reflect uncertainty about demand, economic stability, enforcement of property rights, regulatory risk, etc. So, too, it could be a *technology adoption game*, where firms decide whether to pursue a new network/platform investment that only pays off if it is broadly adopted.[14] Here, the payoff h is the known expected payoff if both firms retain the existing platform, $\theta - c$ is the uncertain payoff if the new technology became the standard (c is an investment cost), and $l + a\theta$, with $0 \leq a < 1$ is the payoff received by a firm that converts to the new technology when the other firm does not, and $l + aE[\theta] < h$ means that the game retains a coordination game structure.

[14] The literature on technology adoption focuses on coordination problems and network externalities—see Farrell and Klemperer (2007) for a review. In Farrell and Simcoe (2012), each of two firms proposes a standard, the quality of which is its private information. It is known that both proposals improve on the status quo, but each firm has a vested interest in its own proposal. They model the firms' strategic interactions as a war of attrition: the proposal of the firm that lasts longer is adopted. Although a firm's strategy contains information about its proposal quality, the nature of learning is very different from our common value setting. Moreover, their interaction does not have the nature of a coordination game.

That is, in expectation, relative to the status quo, a firm expects to lose by investing when its rival does not. Here, $a > 0$ admits the possibility that some portion of the common payoff component is still received when one firm invests but its rival does not. Via renormalization, one can show that this game is strategically equivalent to the one with $a = c = 0$. To see this, observe that the net expected payoffs from taking action 1, equation (1), becomes

$$\Delta(s^i; k^j) = Pr(s^j > k^j | s^i) \ ((1-a)E[\theta | s^j > k^j, s^i] - c - w) - Pr(s^j \leq k^j | s^i) \ \mu.$$

Dividing by $1 - a$ and letting $w' = (c + w)/(1 - a)$ and $\mu' = \mu/(1 - a)$ yields

$$\frac{\Delta(s^i; k^j)}{1-a} = Pr(s^j > k^j | s^i) \ E[\theta | s^j > k^j, s^i] - w') - Pr(s^j \leq k^j | s^i) \ \mu'.$$

The right-hand side is now the same as equation (1). Recall that symmetric finite-cutoff equilibria are the solutions to $\Delta_1(k) = \Delta(k; k) = 0$. Thus, the equilibria of the game in Figure 6 are the same as the equilibria of our initial game (Figure 1) in which μ and w have been replaced by $\frac{\mu}{1-a}$ and $\frac{w+c}{1-a}$, respectively.[15]Observe that increases in a have the same effect as increases in w and μ, i.e., they raise the cutoff $\bar{k}$.

If $a = 0$ then $l = h - \mu, w = h$ would indicate that the investment in the new platform was immediately abandoned and wasted, and that the two firms continue to use the old platform; while $l < w < h$ would indicate that failing to coordinate on a common platform hurts both firms, albeit hurting the firm that invests more, while $l < h < w$ would indicate that a firm gains when its rival invests in a project or adopts a technology standard that fails.

Coup Games. The payoffs below correspond to a coup game between officers who must decide whether to mount a coup based on private signals about the successful coup payoff θ. The coup only succeeds if both officers act. If only one officer acts, the coup fails, the status quo is preserved, and the state sanctions the sole disloyal officer with punishment μ. As footnote 2 notes, this game is strategically identical to one in which players are uncertain about the status quo payoff h, rather than θ. In case of a failed coup attempt, a loyal officer's payoff always exceeds a disloyal one, i.e., $w > h - \mu$. However, even a loyal officer's payoff under the status quo can fall if the ruler increases surveillance of the military or reduces its budget to weaken it, in which case $w < h$ (Geddes 1999). Alternatively, $w > h$

[15]Alternatively, one can replace the distribution $f(x, y, z)$ with the distribution $g(x, y, z)$, where $g(\theta/(1-a), s^A/(1-a), s^B/(1-a)) = f(\theta, s^A, s^B)$.

	no coup	*coup*
no coup	h, h	$w, h-\mu$
coup	$h-\mu, w$	θ, θ

Figure 7: Coup game.

can capture a case where the loyal officer informed the ruler and was rewarded, or the ruler raised the military budget to keep officers happy.[16]

Conflict Games. The payoffs in the left panel of Figure 8 correspond to a conflict game between two countries that simultaneously must decide whether to be peaceful or attack (Chassang and Padró i Miquel 2010). The countries receive signals about the payoff θ from peace, F is the payoff from a surprise attack on a peaceful neighbor (who receives S), and countries receive payoff W if both attack, where $F > W > S$. This payoff structure also captures a *trade game* between countries, where political leaders in each country know the payoffs associated with high tariffs, but are uncertain about the "political economy" payoffs from mutual free trade. Here, "tariff" corresponds to "attack" and "no tariff" corresponds to "peace", and $F > W > S$ captures the fact that one country gains from a unilateral tariff at the other country's expense.

	attack	*peace*
attack	W, W	F, S
peace	S, F	θ, θ

	incumbent	*challenger*
incumbent	h, h	w, l
challenger	l, w	θ, θ

Figure 8: Conflict game (left), and contribution game (right).

Lobbying Games. The right panel in Figure 8 describes a *political contribution game* between interest groups, in which the interest groups must decide whether to support a known incumbent or an unknown challenger, where the challenger only wins if both in-

[16]Our analysis extends directly when, for example, there are three officers, and a coup's success requires all three to act. When the number of agents is small, but exceeds two and θ is obtained if enough (but not all) agents act, then unless one employs a truth or noise signal structure, analysis requires explicit functional form assumptions due to the necessity of calculating pivotal probabilities.

terest groups support him. Here “incumbent” corresponds to “attack” and “challenger” corresponds to “peace”, and $w > h > l$ reflects that if the incumbent wins, he will reward a loyal interest group, and punish a disloyal one.

Relationship Games. Relabeling actions “Incumbent” and “Challenger” as “Stay” and “Breakup”, the payoffs describe a relationship game. Two individuals know the value of their relationships with their current partners; they receive signals about their payoffs if they break those relationships to form a new one together; and $h > l$ captures the fact that breaking an existing relationship is costly if a potential partner does not reciprocate.

8 Conclusion and Discussion

We analyze a class of games with both coordination and information aggregation features. Players have private information about the common value payoff received when they coordinate on an action. As a result, optimal actions convey some of their private information, which is aggregated in equilibrium. Moreover, at least one player incurs a cost when players miscoordinate. These two features are central to a host of economic and political settings.

At the outset, we posed issues in terms of interest groups that receive private signals about a challenger to an incumbent politician. We asked: Do the interest groups share their information? From a welfare perspective, does the challenger receive sufficient support? To maximize his chances of winning should an incumbent punish the disloyal more harshly or reward the loyal more generously? And, does an interest group gain if an incumbent increases its loyalty reward w or reduces the punishment μ for non-support?

We found that (1) when the incumbent rewards support and punishes non-support, no informative cheap talk equilibrium exists in monotone strategies; (2) From a welfare perspective, the challenger receives too little support (if and only if $w > l$); (3) To maximize his chances of winning, the incumbent should raise the punishment for non-support rather than the reward for support if and only if he is sufficiently likely to win the election; Ironically, the incumbent’s favoritism toward an interest group can hurt that interest group: (4) Higher rewards can hurt an interest group when the punishments for non-support are high enough so that actions are strategic complements in equilibrium; and (5) Lower punishments for non-support can hurt an interest group when these punishments are small enough that action are strategic substitutes in equilibrium.

9 Appendix

Proof of Lemma 1: Let $h(k) = Pr(s^j > k|s^i = k)$ and $g(k) = E[\theta|s^j > k, s^i = k]$, where $h'(k) < 0 < g'(k)$, so that $\Delta_1(k) = h(k)[g(k) - w + \mu] - \mu$. $\Delta_1'(k) = h'(k)[g(k) - w + \mu] + h(k)g'(k)$. If $g(k) - w + \mu \le 0$, $\Delta_1(k) < 0 < \Delta_1'(k)$. If $g(k) - w + \mu > 0$, $\Delta_1'(k) = h'(k)[g(k) - w + \mu] + h(k)g'(k) = 0$ if and only if $\frac{h'(k)}{h(k)} = -\frac{g'(k)}{g(k)-w+\mu}$, which has at most one solution if $h(k)$ and $g(k)$ are logconcave. Therefore, if $Pr(s^j > k|s^i = k)$ and $E[\theta|s^j > k, s^i = k]$ are logconcave, then $\Delta_1'(k) = 0$ has at most one solution, which must be a maximum. Thus, $\Delta_1(k)$ is either strictly increasing or single-peaked. □

Proof of Result 1: The proof builds on that in Shadmehr and Bernhardt (2011). To simplify presentation of expected net payoffs we use the following notation:

$$\alpha \equiv \frac{\sigma^2}{\sigma_\nu^2}, \quad b \equiv \frac{\sigma^2}{\sigma^2 + \sigma_\nu^2} = \frac{\alpha}{1+\alpha}, \quad a \equiv \sqrt{\sigma_\nu^2 \frac{1+2\alpha}{1+\alpha}}, \quad c \equiv \frac{\alpha}{1+2\alpha}\ a, \quad f \equiv \frac{1-b}{a}.$$

α is the signal-to-noise ratio, and the other expressions enter conditional distributions and expectations. Recall that (i) given s^i, θ is distributed normally with mean bs^i and variance $\frac{\sigma^2\sigma_\nu^2}{\sigma^2+\sigma_\nu^2} = b\sigma_\nu^2$; (ii) given s^j and s^i, θ is distributed normally with mean $\frac{\sigma^2}{\sigma_\nu^2+2\sigma^2}\ (s^j + s^i) = \frac{\alpha}{1+2\alpha}\ (s^j + s^i)$; and (iii) given s^i, s^j is distributed normally with mean bs^i and variance $\frac{\sigma^2\sigma_\nu^2}{\sigma^2+\sigma_\nu^2} + \sigma_\nu^2 = \sigma_\nu^2 \frac{1+2\alpha}{1+\alpha} = a^2$. Further, if X is normally distributed with mean m and variance v, then $E[X|X > l] = m + \sqrt{v}\ \frac{\phi(\beta)}{1-\Phi(\beta)}$, with $\beta = \frac{l-m}{\sqrt{v}}$, where ϕ and Φ are the normal pdf and cdf, respectively. Thus, the expected value θ given both i's signal and the information contained in j's decision to act is

$$\begin{aligned} E[\theta|s^j > k^j, s^i] &= E\big[E[\theta|s^j, s^i]\big|s^j > k^j, s^i\big] = E\Big[\frac{\alpha}{1+2\alpha}\ (s^i + s^j)\big|s^j > k^j, s^i\Big] \\ &= \frac{\alpha}{1+2\alpha}\ (s^i + E[s^j|s^j > k^j, s^i]) = \frac{\alpha}{1+2\alpha}\left(s^i + b\ s^i + a\frac{\phi\left(\frac{k^j - b\ s^i}{a}\right)}{1-\Phi\left(\frac{k^j - b\ s^i}{a}\right)}\right) \\ &= b\ s^i + c\ \frac{\phi\left(\frac{k^j - b\ s^i}{a}\right)}{1-\Phi\left(\frac{k^j - b\ s^i}{a}\right)}, \end{aligned} \tag{7}$$

where the last line exploits

$$\frac{\alpha}{1+2\alpha}\ (1+b)\ s^i = \frac{\alpha}{1+2\alpha}\left(1 + \frac{\alpha}{1+\alpha}\right) s^i = \frac{\alpha}{1+2\alpha}\ \frac{1+2\alpha}{1+\alpha}\ s^i = \frac{\alpha}{1+\alpha}\ s^i = bs^i.$$

Substituting equation (7) into the expected net payoff, equation (2), yields:

$$\Delta_1(k) = (1 - \Phi(fk))\ (bk - w + \mu) + c\phi(fk) - \mu,$$

where symmetry implies that the arguments of the normal probability terms simplify, as $\frac{k^j - bk^i}{a} = \frac{1-b}{a}\ k = fk$. The asymptotic behavior of $\Delta_1(k)$ follows immediately from inspection. Differentiating the expression for $\Delta_1(k)$ yields:

$$\Delta_1'(k) = -f\phi(fk)(bk - w + \mu) + b(1 - \Phi(fk)) + c(-fk)\phi(fk).$$

It follows that $\Delta_1'(k) = 0$ if and only if

$$(b + cf)k - w + \mu = \frac{b}{f}\frac{1 - \Phi(fk)}{\phi(fk)}.$$

The left-hand side is strictly increasing in k and onto, and the right-hand side is strictly decreasing in k because Φ is logconcave. Thus, there is a unique solution to $\Delta_1'(k) = 0$, call it k_m. Combining this result with the asymptotic properties of $\Delta_1(k)$ yields $\Delta_1'(k) > 0$ if $k < k_m$, and $\Delta_1'(k) < 0$ if $k > k_m$. $\square$

Proof of Result 2: When players receive separate (truth or noise) signals,

$$Pr(s^j > k|s^i) = \begin{cases} 1 - F(k) & ; s^i \leq k. \\ p^2 + (1 - p^2)[1 - F(k)] & ; s^i > k. \end{cases} \tag{8}$$

s^i is informative about s^j if and only if $s^i = s^j = \theta$, which happens with probability p^2. Otherwise, i's signal is a random draw, or j's signal is a random draw, or both. Thus, when $s^i \leq k$, the event $s^j > k$ implies that at least one signals is a random draw, and hence $Pr(s^j > k|s^i) = 1 - F(k)$. When $s^i > k$, then $s^i = s^j = \theta$ with probability p^2; and with the remaining probability $(1 - p^2)$, $Pr(s^j > k|s^i) = 1 - F(k)$. Routine calculations yield:

$$E[\theta|s^j > k, s^i] = p\ (s^i + (1 - p)E[\theta|\theta > k]). \tag{9}$$

From equations (8) and (9),

$$\begin{aligned} \Delta_1(k) &= Pr(s^j > k|s^i = k)\ (E[\theta|s^j > k, s^i = k] - w + \mu) - \mu \\ &= (1 - F(k))\ (p\ (k + (1 - p)E[\theta|\theta > k]) - w + \mu) - \mu \\ &= (1 - F(k))\ (pk - w + \mu) + p(1 - p)\int_k^\infty \theta dF(\theta) - \mu. \end{aligned}$$

Clearly, $\lim_{k\to-\infty}\Delta_1(k)=-\infty$. Log-concavity of f implies that its right tail goes to zero faster than exponential (An 1998, p. 359). Therefore, $\lim_{k\to\infty}(1-F(k))k=0$, and hence $\lim_{k\to\infty}\Delta_1(k)=-\mu$. Differentiating with respect to k yields

$$\begin{aligned}\Delta_1'(k) &= -f(k)\ (pk-w+\mu)+p(1-F(k))-p(1-p)kf(k)\\ &= -(p(2-p)k-w+\mu)\ f(k)+p(1-F(k)).\end{aligned}$$

If $\Delta_1'(k)=0$ has a solution, it must be at some $k>\frac{w-\mu}{p(2-p)}$. Given $k>\frac{w-\mu}{p(2-p)}$, $\Delta_1'(k)=0$ if and only if $\frac{p}{p(2-p)k-w+\mu}=\frac{f(k)}{1-F(k)}$. Observe that $\frac{p}{p(2-p)k-w+\mu}$ is strictly decreasing in $k>\frac{w-\mu}{p(2-p)}$, falling from $+\infty$ to 0, while $\frac{f(k)}{1-F(k)}$ is strictly increasing by logconcavity. Therefore, $\Delta_1'(k)=0$ has a unique solution, which is a maximum. $\square$

Sketch of Proof of Result 3: To calculate the symmetric net expected payoff, $\Delta_1(k)$, we condition the terms on θ, and integrate over θ. Using Bayes Rule to calculate $f(\theta|s^i)$ and $Pr(s^j>k|\theta)=\frac{e^\theta}{e^k+e^\theta}$, we solve for

$$\begin{aligned}\Delta_1(k) &= \int_{-\infty}^{\infty}(\theta-w+\mu)Pr(s^j>k|\theta)\frac{f(s^i|\theta)\ f(\theta)}{\int_{-\infty}^{\infty}f(s^i|\theta)f(\theta)d\theta}d\theta-\mu\\ &= \frac{1}{4}\left(k^2\frac{k+2(\mu-w)}{4-e^k(2-k)^2-k^2}+3\frac{k+2(\mu-w)}{1-e^k}+2\frac{1-k-(\mu-w)}{2-k}\right)-\mu.\end{aligned}$$

The limiting properties of $\Delta_1(k)$ as $k\to\pm\infty$ are revealing about its shape. We have

$$\lim_{k\to-\infty}\Delta_1(k)=\lim_{k\to-\infty}\frac{1}{4}(-k+3k+2)-\mu=-\infty,$$

and as k increases unboundedly, the first two terms go to zero, and if $1-\mu+w\neq 2$ then

$$\lim_{k\to\infty}\Delta_1(k)=\lim_{k\to\infty}\frac{1}{4}\left(2\frac{1-k-(\mu-w)}{2-k}\right)-\mu=\begin{cases}(\frac{1}{2}-\mu)^- & ;w>1+\mu\\ (\frac{1}{2}-\mu)^+ & ;w<1+\mu,\end{cases}$$

where $(\frac{1}{2}-\mu)^-$ means that the function approaches its limit from below (it is increasing), and $(\frac{1}{2}-\mu)^+$ means that the function approaches its limit from above (it is decreasing). To see this, differentiate $\frac{a-k}{b-k}$ with respect to k to get $\frac{a-b}{(b-k)^2}$, which is positive if and only if $a>b$: the limit of $\Delta_1(k)$ is positive if and only if $1-\mu+w>2$, i.e., if and only if $w>1+\mu$. One can also show that $\lim_{k\to\infty}\Delta_1(k)=(\frac{1}{2}-\mu)^-$ when $w=1+\mu$.

The asymptotic behavior of $\Delta_1(k)$ means that when $w>1+\mu$, $\Delta_1(k)$ cannot be single-peaked—it must either be monotone increasing, or it has, at least one maximum and one minimum. In fact, one can show that $\Delta_1(k)$ is monotone increasing when $w > 1+\mu$, and that it is single-peaked when $w < 1+\mu$. $\square$

Proof of Proposition 1: First, we prove two lemmas.

Lemma 5 *The best response to a cutoff strategy is a unique cutoff strategy.*

Proof of Lemma 5: $Pr(s^j > k^j|s^i) > 0$, and $Pr(s^j > k^j|s^i)$ and $E[\theta|s^j > k^j, s^i]$ increase with s^i due to affiliation. Thus, from equation (1), if $\Delta(s^i = x; k^j) = 0$, then $\Delta(s^i; k^j) > 0$ for all $s^i > x$. From Assumption 1, $\lim_{s^i\to-\infty}\Delta(s^i; k^j) < 0 < \lim_{s^i\to+\infty}\Delta(s^i; k^j)$. Thus, for every k^j, there exists a unique $s^i = k^i$ such that $\Delta(k^i; k^j) = 0$. Further, at $s^i = k^i$,

$$\left.\frac{\partial\Delta(s^i; k^j)}{\partial s^i}\right|_{s^i=k^i} > 0. \quad \square \tag{10}$$

Lemma 6 *There exists a k^* such that if $k^j > k^*$, then k^i and k^j are strategic complements, and if $k^j < k^*$, then k^i and k^j are strategic substitutes.*

Proof of Lemma 6:

$$\frac{\partial k^i(k^j)}{\partial k^j} = -\left(\frac{\partial\Delta(k^i; k^j)}{\partial k^i}\right)^{-1} \frac{\partial\Delta(k^i; k^j)}{\partial k^j}. \tag{11}$$

Rewrite equation (1) as

$$\Delta(k^i; k^j) = \int_{k^j}^{\infty} E[\theta|s^j, k^i]\ f(s^j|k^i)\ ds^j + F(k^j|k^i)\ (w-\mu) - w.$$

Recall that $\delta(k^j, k^i(k^j)) \equiv E[\theta|k^j, k^i(k^j)] - w + \mu$. Thus,

$$\frac{\partial\Delta(k^i; k^j)}{\partial k^j} = f(k^j|k^i)\ \big(-E[\theta|k^j, k^i] + w - \mu\big) \equiv -f(k^j|k^i)\ \delta(k^j, k^i), \tag{12}$$

and hence from equation (11) and equation (10), we have $sign\left(\frac{\partial k^i}{\partial k^j}\right) = sign\left(\delta(k^j, k^i)\right)$. Next, we sign δ, establishing its single-crossing properties:

$$\begin{aligned}
\frac{d\delta(k^j, k^i(k^j))}{dk^j} &= \frac{dE[\theta|k^j, k^i(k^j)]}{dk^j} \\
&= \frac{\partial E[\theta|k^j, k^i(k^j)]}{\partial k^j} + \frac{\partial E[\theta|k^j, k^i(k^j)]}{\partial k^i}\ \frac{\partial k^i}{\partial k^j} \\
&= \frac{\partial E[\theta|k^j, k^i(k^j)]}{\partial k^j} - \frac{\partial E[\theta|k^j, k^i(k^j)]}{\partial k^i}\ \left(\frac{\partial\Delta(k^i; k^j)}{\partial k^i}\right)^{-1}\ \frac{\partial\Delta(k^i; k^j)}{\partial k^j} \\
&= \frac{\partial E[\theta|k^j, k^i(k^j)]}{\partial k^j} + \frac{\partial E[\theta|k^j, k^i(k^j)]}{\partial k^i}\ \frac{f(k^j|k^i)\ \delta(k^j, k^i)}{\frac{\partial\Delta(k^i;k^j)}{\partial k^i}},
\end{aligned} \tag{13}$$

where the third equality follows from equation (11) and the fourth from equation (12). Both $\frac{\partial E[\theta|k^j,k^i]}{\partial k^i}$ and $\frac{\partial E[\theta|k^j,k^i]}{\partial k^j}$ are positive because s^i, s^j, and θ are strictly affiliated; and $\frac{\partial \Delta(k^i;k^j)}{\partial k^i} > 0$ from equation (10). Thus, $\frac{d\delta}{dk^j} > 0$ for all $\delta \geq 0$, which implies that $\delta(k^j, k^i(k^j))$ has a single-crossing property as a function of k^j. Next, we show that δ changes sign from negative (strategic substitutes) to positive (strategic complements). From Assumption 1, $\lim_{k^j \to -\infty} k^i(k^j) < \infty$ and $\lim_{k^j \to \infty} k^i(k^j) = \infty > -\infty$. To see the latter, observe that for a given k^i, $\lim_{k^j \to \infty} \Delta(k^i; k^j) = \lim_{k^j \to \infty} \int_{s^j = k^j}^{\infty} E[\theta|s^j, k^i] f(s^j|k^i) ds^j + \lim_{k^j \to \infty} Pr(s^j > k^j|k^i)(-w + \mu) - \mu = -\mu < 0$. Thus, $\lim_{k^j \to \pm\infty} \delta(k^j, k^i(k^j)) = \pm\infty$. $\square$

We now characterize symmetric monotone equilibria. From Assumption 2, $\Delta_1(k)$ has, at most two solutions. First, we show that when μ is sufficiently small, $\Delta_1(k) = 0$ has, at least one solution; and when μ is sufficiently large, $\Delta_1(k) = 0$ does not have any solution. At $\mu = 0$, $\Delta_1(k; \mu = 0) = 0$ if and only if $E[\theta|s^j > k, s^i = k] = w$, which has a unique solution for a given w because $\lim_{k \to \pm\infty} E[\theta|s^j > k, s^i = k] = \pm\infty$. This together with the continuity of $\Delta_1(k; \mu)$ in μ, and assumptions 1 (part c) and 2 imply that $\Delta_1(k) = 0$ has a solution for sufficiently small $\mu > 0$. Moreover, from Assumption 1, $\lim_{k \to \infty} Pr(s^j > k|k)\ E[\theta|s^j > k, k] < \infty$, and from Assumption 2, $\Delta_1(k)$ is either single-peaked or strictly increasing. Hence, $Pr(s^j > k|k)\ E[\theta|s^j > k, k]$ is bounded from above, and hence for sufficiently high w or μ, $\Delta_1(k)$ is uniformly negative.

Rewrite equation (2) as

$$Pr(s^j > k|k)\ E[\theta|s^j > k, k] - Pr(s^j > k|k)\ w - (1 - Pr(s^j > k|k))\ \mu. \tag{14}$$

For any k, $\frac{\partial \Delta_1(k;\mu)}{\partial \mu} < 0$.Therefore, there exists a $\mu^* > 0$ such that $\Delta_1(k)$ has a solution if $\mu < \mu^*$ and does not have a solution when $\mu > \mu^*$.

Next, we prove the stability results. First, we establish that a player's best response function is strictly increasing in μ: $\frac{\partial k^i(k^j;\mu)}{\partial \mu} > 0$, where we have made the dependency of $k^i(k^j)$ on μ explicit. Recall that $\Delta(k^i; k^j, \mu) = 0$ and $\frac{\partial \Delta(k^i;k^j,\mu)}{\partial k^i} > 0$, and hence

$$\frac{\partial k^i(k^j;\mu)}{\partial \mu} = -\frac{\partial \Delta(k^i;k^j,\mu)}{\partial \mu} \Big/ \frac{\partial \Delta(k^i;k^j,\mu)}{\partial k^i}. \tag{15}$$

Moreover, from equation (1), $\frac{\partial \Delta(k^i;k^j,\mu)}{\partial \mu} = -Pr(s^j \leq k^j|s^i = k^i) < 0$, and hence $\frac{\partial k^i(k^j;\mu)}{\partial \mu} > 0$. This together with the continuity of $k^i(k^j, \mu)$ in μ implies that, for $\mu < \mu^*$, at any symmetric equilibrium the best response function $k^i(k^j)$ must cross the 45° line—cannot be tangential. Because $\lim_{k^j \to -\infty} k^i(k^j) > -\infty$, the smallest k^j at which $k^i(k^j)$ crosses the 45° line is from above, and hence the corresponding equilibrium is stable. Moreover, if there exists another equilibrium, it must be associated with a larger k^j at which $k^i(k^j)$ crosses the 45° line form below, and hence it is unstable.

Finally, we prove comparative statics results. Suppose $\mu < \mu^*$. Let $\bar{k}$ be the largest equilibrium, and let $\underline{k}$ be the smallest equilibrium—when it exists. We must have

$$\frac{d\Delta_1(\underline{k})}{dk} < 0 < \frac{d\Delta_1(\bar{k})}{dk}. \tag{16}$$

Making w and μ explicit in the argument, we have $\Delta_1(\bar{k}(w); w) = 0$ and $\Delta_1(\bar{k}(\mu); \mu) = 0$. Together with equation (2), this yields

$$\frac{\partial \bar{k}}{\partial w} = -\frac{\partial \Delta_1(\bar{k}(w); w)}{\partial w} \Bigg/ \frac{\partial \Delta_1(\bar{k}; w)}{\partial \bar{k}}, \quad \frac{\partial \bar{k}}{\partial \mu} = -\frac{\partial \Delta_1(\bar{k}(\mu); \mu)}{\partial \mu} \Bigg/ \frac{\partial \Delta_1(\bar{k}; \mu)}{\partial \bar{k}}. \tag{17}$$

From equation (2),

$$\frac{\partial \Delta_1(k; w)}{\partial w} = -Pr(s^j > k|k) \text{ and } \frac{\partial \Delta_1(k; \mu)}{\partial \mu} = -(1 - Pr(s^j > k|k)). \tag{18}$$

This together with equation (16) implies $\frac{\partial \bar{k}}{\partial w}, \frac{\partial \bar{k}}{\partial \mu} > 0$. When $\underline{k}$ exists, the analogous argument for $\underline{k}$ shows $\frac{\partial \bar{k}}{\partial w}, \frac{\partial \bar{k}}{\partial \mu} < 0$ □.

Proof of Proposition 2: We must show $\frac{\partial \bar{k}}{\partial w} > \frac{\partial \bar{k}}{\partial \mu}$ if and only if players are sufficiently likely to take action 1. From equations (16), (17), and (18), $\frac{\partial \bar{k}}{\partial w} > \frac{\partial \bar{k}}{\partial \mu}$ if and only if $Pr(s^j > k|s^i = k) > 1 - Pr(s^j > k|s^i = k)$, that is, $Pr(s^j > k|s^i = k) > \frac{1}{2}$. □

Proof of Lemma 2: Let $E[U^i|c^i, k^j]$ be player i's ex-ante expected utility when he adopts cutoff c^i and player j adopts cutoff k^j. We begin by establishing that

$$\frac{\partial E[U^i|c^i, k^j]}{\partial c^i} = -g(c^i)\ \Delta(c^i, k^j), \text{ and } \frac{\partial E[U^i|c^i, k^j]}{\partial k^j} = -g(k^j)\ (\Delta(k^j, c^i) + w - l),$$

where $g(\cdot)$ is the pdf of s^i and s^j. We have

$$\begin{aligned}
E[U^i|c^i, k^j] &= Pr(s^i \le c^i, s^j \le k^j)\ h + Pr(s^i \le c^i, s^j > k^j)\ w \\
&\quad + Pr(s^i > c^i, s^j \le k^j)\ l + Pr(s^i > c^i, s^j > k^j) E[\theta | s^i > c^i, s^j > k^j] \\
&= Pr(s^i > c^i, s^j > k^j)(E[\theta | s^i > c^i, s^j > k^j] - w + \mu) \\
&\quad + [Pr(s^i \le c^i, s^j > k^j) + Pr(s^i > c^i, s^j > k^j)]\ w \\
&\quad + [Pr(s^i \le c^i, s^j \le k^j) - Pr(s^i > c^i, s^j > k^j)]\ h \\
&\quad + [Pr(s^i > c^i, s^j \le k^j) + Pr(s^i > c^i, s^j > k^j)]\ l
\end{aligned}$$

$$
\begin{aligned}
&= \int_{c^i}^{\infty}\int_{k^j}^{\infty}(E[\theta|s^i,s^j]-w+\mu)g(s^i,s^j)ds^i ds^j \\
&\quad +Pr(s^j>k^j)\ w+[Pr(s^j\le k^j)-Pr(s^i>c^i)]\ h+Pr(s^i>c^i)\ l \\
&= \int_{c^i}^{\infty}\int_{k^j}^{\infty}(E[\theta|s^i,s^j]-w+\mu)g(s^i,s^j)ds^i ds^j \\
&\quad -Pr(s^i>c^i)\ (h-l)+Pr(s^j>k^j)\ w+Pr(s^j\le k^j)\ h
\end{aligned}
$$

$$
= \int_{c^i}^{\infty}\int_{k^j}^{\infty}(E[\theta|s^i,s^j]-w+\mu)g(s^i,s^j)ds^i ds^j-(1-G(c^i))\ \mu+G(k^j)\ (h-w)+w,
$$

where $g(s^i, s^j)$ is the joint pdf of s^i and s^j. Rewrite equation (1) as $\Delta(s^i,k^j)\equiv\int_{k^j}^{\infty}(E[\theta|s^i,s^j]-w+\mu)\ g(s^j|s^i)ds^j-\mu$. Then,

$$
\begin{aligned}
\frac{\partial E[U^i|c^i,k^j]}{\partial c^i} &= -\int_{k^j}^{\infty}(E[\theta|c^i,s^j]-w+\mu)\ g(c^i,s^j)ds^j+g(c^i)\mu \\
&= -g(c^i)\Big(\int_{k^j}^{\infty}(E[\theta|c^i,s^j]-w+\mu)\ g(s^j|c^i)ds^j-\mu\Big) \\
&= -g(c^i)\ \Delta(c^i,k^j)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E[U^i|c^i,k^j]}{\partial k^j} &= -\int_{c^i}^{\infty}(E[\theta|s^i,k^j]-w+\mu)\ g(s^i,k^j)ds^i+g(k^j)\ (h-w) \\
&= -g(k^j)\ \left(\int_{c^i}^{\infty}(E[\theta|s^i,k^j]-w+\mu)\ g(s^i|k^j)ds^i-(h-w)\right) \\
&= -g(k^j)\ (\Delta(k^j,c^i)+\mu-(h-w)) \\
&= -g(k^j)\ (\Delta(k^j,c^i)+w-l). \qquad (19)
\end{aligned}
$$

In equilibrium, $\Delta(c^i,k^j)=\Delta(k^j,c^i)=0$, so $sign(\frac{\partial E[U^i|c^i,k^j]}{\partial k^j})=sign(l-w)$. $\square$

Proof of Proposition 3: The symmetric socially optimal cutoffs (c^i, c^j) solve

$$
\max_{c^i,c^j} E[U^i(c^i,c^j)]+E[U^j(c^i,c^j)] \ \text{ s.t. } \ c^i=c^j.
$$

Equivalently, using symmetry, they solve

$$
2\max_{c^i,c^j} E[U^i(c^i,c^j)] \ \text{ s.t. } \ c^i=c^j. \qquad (20)
$$

Denote the maximand when it exists, by k^s. The first-order condition is $\nabla E[U^i(k^s,k^s)]=0$:

$$
\begin{aligned}
\nabla E[U^i(k^s,k^s)] &= \frac{\partial E[U^i(k^s,k^s)]}{\partial c^i}+\frac{\partial E[U^i(k^s,k^s)]}{\partial c^j} \\
&= -g(k^s)\ \Delta(k^s,k^s)-g(k^s)\ (\Delta(k^s,k^s)+w-l) \\
&= -g(k^s)(2\Delta_1(k^s)+w-l). \qquad (21)
\end{aligned}
$$

Thus, the first-order condition becomes $-g(k^s)(2\Delta_1(k^s) + w - l) = 0$ or

$$\Delta_1(k^s) = \frac{l - w}{2}. \tag{22}$$

By Assumption 2, $\Delta_1(k)$ is either single-peaked or strictly increasing. From equations (21) and (22), when the optimization problem (20) has solutions, the largest one is a (local or global) maximum. The finite-cutoff equilibria of the game are solutions to $\Delta_1(k) = 0$. Thus, whenever the optimization problem (20) has an interior solution k_s, then $k_s < \bar{k}$ if $w > l$, $k_s > \bar{k}$ if $w < l$, and $k_s = \bar{k}$ if $w = l$.

Finally, $\lim_{k\to-\infty} E[U^i|k,k] = E[\theta]$ and $\lim_{k\to\infty}[U^i|k,k] = h$, and $\lim_{k\to-\infty}[U^i|k,k] < E[U^i|k^s,k^s]$. Hence a sufficient condition for (k^s,k^s) to be the unique global maximum is that $h < E[\theta]$. $\square$

Proof of Lemma 3: First, consider the $\bar{k}$ equilibrium. From Lemma 6, it features strategic substitutes if and only if $\delta(\bar{k},\bar{k}) = E[\theta|s^j = \bar{k}, s^i = \bar{k}] - w + \mu < 0$. From Proposition 1, $\frac{\partial \bar{k}}{\mu} > 0$, and hence, $\delta(\bar{k}(\mu),\bar{k}(\mu);\mu)$ is strictly increasing in μ. At $\mu = 0$, $w = E[\theta|s^j > \bar{k}, s^i = \bar{k}] > E[\theta|s^j = \bar{k}, s^i = \bar{k}]$, and hence $\delta(\mu = 0) < 0$. Therefore, by continuity, $\delta(\mu) < 0$ for sufficiently small μ. That best response functions feature a unique switch from strategic substitutes to complements together with continuity ensures that $\bar{k}$ features strategic complements when μ is close to $\mu^*(w)$. Moreover, the shape of best response functions ensure that when $\underline{k}$ exists, best response function must cross the 45° line from below, and hence have positive slopes. $\square$

Proof of Lemma 4. We proceed in three steps.

Step 1: Prove that $\pi(\theta; k_L^j, k_R^j)$ has at most two sign changes, and that there exists a w_I such that if $w < w_I$, then $\pi(\theta; k_L^j, k_R^j)$ has exactly two sign changes.

Proof of Step 1: $[\theta-(w-\mu)]\ Pr(k_L^j < s^j < k_R^j|\theta)$ has a unique root at $w-\mu$. Let $g(\theta) \equiv Pr(k_L^j < s^j < k_R^j|\theta)$. Differentiating $[\theta-(w-\mu)]\ Pr(k_L^j < s^j < k_R^j|\theta)$ with respect to θ yields $g(\theta)+[\theta-(w-\mu)]\ g'(\theta)$. Thus, $g(\theta)+[\theta-(w-\mu)]\ g'(\theta) > 0$ if and only if $g(\theta) > -[\theta-(w-\mu)]\ g'(\theta)$. If $\theta > w-\mu$, this inequality is equivalent to $-\frac{1}{\theta-(w-\mu)} < \frac{g'(\theta)}{g(\theta)}$. The left-hand side is strictly increasing and the right-hand side is strictly decreasing because $g(\theta)$ is log-concave. Thus, they can cross, at most once. Moreover, $\lim_{\theta\to\infty}[\theta-(w-\mu)]\ g(\theta) = 0$ because of log-concavity (An 1998). Thus, the crossing happens exactly once, and its a maximum.

Therefore, $(\theta-(w-\mu))\ Pr(k_L^j < s^j < k_R^j|\theta)$ has a unique maximum for $\theta \in (w-\mu,\infty)$, and is negative for $\theta \in (-\infty, w-\mu)$. Further, $\lim_{\theta\to\infty}(\theta-(w-\mu))\ Pr(k^j_L < s^j < k^j_R|\theta) = 0$. Since $\pi(\theta; k^j_L, k^j_R)$ equals $(\theta-(w-\mu))\ Pr(k^j_L < s^j < k^j_R|\theta)$ minus $\mu > 0$, it inherits the shape and its optima (see Figure 5). From equation (5), for any θ, there exists a w_θ such that $\pi(\theta; k^j_L, k^j_R) > 0$ if and only if $w < w_\theta$. Let $\hat{\theta}$ be the unique maximand of π, then it follows that there exists a $w_{\hat{\theta}}$ such that π has exactly two sign changes if and only if $w < w_{\hat{\theta}}$. $\square$

One can establish an analogous result by bounding $\mu < \mu_{\hat{\theta}}$ for some $\mu_{\hat{\theta}}$.

STEP 2: Prove that best response to a bounded interval strategy *cannot* take a cutoff form with a *finite* cutoff. In particular, $\Gamma(s^i; k^j_L, k^j_R)$ has either 2 sign changes or no sign change. If it has no sign changes, then player i's best response (to a bounded interval strategy) is to never take action 1, i.e., it takes a cutoff form with associated cutoff ∞. If it has two sign changes, then it takes a bounded interval form.

PROOF OF STEP 2: Step 1 established that $\pi(\theta; k^j_L, k^j_R)$ has at most two sign changes. By Karlin's theorem, $\Gamma(s^i; k^j_L, k^j_R)$ has *at most* two sign changes. Moreover, $\lim_{s^i\to\pm\infty}\Gamma[s^i; k^j_L, k^j_R] < 0$. This implies that (i) $\Gamma(s^i; k^j_L, k^j_R)$ either has no sign change or two sign changes, and (ii) *if* it has no sign change, then $\Gamma(s^i; k^j_L, k^j_R) < 0$ for all s^i, i.e., player i never takes action 1.

Moreover, if $\Gamma(s^i; k^j_L, k^j_R)$ has two sign changes, then by Karlin's theorem and the pattern of sign changes in $\pi(\theta; k^j_L, k^j_R)$, $\Gamma(s^i; k^j_L, k^j_R)$ is first negative, then positive, and then negative again, which implies a bounded interval strategy. $\square$

STEP 3: Fix a signal s^i. From inspection of (4), there exists a w_{s^i} such that $\Gamma(s^i; k^j_L, k^j_R) > 0$ if $w < w_{s^i}$. That $\Gamma(s^i; k^j_L, k^j_R) > 0$ for some s^i is inconsistent with never taking action 1. This together with Step 1, implies that if $w < \min\{w_{\hat{\theta}}, w_{s^i}\}$, then the best response to bounded interval strategy (k^j_L, k^j_R) is a bounded interval strategy. $\square$

Proof of Corollary 1: The "only if" part is immediate from the continuity of $\Gamma[s^i; k_L, k_R]$ in s^i for $i \in \{A, B\}$. Next, we prove the "if" part. If $\Gamma[s^i; k_L, k_R]$ changes sign at both k_L and k_R, then from Lemma 4, $\Gamma[s^i; k_L, k_R] > 0$ if and only if $s^i \in (k_L, k_R)$. Otherwise, Γ does not change sign at k_L or k_R or both. We consider two cases:

Case I: Suppose Γ changes sign at only one of k_L and k_R. WLOG, suppose Γ changes sign at k_R, but not k_L. Then k_L must be a local maximum (minimum). Adding a small positive (negative) constant ϵ to π in equation (6) adds a constant to Γ,

$$\int_{\theta=-\infty}^{\infty} (\pi(\theta; k^j_L, k^j_R) + \epsilon)\ f(\theta|s^i)\ d\theta = \Gamma(s^i; k^j_L, k^j_R) + \epsilon,$$

and hence creates k_{Ll} and k_{Lr} at which Γ changes sign and $k_{Ll} < k_L < k_{Lr} < k_R$. Thus, Γ changes sign at least three times: at k_{Ll}, k_{Lr}, and k_R. But from the proof of Lemma 4, $\pi + \epsilon$ has at most two sign changes, which together with the TP_3 property of $f(\theta|s^i)$ implies that $\Gamma + \epsilon$ has at most two sign changes, which is a contradiction.

Case II: Suppose Γ does not change sign at k_L and k_R. If k_L and k_R are both local maxima or both local minima, an argument similar to Case I leads to a contradiction. If one is a local maximum and the other is a local minimum, then there exists a k_M with $k_L < k_M < k_R$ at which Γ changes sign. Now apply the argument in Case I with k_M instead of k_R. $\square$

Proof of Proposition 6: From Corollary 1, $\Gamma[k_L; k_L, k_R] = 0$ and $\Gamma[k_R; k_L, k_R] = 0$. From equation (4), if $s^i \in \{k_L, k_R\}$, then

$$Pr(k_L < s^j < k_R|s^i)\ (E[\theta|s^i, k_L < s^j < k_R] - w + \mu) = \mu > 0,$$

and hence $(E[\theta|s^i, k_L < s^j < k_R] - w + \mu) > 0$.

Observe that $Pr(k_L^j < s^j < k_R^j|s^i)$ and $E[\theta|s^i, k_L^j < s^j < k_R^j]$ are strictly increasing in k_R^j. Thus, from equation (4), if $\Gamma[s^i; k_L^j, k_R^j] \geq 0$, then $\Gamma[s^i; k_L^j, \gamma_R'] > 0$ for all $\gamma_R' > k_R^j$. Because (k_L, k_R) is an equilibrium, we have $\Gamma[k_L; k_L, k_R] = 0 < \Gamma[k_L; k_L, \infty] = \Delta_1(k_L; k_L)$, and hence $\Delta_1(k, k)$ has at least one solution and finite-cutoff equilibria exist. Define $BR_L^i(k_L^j, k_R^j) = \min\{s^i : \Gamma[s^i; k_L^j, k_R^j] = 0\}$ when this minimum exists, and observe that $k_L = BR_L^i(k_L, k_R)$. Thus, $BR_L^i(k_L^j, k_R^j) > BR_L^i(k_L^j, \gamma_R')$ for all $k_R^j < \gamma_R'$ for which $BR_L^i(k_L^j, k_R^j)$ and $BR_L^i(k_L^j, \gamma_R')$ exist. Therefore, $k_L = BR_L^i(k_L, k_R) > BR_L^i(k_L, \infty) \in \mathbb{R}$.

Suppose $k_L \leq \bar{k}$. If $k_L = \bar{k}$, then $\bar{k} = k_L = BR_L^i(k_L, k_R) > BR_L^i(k_L, \infty) = BR_L^i(\bar{k}, \infty) = \bar{k}$, a contradiction. If $k_L < \bar{k}$, then by the stability of the $\bar{k}$ equilibrium, $k_L < BR_L^i(k_L, \infty)$, a contradiction. The proof for $k_L < \underline{k}$ follows analogously using the instability of $\underline{k}$. $\square$

Proof of Proposition 7: From Corollary 1, (k_L, k_R) is a symmetric bounded interval equilibrium strategy if and only if $\Gamma[k_L; k_L, k_R] = \Gamma[k_R; k_L, k_R] = 0$. From equation (4),

$$-w = \frac{\mu}{Pr(\rho_j = 1|k_L)} - \mu - E[\theta|k_L, \rho_j = 1] = \frac{\mu}{Pr(\rho_j = 1|k_R)} - \mu - E[\theta|k_R, \rho_j = 1]. \quad (23)$$

Rearranging the equations yields,

$$\mu\left\{\frac{1}{Pr(\rho_j = 1|k_L)} - \frac{1}{Pr(\rho_j = 1|k_R)}\right\} = E[\theta|k_L, \rho_j = 1] - E[\theta|k_R, \rho_j = 1],$$

which implies

$$\frac{\mu}{Pr(\rho_j = 1|k_L)\ Pr(\rho_j = 1|k_R)} = \frac{E[\theta|k_L, \rho_j = 1] - E[\theta|k_R, \rho_j = 1]}{Pr(\rho_j = 1|k_R) - Pr(\rho_j = 1|k_L)}. \tag{24}$$

The left-hand side is positive. Therefore, a necessary condition for the equilibrium to exist is that the right-hand side be positive. Since $E[\theta|k_R, \rho_j = 1] > E[\theta|k_L, \rho_j = 1]$, existence requires that $Pr(s^j \in (k_L, k_R)|s^i = k_L) > Pr(s^j \in (k_L, k_R)|s^i = k_R)$. With normality, $Pr(s^j \in (k_L, k_R)|s^i = k_L) > Pr(s^j \in (k_L, k_R)|s^i = k_R)$ if and only if $|bk_L - (k_L + k_R)/2| < |bk_R - (k_L + k_R)/2|$, i.e., if and only if $|k_L| > |k_R|$. Further, $|k_L| > |k_R|$ and $k_L < k_R$ imply $k_L < 0$. This proves the "only if" part. To prove the "if" part, fix σ^2, σ_ν^2, and (k_L, k_R) with $k_L < E[\theta] = 0$ and $|k_L| > |k_R|$. The right-hand side of the equation (24) is positive. Therefore, there exists a $\mu > 0$ such that this equation holds. Finally, substitute that μ into equation (23). There exists a w that satisfies this equation. $\square$

Proof of Proposition 8: In the text, we prove the result whenever player j's equilibrium messages have a maximal index. Now, suppose the players' equilibrium messages do not have a maximal index. From Assumption 1, for every x, $\lim_{s^i \to -\infty} E[\theta|s^j > x, s^i] = -\infty$ and $\lim_{s^i \to -\infty} Pr(s^j > x|s^i) = 0$. Thus, $\lim_{s^i \to -\infty} Pr(w > E[\theta|s^i, m^j]) = 1$, implying that i expects to take action 0 with probability going to one, with expected payoff $\lim_{s^i \to -\infty} h + Pr(a^j(s^j, m^i, m^j) = 1|s^i)(w - h)$. But then when i receives such a sufficiently negative signal, his payoff increases in the index of the message that he sends, since $Pr(a^j(s^j, m^i, m^j) = 1|s^i)$ weakly increases in that index. Further, if there is no maximal index on i's messages, then for any s^j, there exist sufficiently high indexed messages that i can send such that for all messages with index k exceeding some $k(s^j)$, $E[\theta|s^j, m^k] > w$, so that higher indexed messages do induce j to take action 1 with probability going to one, yielding i an expected payoff exceeding that with the message associated with his negative signal. But then there must be a maximal index on i's messages, else a contradiction obtains. But, we have ruled out informative equilibria when a player's equilibrium messages have a maximal index. $\square$

10 References

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